MONOGRAPHS IN INEQUALITIES 14

Weighted Energy Estimates for Convex Functions, Convex Vectors and Subsolution of Partial Differential Equation

Selected topics in reverse Poincare type inequalities

Julije Jakšetić, Josip Pečarić and Muhammad Shoaib Saleem

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Preface

The role of convex sets, convex function and their generalizations get rapid development due to its enormous use in applied mathematics specially in non-linear programming, optimization theory and hedging strategies.

Due to the advancement in applied mathematics there was necessity to extend the notion of convexity. Recently several extensions have been made for the concept of convexity among which few are pseudo convex , quasi convex, invex function , φ-convex function, s-convex function , h-convex function, half convex and exponentially convex functions. The definition of convexity have deep relations with the theory of inequalities because the definition of convex function is itself an inequality and many important inequalities follows from convexity.

Usually the payoff function of the various options (for example, European and American options) in mathematical finance is convex and this property leads to the corresponding value function to be convex with respect to the underlying stock price. Traders and practitioners dealing with real-world financial markets use the value function to construct an optimal hedging process of the options. When the value function is unknown, they use the above property to construct uniform approximations to the unknown optimal hedging process. In this construction one has to pass some weighted integrals involving weak partial derivative of the value function.

The regularity theory for solutions of certain parabolic partial differential equations is a well developed topic, but when it comes to subsolutions and supersolutions a lot remains to be done. Subsolutions are often auxiliary tools as in the celebrated Perron method. They appear as solutions to obstacle problems and variational inequalities. Weak subsolutions and weak supersolutions are not assumed to be differentiable in any sense- part of the theory is to prove that they have Sobolev derivatives. The Sobolev regularity of the weak subsolution in case of the Laplace operator is well-known classical result. The existence of the Sobolev derivatives enables one to establish the reverse Poincare inequality (or Caccioppoli type inequality) for the weak subsolutions and supersolutions of various elliptic and parabolic equations. The reverse Poincare type inequalities represent an important tool in the study of qualitative properties of solution of elliptic as well as parabolic partial differential equations. The natural generalization of univariate convex functions is the case of several variables are subharmonic functions that share many convenient attributes of the former functions.

The book is organized as follows:

In the first chapter we overview results from convex analysis that we need in the next four chapters of the book.

In the second chapter we develop the inequalities for convex functions, 4-convex function and 6-convex function. The important part of the chapter is to approximate arbitrary convexity or generalized convexity by the smooth functions, using classical mollification technique. We close this chapter with weighted energy estimates for $(2,2)-$convex functions.

In the third chapter we first prove reversed Poincaré inequality for the difference of vectors that belong to the class $\chi_{[1,j]}^{[j+1,n]}[a,b]$, then we prove that an arbitrary convex vector has weak derivative. Using mollification, we give energy estimate for two arbitrary $4-$convex vectors that belongs to $\Upsilon_{[j+1,n]}^{[1,j]}[a,b]$.

In the fourth chapter we give weighted energy estimate for a difference of subharmonic function over smooth domain. We prove existence of Sobolev gradient and its square integrability with respect to the weight function on the ball. Then we give, weighted estimate for the smooth subsolution of the heat and telegraph equation, and the approximation of weak subsolutions by smooth ones. The weighted reverse Poincaré type inequalities are obtained in case of: subharmonic functions, wave equation, elliptic subsolutions, parabolic subsolutions and bounded smooth domains.

In the fifth chapter, we deal with higher order partial differential equations such as $n-$dimensional beam equation and fourth order Laplace equation with n variables.

Contents

Chapter 1

Basic results on convexity

1.1 Different types of convexity

In this section we give definitions and some properties of various types of convexity that are used in this book. Most of these material can be found in [53].

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \tag{1.1}$$

for every $x, y \in I$ and every $\lambda \in [0,1]$. If the inequality (1.1) *is reversed, then f i said to be concave.*

Definition 1.2 *Let f a real function defined on $[a,b]$. The $n-th$ divided difference of f at mutually different knots $x_0, x_1, x_2, \ldots, x_n \in [a,b]$ is defined recursively by*

$$[x_i]f = f(x_i) \; i = 0, 1, \ldots, n,$$

and

$$[x_0, x_1, \ldots, x_k]f = \frac{[x_1, x_2, \ldots, x_n]f - [x_0, x_1, \ldots, x_{n-1}]f}{x_n - x_0}.$$

Definition 1.3 *Let $n \in \mathbb{N}_0$. A function $f : [a,b] \to \mathbb{R}$ is said to be $n-$convex on $[a,b]$ if and only if for every choice of $n+1$ distinct knots $x_0, x_1, x_2, \ldots, x_n \in [a,b]$*

$$[x_0, x_1, \ldots, x_k]f \geq 0. \tag{1.2}$$

If the inequality in (1.2) *is reversed, the function f is said to be $n-$concave on $[a,b]$.*

Remark 1.1 *Particulary, $0-$convex functions are nonnegative functions, $1-$convex functions are nondecreasing functions, $2-$convex functions are convex functions.*

Theorem 1.1 *If $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \geq 0$.*

Theorem 1.2 *If $f^{(n)}$ is $n-$convex on $[a,b]$, for $n \geq 2$, then $f^{(k)}$ exists and is $(n-k)-$convex for $1 \leq k \leq n-2$.*

Definition 1.4 *Let $I_1 = [a,b]$, $I_2 = [c,d]$. The $(n,m)-$divided difference of a function $f : I_1 \times I_2 \to \mathbb{R}$ at mutually different knots $x_0, x_1, \ldots, x_n \in I$ and $y_0, y_1, \ldots, y_m \in J$ is defined by*

$$\begin{aligned}\begin{bmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_m \end{bmatrix} f &= [x_0, x_1, \ldots, x_n]([y_0, y_1, \ldots, y_m]f) \\ &= [y_0, y_1, \ldots, y_m]([x_0, x_1, \ldots, x_n]f) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{f(x_i, x_j)}{\omega'(x_i)\omega'(y_j)}.\end{aligned}$$

where,

$$\omega(x) = \prod_{i=0}^{n}(x - x_i); \; \omega(y) = \prod_{j=0}^{m}(y - y_j).$$

Definition 1.5 *A function $f : I_1 \times I_2 \to \mathbb{R}$ is said to be $(n,m)-$convex or convex of order (n,m) if at mutually different knots $x_0, x_1, \ldots, x_n \in I$ and $y_0, y_1, \ldots, y_m \in J$*

$$\begin{bmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_m \end{bmatrix} f \geq 0.$$

Theorem 1.3 *If the partial derivative $f^{(n+m)}_{x^n y^m}$ of f exists, then f is (n,m)-convex if and only if $f^{(n+m)}_{x^n y^m} \geq 0$.*

Definition 1.6 *Let I be an interval in $\mathbb{R}$. The n-dimensional vector $F : I \to \mathbb{R}^n$*

$$F(x) = \big(f_1(x), f_2(x), \ldots, f_n(x)\big) \tag{1.3}$$

is called convex if

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f(y)$$

for all $i = 1, 2, \ldots, n$, $\lambda \in [0,1]$ and all $x, y \in I$.

Definition 1.7 *The n-dimensional vector $F : I \to \mathbb{R}^n$ is called smooth convex if*

$$\frac{d^2}{dx^2} f_i(x) \geq 0, \; \textit{for all } \; i = 1, 2, \ldots, n.$$

The vector addition and scalar multiplication is defined in the usual way:
if

$$F(x) = \big(f_1(x), f_2(x), \ldots, f_n(x)\big)$$

and

$$G(x) = \big(g_1(x), g_2(x), \ldots, g_n(x)\big),$$

then the vector addition is defined as

$$F(x) + G(x) = \Big(f_1(x) + g_1(x), f_2(x) + g_2(x), \ldots, f_n(x) + g_n(x)\Big)$$

and scalar multiplication as

$$\alpha F(x) = \Big(\alpha f_1(x), \alpha f_2(x), \ldots, \alpha f_n(x)\Big).$$

The vector composition is defined as follows

$$F \circ G(x) = F(G(x)) = \Big(f_1(g_1(x)), f_2(g_2(x)), \ldots, f_n(g_n(x))\Big).$$

Definition 1.8 *The vector F is said to be increasing vector if f_i are increasing functions for all $i = 1, 2, \ldots, n$.*
The vector F is said to be decreasing vector if f_i are decreasing functions for all $i = 1, 2, \ldots, n$.

Let $\chi_{[1,j]}^{[j+1,n]}[a,b]$ be the class of vectors having convex function on its first j components and remaining $n-j$ components are concave on the interval $[a,b]$ and let $\chi_{[j+1,n]}^{[1,j]}[a,b]$ be the class of vectors having concave functions on its first j components and remaining are convex on the interval $[a,b]$. It is obvious that if $F \in \chi_{[1,j]}^{[j+1,n]}[a,b]$ then $-F(x) \in \chi_{[j+1,n]}^{[1,j]}[a,b]$.

The proofs of two following propositions can be found in [51] and [53].

Proposition 1.1 *For convex vectors, we have*

(i) Adding two convex vectors, we obtain also a convex vector.

(ii) Multiplying a convex vector by a positive scalar is also a convex vector.

(iii) If $F : I \to \mathbb{R}$ is a convex vector and $G : \mathbb{R} \to \mathbb{R}$ is increasing vector then GoF is also convex vector.

Proposition 1.2 *Let F ,$G \in \chi_{[1,j]}^{[j+1,n]}[a,b]$ then*

(i) $F + G \in \chi_{[1,j]}^{[j+1,n]}[a,b]$.

(ii) For any positive scalar α

$$\alpha F \in \chi_{[1,j]}^{[j+1,n]}[a,b].$$

(iii) Let $F \in \chi_{[1,j]}^{[j+1,n]}[a,b]$*, and* G *is the vector such that* f_i *are increasing function,* $i = 1, \ldots, j$*, and* f_i *are decreasing functions for all* $i = j+1, \ldots, n$*. Then*

$$G \circ F \in \chi_{[1,j]}^{[j+1,n]}[a,b].$$

1.2 Convexity of a mollification

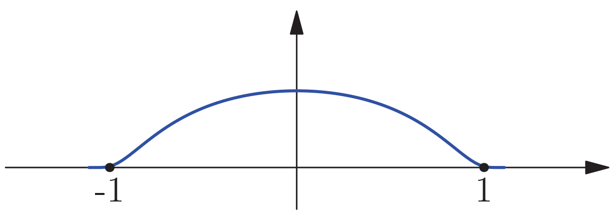
In this book we rely heavily on mollification technique. This is just tool that will allow us to build smooth approximations to given functions.

Definition 1.9 *The function* $\eta \in C^\infty(\mathbb{R})$*,*

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{x^2-1}\right), & x \leq 1, \\ 0, & x > 1, \end{cases}$$

where C *is a constant such that* $\int\limits_{\mathbb{R}} \eta(x)dx = 1$*, is called standard mollifier.*

The graph of this function is shown below.



For each $\varepsilon > 0$, let

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right),$$

and

$$I_\varepsilon = \{x \in I | \ \operatorname{dist}(x, \partial I) > \varepsilon\}.$$

Definition 1.10 *Let* I *be an open interval in* $\mathbb{R}$*. For a locally integrable function* $f : I \to \mathbb{R}$ *its mollification is*

$$f_\varepsilon(x) = (\eta_\varepsilon * f)(x), \ \ x \in I_\varepsilon,$$

i.e.

$$f_\varepsilon(x) = \int\limits_{-\varepsilon}^{\varepsilon} f(x-y)\eta_\varepsilon(y)dy = \int\limits_{I} f(y)\eta_\varepsilon(x-y)dy, \ \ x \in I_\varepsilon.$$

Proof of the next theorem can be found in [14].

Theorem 1.4

(i) $f_\varepsilon \in C^\infty(I_\varepsilon)$.

(ii) $f_\varepsilon \to f$ *a.e. as* $\varepsilon \to 0$.

(iii) *If* $f \in C(I)$, *then* $f_\varepsilon \to f$ *uniformly on compact subsets of* I.

(iv) *If* $1 \le p < \infty$ *and* $f \in L^p_{loc}(I)$, *then* $f_\varepsilon \to f$ *in* $L^p_{loc}(I)$.

Theorem 1.5 *If function* f *is convex, then its mollification* f_ε *is also convex.*

Proof. For $x_1,\ x_2 \in I_\varepsilon,\ \lambda \in [0,1]$, we have

$$\begin{aligned}
f_\varepsilon(\lambda x_1 + (1-\lambda)x_2) &= \int_{-\varepsilon}^{\varepsilon} f(\lambda x_1 + (1-\lambda)x_2 - y)\,\eta_\varepsilon(y)dy \\
&= \int_{-\varepsilon}^{\varepsilon} f(\lambda(x_1-y) + (1-\lambda)(x_2-y))\,\eta_\varepsilon(y)dy \\
&\le \int_{-\varepsilon}^{\varepsilon} [\lambda f(x_1-y) + (1-\lambda)f(x_2-y)]\,\eta_\varepsilon(y)dy \\
&= \int_{-\varepsilon}^{\varepsilon} \lambda f(x_1-y)\eta_\varepsilon(y)dy + \int_{-\varepsilon}^{\varepsilon} (1-\lambda)f(x_2-y)\eta_\varepsilon(y)dy \\
&= \lambda f_\varepsilon(x_1) + (1-\lambda)f_\varepsilon(x_2).
\end{aligned}$$

□

Chapter 2

The weighted energy inequalities for convex functions

2.1 The weighted square integral inequalities for the first derivative of the function of a real variable

We consider the pair of twice continuously differential functions f and g defined on the closed bounded interval $[a,b]$. We assume that the function g is convex and the following requirement is satisfied:

$$|f''(x)| \le g''(x), \quad a \le x \le b. \tag{2.1}$$

Let us introduce a family of nonnegative twice continuously differentiable weight functions $H : [a,b] \to \mathbb{R}$ which satisfy the following conditions

$$H(a) = H(b) = 0, \quad H'(a) = H'(b) = 0. \tag{2.2}$$

Theorem 2.1 *Let $f,\ g : [a,b] \to \mathbb{R}$ be two twice continuously differentiable functions which satisfy the requirement* (2.1) *and let $H : [a,b] \to \mathbb{R}$ be arbitrary nonnegative weight* *function such that condition* (2.2) *is fulfilled. Then the following inequality is valid*

$$\int_a^b \left(f'(x)\right)^2 H(x)dx \le \int_a^b \left[\left(\frac{f(x)}{2}\right)^2 + \left(\sup_{a\le t\le b} |f(t)|\right) g(x)\right] |H''(x)|dx. \tag{2.3}$$

Proof. Using the integration by parts

$$\begin{aligned}\int_a^b \left(f'(x)\right)^2 H(x)dx &= f(x)f'(x)H(x)\Big|_a^b - \int_a^b (f'H)'(x)f(x)dx \\ &= -\int_a^b f(x)f'(x)H'(x)dx - \int_a^b f(x)f''(x)H(x)dx \\ &= -\frac{1}{2}\int_a^b (f^2)'(x)H'(x)dx - \int_a^b f(x)f''(x)H(x)dx. \end{aligned} \tag{2.4}$$

Similarly, using $H'(a) = H'(b) = 0$,

$$\int_a^b (f^2)'(x)H'(x)dx = -\int_a^b f^2(x)H''(x)dx.$$

Now (2.4) becomes

$$\begin{aligned}\int_a^b \left(f'(x)\right)^2 H(x)dx &= \frac{1}{2}\int_a^b f^2(x)H''(x)dx - \int_a^b f(x)f''(x)H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \int_a^b |f(x)||f''(x)|H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b} |f(t)| \int_a^b |f''(x)|H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b} |f(t)| \int_a^b g''(x)H(x)dx \\ \text{(repeated int. by parts)} &= \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b} |f(t)| \int_a^b g(x)H''(x)dx. \end{aligned}$$

□

Corollary 2.1 *Under the same conditions as in the Theorem* 2.1, *the following bound is valid*

$$\int_a^b \left(f'(x)\right)^2 H(x)dx \leq \|f\|_\infty \left(\frac{1}{2}\|f\|_p + \|g\|_p\right) \|H''\|_q \tag{2.5}$$

where $1 \leq p \leq \infty$, *and* p *and* q *are conjugate exponents.*

Proof. We apply Hölder inequality to the right-hand side of estimate (2.3).

Remark 2.1 *Let us notice that dominance* (2.1) *is equivalent to the existence of decomposition of the function* f *as the difference of two twice continuously differentiable convex functions,* f_1 *and* f_2, *such that,* $f(x) = f_1(x) - f_2(x)$, $a \leq x \leq b$ *and* $g(x) = f_1(x) + f_2(x)$. *Indeed,* $|f''(x)| \leq g''(x)$ *is equivalent* $-g''(x) \leq f''(x) \leq g''(x)$, *that is,*

$$f''(x) + g''(x) \geq 0, \quad g''(x) - f''(x) \geq 0.$$

The latter means that the functions

$$f_1(x) = \frac{1}{2}(f(x) + g(x)), \qquad f_2(x) = \frac{1}{2}(g(x) - f(x))$$

are convex functions such that

$$f(x) = f_1(x) - f_2(x), \qquad g(x) = f_1(x) + f_2(x). \tag{2.6}$$

Conversely, if f_1 *and* f_2 *are two twice continuously differentiable convex such that* (2.6) *is valid, then it is obvious that we have dominance* (2.1).

This remark suggests to write inequality (2.5) in a different form:

$$\begin{aligned}\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \ &\leq \ \|f_1 - f_2\|_\infty \left[\frac{1}{2}\|f_1 - f_2\|_p\right. \\ &+ \ \|f_1 + f_2\|_p] \, \|H''\|_q, \end{aligned} \tag{2.7}$$

where $1 \leq p \leq \infty$.

Corollary 2.2 *Let* f_1 *and* f_2 *be twice continuously differentiable convex functions defined on a closed bounded interval* $[a,b]$ *and let the weight function* H *be equal to*

$$H(x) = (x-a)^2(b-x)^2, \quad a \leq x \leq b.$$

Then the following estimate holds

$$\begin{aligned}\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \ &\leq \ \|f_1 - f_2\|_\infty [\frac{4\sqrt{3}}{9}\|f_1 + f_2\|_\infty \\ &+ \ \frac{2\sqrt{3}}{9}\|f_1 - f_2\|_\infty](b-a)^3. \end{aligned} \tag{2.8}$$

Proof. We have

$$H''(x) = 12x^2 - 12(a+b)x + 2(a^2 + 4ab + b^2),$$

and then,

$$\int_a^b |H''(x)| = 2(b-a)^3 \int_0^1 |6u^2 - 6u + 1| du = \frac{4\sqrt{3}}{9}(b-a)^3.$$

Finally, taking into account the latter expression in estimate (2.7), we come to the desired inequality (2.8). □

Remark 2.2 *Comparing the result stated in Corollary 2.2 with Theorem 2.1 from K. Shashiashvili and M. Shashiashvili [50], we come to the conclusion that the constant factor* $\frac{4\sqrt{3}}{9}$ *is twice less than the constant factor obtained in the latter paper.*

2.1.1 The weighted square integral estimates for the difference of derivatives of two convex functions

Now we consider two arbitrary bounded convex functions f and g on an infinite interval $[0,\infty)$. It is well known that they are continuous and have finite left and right hand derivatives $f'(x-)$, $f'(x+)$ and $g'(x-)$, $g'(x+)$ inside the open interval $(0,\infty)$. We will assume that there exists a positive number A such that if $x \geq A$, we have

$$|f'(x-)| \leq C, \quad |g'(x-)| \leq C \tag{2.9}$$

where C is a certain positive constant.
Let us assume also that the difference of the functions f and g is bounded on the infinite interval $[0,\infty)$:

$$\sup_{x\geq 0} |f(x) - g(x)| < \infty. \tag{2.10}$$

Introduce now the family of nonnegative twice continuously differentiable weight functions $H(x)$ defined on the open interval $(0,\infty)$, which satisfy the following conditions:

$$\lim_{x\to 0+} H(x) = 0, \quad \lim_{x\to\infty} H(x) = 0, \quad \lim_{x\to 0+} H'(x) = 0, \quad \lim_{x\to\infty} H'(x) = 0, \tag{2.11}$$

and

$$\int_0^\infty (|f(x)| + |g(x)|)|H''(x)|dx < \infty. \tag{2.12}$$

Theorem 2.2 *For arbitrary bounded convex functions f and g defined on $[0,\infty)$ satisfying conditions* (2.9) *and* (2.10) *and for any nonnegative twice continuously differentiable* *weight function H, $0 < x < \infty$, which satisfy conditions* (2.11) *and* (2.12), *the following energy estimate is valid:*

$$\int_0^\infty (f'(x-) - g'(x-))^2 H(x)dx \leq \frac{3}{2} \sup_{x \geq 0} |f(x) - g(x)| \int_0^\infty (|f(x)| + |g(x)|)|H''(x)|dx. \quad (2.13)$$

Proof. We will prove the theorem in two stages. In the first stage, we verify the validity of the statement for twice continuously differentiable convex functions satisfying conditions (2.9) and (2.10), and on second stage we approximate arbitrary convex functions satisfying the same conditions by smooth ones inside the interval $(0, \infty)$ in an appropriate manner. Afterwards we will pass with a limit in the previously established estimate.
Let the function F be defined as

$$F(x) = f(x) - g(x) \qquad 0 \leq x < \infty.$$

Then F is twice continuously differentiable inside the infinite interval $(0, \infty)$ and at point zero, it has finite limit $F(0+)$.
Consider the following integral on a finite interval $[\delta, b]$ and use in it the integration by parts formula (here δ and b are arbitrary strictly positive numbers),

$$\begin{aligned}\int_\delta^b F'(x)(FH)'(x)dx &= F'(x)F(x)H(x)\Big|_\delta^b - \int_\delta^b F''(x)(F(x)H(x))dx \\ &= F(b)F'(b)H(b) - F(\delta)F'(\delta)H(\delta) - \int_\delta^b F''(x)F(x)H(x)dx.\end{aligned} \quad (2.14)$$

The absolute value of the last integral

$$\begin{aligned}\left|\int_\delta^b F''(x)F(x)H(x)dx\right| &\leq \sup_{\delta \leq x \leq b} |F(x)| \int_\delta^b |f''(x) - g''(x)|H(x)dx \\ &\leq \sup_{\delta \leq x \leq b} |F(x)| \int_\delta^b (f''(x) + g''(x))H(x)dx \end{aligned} \quad (2.15)$$

since $f''(x) \geq 0, \quad g''(x) \geq 0$, for $0 < x < \infty$.
Transforming the integral on the right-hand side of inequality (2.15),

$$\begin{aligned}\int_\delta^b (f''(x) + g''(x))H(x)dx &= (f'(x) + g'(x))H(x)\Big|_\delta^b - \int_\delta^b (f'(x) + g'(x))H'(x)dx \\ &= (f'(x) + g'(x))H(x)\Big|_\delta^b - (f(x) + g(x))H'(x)\Big|_\delta^b + \int_\delta^b (f(x) + g(x))H''(x)dx.\end{aligned}$$

If we substitute the above expression in inequality (2.15), we obtain the estimate

$$\begin{aligned}\left|\int_\delta^b F''(x)F(x)H(x)dx\right| &\le \sup_{\delta\le x\le b}|F(x)|\Big\{|f'(b)+g'(b)|H(b)\\ &+ |f'(\delta)+g'(\delta)|H(\delta)+|f(b)+g(b)||H'(b)|\\ &+ \int_\delta^b |f(x)+g(x)||H''(x)|dx\Big\}.\end{aligned}$$

Thus, from equality (2.14), we come to the following bound:

$$\begin{aligned}\left|\int_\delta^b F'(x)(FH)'(x)dx\right| &\le |F(b)F'(b)|H(b)+|F(\delta)F'(\delta)|H(\delta)\\ &+\sup_{\delta\le x\le b}|F(x)|\Big\{|f'(b)+g'(b)|H(b)+|f'(\delta)+g'(\delta)|H(\delta)\\ &+|f(b)+g(b)||H'(b)|+\int_\delta^b |f(x)+g(x)|\cdot|H''(x)|dx\Big\}.\end{aligned}\tag{2.16}$$

On the other hand, since

$$\int_\delta^b F'(x)(FH)'(x)dx=\int_\delta^b (F'(x))^2H(x)dx+\int_\delta^b F(x)F'(x)H'(x)dx,$$

we have

$$\begin{aligned}&\int_\delta^b (F'(x))^2H(x)dx=\int_\delta^b F'(x)(FH)'(x)dx-\frac{1}{2}\int_\delta^b (F^2)'(x)H'(x)dx\\ &=\int_\delta^b F'(x)(FH)'(x)dx-\frac{1}{2}\Big\{F^2(x)H'(x)\Big|_\delta^b-\int_\delta^b F^2(x)H''(x)dx\Big\}\\ &=\int_\delta^b F'(x)(FH)'(x)dx-\frac{1}{2}F^2(b)H'(b)|+\frac{1}{2}F^2(\delta)H'(\delta)dx+\frac{1}{2}\int_\delta^b F^2(x)H''(x)dx\end{aligned}\tag{2.17}$$

Using inequality (2.16) in the expression (2.17), we arrive to the estimate

$$\int_\delta^b (F'(x))^2H(x)dx\le\frac{1}{2}F^2(b)|H'(b)|+\frac{1}{2}F^2(\delta)|H'(\delta)|+|F(b)F'(b)|H^(b)$$

$$
\begin{aligned}
&+|F^2(\delta)|F'(\delta)|H(\delta)+\sup_{\delta\le x\le b}|F(x)|\cdot\Big\{\frac{3}{2}\int_{\delta}^{b}(|f(x)|+|g(x)|(|H)''(x)|dx)\\
&+|f'(b)+g'(b)|H(b)+|f'(\delta)+g'(\delta)|H(\delta)\\
&+|f(b)+g(b)||H'(b)|+|f(\delta)+g(\delta)H'(\delta)\Big\}. \qquad (2.18)
\end{aligned}
$$

It is well known that any convex function is locally absolutely continuous (see, e.g., [59] Proposition 17 of Chapter 5) that is,

$$
f(x_2)-f(x_1)=\int_{x_1}^{x_2} f'(u-)du, \quad 0<x_1<x_2<\infty. \qquad (2.19)
$$

As the lefthand derivative $f'(x-)$ of the convex function f is nondecreasing function, we have

$$
f'(x_1-)\le f'(u-)\le f'(x_2-), \quad \text{if} \quad 0<x_1<u<x_2<\infty.
$$

Therefore, from (2.19), we find that

$$
f'(x_1-)(x_2-x_1)\le f(x_2)-f(x_1)\le f'(x_2-)(x_2-x_1), \qquad (2.20)
$$

where $0<x_1<x_2<\infty$.
Taking $x_1=x,\ x_2=2x$, we get

$$
f'(x-)x\le f(2x)-f(x) \quad \text{for } x>0.
$$

On the other hand, letting $x_1\searrow 0$ in inequality (2.20), we have

$$
f(x_2)-f(0+)\le f'(x_2-)x_2,
$$

that is,

$$
f(x)-f(0+)\le f'(x-)x, \quad x>0.
$$

Ultimately, we obtain the two-sided inequality

$$
f(x)-f(0+)\le f'(x-)x\le f(2x)-f(x) \quad \text{for } x>0,
$$

which gives (also for the function g)

$$
\lim_{x\to 0+} xf'(x-)=0 \quad \text{and} \quad \lim_{x\to 0+} xg'(x-)=0. \qquad (2.21)
$$

By equality (2.19) and using condition (2.9), we obtain the bound

$$
|f(b)|\le|f(A)|\le C(b-A)\le|f(A)|+Cb \quad A\le b.
$$

But since

$$
|f(A)|\le\frac{|f(A)|}{A}b \quad \text{if } A\le b.
$$

Therefore we can write, if $A \leq b$

$$|f(b)H'(b)| \leq |f(A)H'(b)| + Cb|H'(b)| \leq \left(\frac{|f(A)|}{A} + C\right) b|H'(b)| \tag{2.22}$$

and similarly, if $A \leq b$

$$|g(b)H'(b)| \leq \left(\frac{|g(A)|}{A} + C\right) b|H'(b)| \quad \text{for } A \leq b. \tag{2.23}$$

Using condition (2.11) and bounds (2.22) and (2.23), we get

$$\overline{\lim_{b\to\infty}} F^2(b)|H'(b)| \leq \sup_{0\leq x<\infty} |F(x)| \overline{\lim_{b\to\infty}} (|f(b)+g(b)|)(|H'(b)|) = 0,$$

since

$$\overline{\lim_{b\to\infty}} (|f(b)+g(b)|)(|H'(b)|) = 0.$$

Moreover, from conditions (2.9) and (2.11), we find

$$\overline{\lim_{\delta\to 0+}} F^2(\delta)|H'(\delta)| = (|f(0+)-g(0+)|)^2 \overline{\lim_{\delta\to 0+}} |H'(\delta)| = 0,$$

$$\begin{aligned}\overline{\lim_{b\to\infty}} |F(b)F'(b-)|H(b) &\leq \sup_{0\leq x<\infty} |F(x)| \overline{\lim_{b\to\infty}} (|f'(b-)| + |g'(b-)|)(|H(b)|) \\ &\leq 2C \sup_{0\leq x<\infty} |F(x)| \overline{\lim_{b\to\infty}} H(b) = 0,\end{aligned} \tag{2.24}$$

$$\overline{\lim_{b\to\infty}} (|f'(b-)| + |g'(b-)|)(|H(b)|) \leq 2C \overline{\lim_{b\to\infty}} H(b) = 0,$$

$$\overline{\lim_{\delta\to 0+}} |f(\delta)+g(\delta)||H'(\delta)| = |(f(0+)+g(0+))| \overline{\lim_{\delta\to 0+}} |H'(\delta)| = 0.$$

Using the mean value theorem, we have

$$\frac{H(\delta)}{\delta} = \frac{H(\delta)-H(0+)}{\delta} = H'(\upsilon_\delta), \quad \text{where } 0 < \upsilon_\delta < \delta,$$

therefore from condition (2.11), we deduce

$$\overline{\lim_{\delta\to 0+}} \frac{H(\delta)}{\delta} = 0. \tag{2.25}$$

Using the limit relations above and (2.21), we find

$$\overline{\lim_{\delta\to 0+}} |F(\delta)F'(\delta-)|H(\delta) \leq \sup_{0\leq x<\infty} |F(x)|) \overline{\lim_{\delta\to 0+}} |f'(\delta-) - g'(\delta-)|H(\delta)$$

$$\leq \sup_{0 \leq x < \infty} |F(x)|) \overline{\lim_{\delta \to 0+}} \left(|\delta f'(\delta -)| \frac{H(\delta)}{\delta} + |\delta g'(\delta -)| \frac{H(\delta)}{\delta} \right) = 0, \tag{2.26}$$

and similarly

$$\overline{\lim_{\delta \to 0+}} |f'(\delta -) + g'(\delta -)| H(\delta) = 0. \tag{2.27}$$

Now, in inequality (2.18), we pass with limit when $b \to \infty$ and $\delta \to 0$. Obviously, the left-hand side of the inequality increases and the right-hand side is bounded, when $b \to \infty$, $\delta \to 0$, therefore the left-hand side also converges to finite limit, so we come to the required estimate (2.13).

Next we move to the second stage of the proof. Consider two arbitrary convex functions f and g defined on $[0, \infty)$, satisfying conditions (2.9) and (2.10). We have to construct the sequences of twice continuously differentiable (in the open interval $(0, \infty)$) convex functions f_n and g_n approximating, respectively, the functions f and g inside the interval $[0, \infty)$ in an appropriate manner. To construct such sequences, we will use the following smoothing function:

$$\rho(x) = \begin{cases} C\exp[\frac{1}{x(x-2)}]; & 0 < x < 2, \\ 0; & \text{otherwise}, \end{cases}$$

where the factor C is chosen to satisfy the equality

$$\int_0^2 \rho(x) dx = 1.$$

Define for $x \in [0, \infty)$, $n \in \mathbb{N}$

$$f_n(x) = \int_0^\infty n\rho(n(x-y)) f(y) dy,$$

$$g_n(x) = \int_0^\infty n\rho(n(x-y)) g(y) dy. \tag{2.28}$$

For arbitrary fixed $\delta > 0$ consider the restriction of functions f_n and g_n on the interval $[\delta, b]$ and let $n \geq 4/\delta$. Then $nx \geq 4$ for $x \in [\delta, b]$.

After we perform in (2.28) the change of variable $z = n(x-y)$, then we find

$$f_n(x) = \int_{-\infty}^{nx} \rho(z) f\left(x - \frac{z}{n}\right) dz,$$

$$g_n(x) = \int_{-\infty}^{nx} \rho(z) g\left(x - \frac{z}{n}\right) dz,$$

Since the function ρ is equal to zero outside the interval $(0,2)$, we can write

$$f_n(x) = \int_0^2 \rho(z) f\left(x - \frac{z}{n}\right) dz,$$
$$g_n(x) = \int_0^2 \rho(z) g\left(x - \frac{z}{n}\right) dz, \tag{2.29}$$

if $x \in [\delta, b], n \geq 4/\delta$.
From definition (2.28), it is obvious that the functions f_n and g_n are infinitely differentiable, while their convexity follows from the expressions (2.29).
Now we show the uniform convergence of the sequence of functions f_n to the function f on the interval $[\delta, b]$ (similarly, the uniform convergence of g_n to g). For this purpose, we use the uniform continuity of the function f on the interval $[\frac{\delta}{2}, b]$. For fixed $\varepsilon > 0$ there exists $\widehat{\delta} > 0$ such that we have

$$|f(x_2) - f(x_1)| \leq \varepsilon \quad \text{if } |x_2 - x_1| < \widehat{\delta}, \quad x_1, x_2 \in \left[\frac{\delta}{2}, b\right].$$

Take $n \geq \max\{\frac{4}{\delta}, \frac{4}{\widehat{\delta}}\}$. Then for $0 \leq z \leq 2$ and $x \in [\delta, b]$, we get

$$\frac{z}{n} \leq \min\left\{\frac{\delta}{\widehat{\delta}}\right\}, \qquad x - \frac{z}{n} \geq \frac{\delta}{2}.$$

Hence

$$\left|f\left(x - \frac{z}{n}\right) - f(x)\right| \leq \varepsilon \quad \text{for } n \geq \max\left\{\frac{4}{\delta}, \frac{4}{\widehat{\delta}}\right\}$$

and consequently

$$|f_n(x) - f(x)| = \left|\int_0^2 \rho(z)\left(f\left(x - \frac{z}{n}\right) - f(x)\right) dz\right| \leq \varepsilon \tag{2.30}$$

$$\text{for } \in [\delta, b] \text{ and } n \geq \max\left\{\frac{4}{\delta}, \frac{4}{\widehat{\delta}}\right\}. \tag{2.31}$$

Next we need to differentiate (2.29). For this purpose, we will use the following inequality ([18], page 114) concerning convex function $f(x)$ and its left-derivative $f'(x-)$

$$f'(x_1-) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2-), \qquad 0 < x_1 < x_2 < \infty.$$

Now, if we subsitute

$$x_1 = \left(x - \frac{z}{n}\right) - h, \quad x_2 = x - \frac{z}{n},$$

where $0 < h < \frac{\delta}{4}$, we have

$$f'\left(\left((x - \frac{z}{n} - h\right) - \right) \leq \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} \leq f'\left(\left((x - \frac{z}{n}\right) - \right)$$

for $x \in [\delta, b]$, $0 \leq z \leq 2$, $0 < h < \frac{\delta}{4}$, and $n \geq \frac{4}{\delta}$.
It is well known that the left derivative of the convex function is nondecreasing and, since,

$$x - \frac{z}{n} - h \geq \frac{\delta}{4}, \qquad x - \frac{z}{n} \leq b$$

we can write

$$f'\left(\frac{\delta}{4} - \right) \leq \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} \leq f'(b-).$$

This shows that the family of functions

$$\Phi_h^{n,x}(z) = \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h}$$

is uniformly bounded by the constant $D = |f'(b-)| + |f'(\frac{\delta}{4}-)|$ if $x \in [\delta, b]$, $0 \leq z \leq 2$, $0 < h < \frac{\delta}{4}$, and $n \geq (\frac{4}{\delta})$.
Using expression (2.29), we can write

$$\frac{f_n(x) - f_n(x-h)}{h} = \int_0^2 \rho(z) \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} dz.$$

Taking limit as h tends to zero and using dominated convergence theorem, we obtain the formula

$$f_n'(x) = \int_0^2 \rho(z) f'\left(\left(x - \frac{z}{n} - \right)\right) dz \tag{2.32}$$

for $x \in [\delta, b]$ and $n \geq \frac{4}{\delta}$.
Using (2.32) let us show that for fixed $x \in [\delta, b]$, the sequence $f_n'(x)$ converges to the left-derivative $f'(x-)$.
We have

$$f_n'(x) - f'(x-) = \int_0^2 \rho(z) \left(f'\left(\left(x - \frac{z}{n} - \right)\right) - f'(x-) \right) dz, \tag{2.33}$$

where $n \geq \frac{4}{\delta}$. Choose arbitrary $\varepsilon > 0$. Since the left-derivative $f'(x-)$ is left continuous, we can find $N(\varepsilon)$ such that (for $0 \leq z \leq 2$):

$$\left| f'\left(\left(x - \frac{z}{n} - \right)\right) - f'(x-) \right| \leq \varepsilon \quad \text{if } n \geq N(\varepsilon).$$

Then we have

$$f_n'(x)-f'(x-)=\int_0^2\rho(z)\varepsilon dz=\varepsilon \qquad \text{if } x\in[\delta,b],\quad n\ge\max\left\{\frac{4}{\delta'},N(\varepsilon)\right\},$$

that is,

$$\lim_{n\to\infty}f_n'(x)=f'(x-),\quad x\in[\delta,b].$$

Similarly,

$$\lim_{n\to\infty}g_n'(x)=g'(x-)\text{if } x\in[\delta,b]. \tag{2.34}$$

Now we apply (2.18) estimate for the function $F_n(x)=f_n(x)-g_n(x)$ on $[\delta,b]$,

$$\begin{aligned}\int_\delta^b(F_n'(x))^2H(x)dx \le\ & \frac{1}{2}F_n^2(b)|H'(b)|+\frac{1}{2}F_n^2(\delta)|H'(\delta)|+|F_n(b)F_n'(b)|H^(b)\\ +\ & |F_n^2(\delta)|F_n'(\delta)|H(\delta)+\sup_{\delta\le x\le b}|F_n(x)|\\ \times\ & \{\frac{3}{2}\int_\delta^b(|f_n(x)|+|g_n(x)|(|H)''(x)|dx)+|f_n'(b)+g_n'(b)|H(b)\\ +\ & |f_n'(\delta)+g_n'(\delta)|H(\delta)+|f_n(b)+g_n(b)||H'(b)|\\ +\ & |f_n(\delta)+g_n(\delta)H'(\delta)\}.\end{aligned} \tag{2.35}$$

For $x\in[\delta,b],\ 0\le z\le 2$ and $n\ge\frac{4}{\delta}$, we have

$$f'\left(\frac{\delta}{2}-\right)\le f'\left(\left(x-\frac{z}{n}-\right)\right)\le f'(b-).$$

Multiplying this inequality by $\rho(z)$ and integrating by z over $(0,2)$ using (2.32), we have

$$f'\left(\frac{\delta}{2}-\right)\le f_n'(x)\le f'(b-),$$

and then

$$|f_n'(x)|\le|f'(b-)|+\left|f'\left(\frac{\delta}{2}-\right)\right|,\quad \text{if } x\in[\delta,b],\ n\ge\frac{4}{\delta}.$$

Similarly, for the functions $g_n(x)$, we have

$$|g_n'(x)|\le|g'(b-)|+\left|g'\left(\frac{\delta}{2}-\right)\right|,$$

From the latter bounds, we obtain

$$|F_n'(x)| \leq |f'(b-)| + |g'(b-)| + \left|f'\left(\frac{\delta}{2}-\right)\right| + \left|g'\left(\frac{\delta}{2}-\right)\right|$$

if $x \in [\delta, b]$ and $n \geq \frac{4}{\delta}$.
Hence the sequence of the functions F_n' is uniformly bounded on the interval $[\delta, b]$ for $n \geq \frac{4}{\delta}$. Thus we can apply the bounded convergence theorem in the left-hand side of inequality (2.35). Letting n to infinity, we will have

$$\begin{aligned}
\int_{\delta}^{b} (F'(x-))^2 H(x)dx \leq\ & \frac{1}{2}F^2(b)|H'(b)| + \frac{1}{2}F^2(\delta)|H'(\delta)| + |F(b)F'(b-)|H^(b) \\
+\ & |F^2(\delta)|F'(\delta-)|H(\delta) + \\
+\|f\|_{L^\infty} \times\ & \{\frac{3}{2}\int_{\delta}^{b}(|f(x)| + |g(x)|(|H)''(x)|dx) + |f'(b-) + g'(b-)|H(b) \\
+\ & |f'(\delta-) + g'(\delta-)|H(\delta) + |f(b) + g(b)||H'(b)| \\
+\ & |f(\delta) + g(\delta)H'(\delta)\}. \qquad (2.36)
\end{aligned}$$

The left-hand side of inequality (2.36) obviously increases when $b \to \infty$ and $\delta \to 0$ and the right-hand side is bounded by the assumption (2.12) and the limit relations (2.24)-(2.27). Therefore passing onto limit $b \to \infty$ and $\delta \to 0$ in inequality (2.36), we arrive to the desired estimate (2.13). □

2.2 Weighted integral inequality for the second derivative of 4-convex function

2.2.1 The case of smooth 4-convex functions and mollification of an arbitrary 4-convex function

Theorem 2.3 *Let $f_i \in C^4(I),\ i = 1,2,$ be the two convex and also $4-$convex functions, and let $h : I \to \mathbb{R}$ be the non-negative concave, weight, function having the following properties*

$$h(x) = h'(x) = h''(x) = h'''(x) = 0,\ x \in \partial I.$$

Then the following energy estimate is valid

$$\int_I \left| f_2''(x) - f_1''(x) \right|^2 h(x)dx \leq \int_I \left[\frac{(f_2(x) - f_1(x))^2}{2} - \|f_2 - f_1\|_{L^\infty} (f_1(x) + f_2(x)) \right] h^{(4)}(x)dx. \tag{2.37}$$

Proof. Let $f(x) = f_2(x) - f_1(x)$.
Then

$$\begin{aligned}
\int_I (f''(x))^2 h(x)dx &= \int_I [f''(x)][f''(x)h(x)]ds \\
&= -\int_I f'(x)f'''(x)h(x)dx - \int_I f'(x)f''(x)h'(x)dx \\
&= \int_I f(x)f^{(4)}(x)h(x)dx + \int_I f(x)f'''(x)h'(x)dx - \int_I f'(x)f''(x)h'(x)dx \\
&= \int_I f(x)f^{(4)}(x)h(x)dx - 2\int_I f'(x)f''(x)h'(x)dx - \int_I f(x)f''(x)h''(x)dx \\
&= \int_I f(x)f^{(4)}(x)h(x)dx - \int_I [f'(x)^2]'h'(x)dx - \int_I f(x)f''(x)h''(x)dx \\
&= \int_I f(x)f^{(4)}(x)h(x)dx + \int_I (f'(x))^2 h''(x)dx - \int_I f(x)f''(x)h''(x)dx \\
&= \int_I f(x)f^{(4)}(x)h(x)dx + \int_I f'(x)\left(f'(x)h''(x)\right)dx - \int_I f(x)f''(x)h''(x)dx \\
&= \int_I f(x)f^{(4)}(x)h(x)dx - \int_I f(x)f''(x)h''(x)dx - \int_I f(x)f'(x)h'''(x)dx \\
&= \int_I f(x)f^{(4)}(x)h(x)dx - 2\int_I f(x)f''(x)h''(x)dx - \frac{1}{2}\int_I [f^2(x)]'h'''(x)dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_I f(x) f^{(4)}(x) h(x) dx - 2 \int_I f(x) f''(x) h''(x) dx + \frac{1}{2} \int_I f^2(x) h^{(4)}(x) dx \\
&\leq \|f\|_{L^\infty} \int_I \left| f^{(4)}(x) \right| h(x) dx + 2 \|f\|_{L^\infty} \int_I \left| f''(x) \right| h''(x) dx \\
&+ \frac{1}{2} \int_I f^2(x) h^{(4)}(x) dx \qquad (2.38)
\end{aligned}
$$

Since

$$
\left| f^{(4)}(x) \right| = \left| f_1^{(4)}(x) - f_2^{(4)}(x) \right| \leq \left| f_1^{(4)}(x) \right| + \left| f_2^{(4)}(x) \right| = f_1^{(4)}(x) + f_2^{(4)}(x)
$$

and

$$
f''(x) = \left| f_1''(x) - f_2''(x) \right| \leq \left| f_1''(x) \right| + \left| f_2''(x) \right| = f_1''(x) + f_2''(x).
$$

Also $h''(x) \leq 0$ so $|h''(x)| = -h''(x)$. Hence, (2.38) is less or equal to

$$
\|f\|_{L^\infty} \int_I (f_2(x) + f_1(x))^{(4)} h(x) dx - 2 \|f\|_{L^\infty} \int_I (f_2(x) + f_1(x))^{(4)} h''(x) dx + \frac{1}{2} \int_I f^2(x) h^{(4)}(x) dx
$$

Further, applying integration by parts four times on the first and twice on the second integral, we get

$$
\begin{aligned}
&\|f\|_{L^\infty} \int_I (f_2(x) + f_1(x)) h^{(4)}(x) dx - 2 \|f\|_{L^\infty} \int_I (f_2(x) + f_1(x)) h^{(4)}(x) dx + \frac{1}{2} \int_I f^2(x) h^{(4)}(x) dx \\
&= \int_I \left[\frac{(f_2(x) - f_1(x))^2}{2} - \|f_2 - f_1\|_{L^\infty} (f_1(x) + f_2(x)) \right] h^{(4)}(x) dx.
\end{aligned}
$$

□

Remark 2.3 *If we take supremum in* (2.37), *we will obtain*

$$
\int_I \left| f_2''(x) - f_1''(x) \right|^2 h(x) dx \leq \left[\frac{1}{2} \|f_2 - f_1\|_{L^\infty}^2 + \|f_2 - f_1\|_{L^\infty} \left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty} \right) \right] \cdot \int_I \left| h^{(4)}(x) \right| dx.
$$

Now we take the mollification of arbitrary 4-convex function. Let $f : I \to \mathbb{R}$ be arbitrary $4-$convex function that is also $2-$convex function. Then by the property of the differentiability of the $4-$convex function f is twice continuously differentiable.
We take function, mollifier, $\theta_\varepsilon \in C^\infty(\mathbb{R})$ that has support on interval $I_\varepsilon = (-\varepsilon, \varepsilon)$

$$
\theta_\varepsilon(x) = \begin{cases} C\, exp \frac{1}{x^2 - \varepsilon^2}, & x \leq \varepsilon \\ 0, & x > \varepsilon, \end{cases}
$$

where C is a constant such that

$$
\int_{\mathbb{R}} \theta_\varepsilon(x) dx = 1.
$$

Now using θ_ε as a kernel, we define the $\varepsilon-$approximation of f on I as

$$f_\varepsilon(x) = \int\limits_{\mathbb{R}} f(x-y)\theta_\varepsilon(y)dy = \int\limits_{\mathbb{R}} f(y)\theta_\varepsilon(x-y)dy.$$

Since $\theta_\varepsilon \in C^\infty(\mathbb{R})$, so $f_\varepsilon \in C^\infty(\mathbb{R})$.
If f is a continuous, then f_ε converges uniformly to f in any compact subset $K \subseteq I$, as $\varepsilon \to 0$.
Also convexity of f implies the convexity of f_ε, as is showed in Theorem 1.5.

Even more simply, in this case, convexity of f_ε follows from

$$f_\varepsilon''(x) = \int\limits_{\mathbb{R}} f''(x-y)\theta_\varepsilon(y)dy \geq 0.$$

Quite similarly, if f is $4-$convex, then f_ε is also $4-$convex.

2.2.2 The case of an arbitrary 4-convex function

Now we will prove that the second derivative of continous 4-convex function are square integrable with respect to weight function.

Theorem 2.4 *Let $f: I \to \mathbb{R}$ be convex and $4-$convex function and let the weight function $h: I \to \mathbb{R}$ as in Theorem 2.3. Then the followinng hold*

$$\int\limits_I \left|f''(x)\right|^2 h(x)dx < \infty$$

Proof. Let $I_k \subset I,\ k \in \mathbb{N}$ be an increasing sequence of subintervals such that $\cup_{k\geq 1} I_k = I$. Now we apply the inequality (2.37) for increasing sequence of intervals $I_k \subset I$ such that $\cup_{k\geq 1} I_k = I$ and for the functions $f_1(x) = 0,\ f_2(x) = f_m(x),\ h = h_m$ where f_m and h_m are the approximations of f and h on I_m, respectively.
First, for all $k,\ l \geq 1$.

$$\int\limits_{I_{k+l}} \left|f_m''(x)\right|^2 h_{k+l}(x)dx \leq \frac{3}{2}\|f_m\|^2_{L^\infty(I_{k+l})} \int\limits_I \left|h_{k+l}^{(4)}(x)\right| dx$$

Taking limit $m \to \infty$

$$\int\limits_{I_{k+l}} \left|f''(x)\right|^2 h_{k+l}(x)dx \leq \frac{3}{2}\|f\|^2_{L^\infty(I_{k+l})} \int\limits_I \left|h_{k+l}^{(4)}(x)\right| dx.$$

Since $I_k \subset I_{k+l}$, so writing left hand integral on the smaller interval I_k, we get

$$\int\limits_{I_k} \left|f''(x)\right|^2 h_{k+l}(x)dx \leq \frac{3}{2}\|f\|^2_{L^\infty(I_{k+l})} \int\limits_I \left|h_{k+l}^{(4)}(x)\right| dx.$$

If we let $l \to \infty$

$$\int_{I_k} \left|f''(x)\right|^2 h(x)dx \leq \frac{3}{2}\left|f(x)\right|^2_{L^\infty(I)} \int_I \left|h^{(4)}(x)\right| dx.$$

Left hand side is increasing and bounded, so it has finite limit

$$\int_I \left|f''(x)\right|^2 h(x)dx < \infty$$

□

Corollary 2.3 *Let f_1, f_2 be both convex and 4-convex functions on $[a,b]$. Let the weight function h satisfies the conditions of Theorem 2.3. Then we have*

$$\int_a^b |f_2''(x) - f_1''(x)|^2 h(x)dx \leq \left(\frac{1}{2}\|f_2 - f_1\|^2_{L^\infty} + \|f_2 - f_1\|_{L^\infty}(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty})\right)\|h^{(4)}\|_{L^1}.$$

Proof. The proof follows from Theorem 2.4, using mollification technique, and Remark 2.3.

□

2.3 The weighted energy estimates for the third derivative of 6-convex function

2.3.1 The case of smooth 6-convex functions and molification of an arbitrary 6-convex function

Let $h : [a,b] \to \mathbb{R}$ be the weight function which is non-negative and twice continuously differentiable and satisfying

$$h(a) = h(b) = 0,\ h'(a) = h'(b) = 0. \tag{2.39}$$

The proof of the following four lemmas can be found in the paper of Hussain, Pečarić, and Shashiashvili [32].

Lemma 2.1 *For smooth convex function $f : I \to \mathbb{R}$ and non-negative weight function $h : I \to \mathbb{R}$, which satisfies* (2.39), *we have*

$$\int_I \left(f'(x)\right)^2 h(x)dx \leq \int_I \left[\frac{(f(x))^2}{2} + \|f\|_{L^\infty}\left(f(x)\right)\right]|h''(x)|dx.$$

Lemma 2.2 *For smooth concave function $f : I \to \mathbb{R}$ and non-negative weight function $h : I \to \mathbb{R}$, which satisfies* (2.39), *we have*

$$\int_I \left(f'(x)\right)^2 h(x)dx \leq \int_I \left[\frac{(f(x))^2}{2} - \|f\|_{L^\infty}\right] |h''(x)|dx.$$

Lemma 2.3 *Let $f : I \to \mathbb{R}$ be a convex and 4−convex and let $h : I \to \mathbb{R}$ be the non negative smooth weight function as defined in* (2.39) *and satisfying the condition*

$$h''(x) \leq 0 \;\; \forall\, x \in I \text{ and } h'(x) = h''(x) = h'''(x) = 0 \;\; \forall\, x \in \partial I.$$

Then the following estimate holds

$$\int_I \left(|f''(x)|\right)^2 h(x)dx \leq \int_I \left(\frac{\left(f(x)\right)^2}{2} - \|f\|_{L^\infty}(f(x))\right) h^{(4)}(x)dx.$$

Lemma 2.4 *Let $f : I \to \mathbb{R}$ be a concave and 4−concave and let $h : I \to \mathbb{R}$ be the non negative smooth weight function as defined in* (2.39) *and satisfying the conditions $h''(x) \leq 0$, $x \in I$ and $h'(x) = h''(x) = h'''(x) = 0$, $x \in \partial I$. Then the following estimate holds*

$$\int_I \left(|f''(x)|\right)^2 h(x)dx \leq \int_I \left(\frac{\left(f(x)\right)^2}{2} + \|f\|_{L^\infty}(f(x))\right) h^{(4)}(x)dx.$$

we will start by the following theorem:

Theorem 2.5 *Let $f,\ F \in C^6[a,b]$ and F is a convex, 4−convex and 6−convex function such that the condition*

$$\left.\begin{array}{ll} |f''(x)| \leq F''(x), & x \in (a,b) \\ |f^{(4)}(x)| \leq F^{(4)}(x), & x \in (a,b) \\ |f^{(6)}(x)| \leq F^{(6)}(x), & x \in (a,b) \end{array}\right\} \tag{2.40}$$

are fulfilled. Let $h : I \to \mathbb{R}$ be a non negative 2−concave, 4−convex weight function satisfying

$$\left.\begin{array}{l} h^{(4)}(x) \geq 0 \quad if \quad x \in I \\ h''(x) \leq 0 \quad if \quad x \in I \\ h(x) = h'(x) = h''(x) = h'''(x) = h^{(4)}(x) = h^{(5)}(x) = 0, \quad x \in \partial I. \end{array}\right\} \tag{2.41}$$

Then the following energy estimate is valid:

$$\int_I (f'''(x))^2 h(x)dx \leq \int_I \left(\frac{5}{4}(f(x))^2 + \frac{5}{2}\|f\|_{L^\infty} F(x)\right) |h^{(4)}(x)|dx.$$

Proof. We apply integration by parts on

$$I = \int_I \left(f'''(x)\right)^2 h(x)dx = \int_I f'''(x)\left(f'''(x)h(x)\right)dx,$$

$$\int_I (f'''(x))^2 h(x)dx = -\int_I f''(x)f^{(4)}(x)h(x)dx - \int_I f''(x)f'''(x)h'(x)dx. \tag{2.42}$$

We proceed with integration by parts on the first integral of (2.42)

$$I = \int_I f'(x)f^{(5)}(x)h(x)dx + \int_I f'(x)f^{(4)}(x)h'(x)dx - \int_I f''(x)f'''(x)h'(x)dx.$$

Now we consider the second and the third integral on the right side of the latter expression (2.41), we get

$$\begin{aligned} I = &- \int_I f(x)f^{(6)}(x)h(x)dx - \int_I f(x)f^{(5)}(x)h'(x)dx - \int_I f(x)f^{(5)}(x)h'(x)dx \\ &- \int_I f(x)f^{(4)}(x)h''(x)dx - \int_I f''(x)f'''(x)h'(x)dx \end{aligned}$$

Proceeding in the similar way and using condition (2.41) and the definition of weight function, we obtain

$$\begin{aligned} I = &- \int_I f(x)f^{(6)}(x)h(x)dx + \int_I f(x)f^{(4)}(x)h''(x)dx \\ &+ \frac{7}{2}\int_I \left(f''(x)\right)^2 h''(x)dx - \int_I \left(f'(x)\right)^2 h^{(4)}(x)dx \\ \le &\ \|f\|_{L^\infty} \int_I f^{(6)}(x)h(x)dx + \|f\|_{L^\infty} \int_I f^{(4)}(x)h''(x)dx \\ &+ \frac{7}{2}\int_I \left(f''(x)\right)^2 h''(x)dx - \int_I \left(f'(x)\right)^2 h^{(4)}(x)dx \end{aligned} \tag{2.43}$$

Using the integration by parts formula six and four times respectively on the first and second integral of (2.43) respectively, we have

$$\begin{aligned} I \le &\|f\|_{L^\infty} \int_I f(x)h^{(6)}(x)dx + \|f\|_{L^\infty} \int_I f(x)h^{(6)}(x)dx \\ &+ \frac{7}{2}\int_I \left(f''(x)\right)^2 h''(x)dx - \int_I \left(f'(x)\right)^2 h^{(4)}(x)dx. \end{aligned} \tag{2.44}$$

Now we use Lemma (2.3), by replacing h with h'', and apply it on

$$\int_I \left(f''(x)\right)^2 h''(x)dx.$$

We get

$$\int_I \left[\frac{\left(f(x)\right)^2}{2} - \|f\|_{L^\infty}\left(f(x)\right)\right] h^{(6)}(x)dx \tag{2.45}$$

Now by taking the last integral of (2.43) and using the Lemma 2.3 by replacing h with $h^{(4)}$, we get

$$\int_I \left[\frac{\left(f(x)\right)^2}{2} + \|f\|_{L^\infty}\left(f(x)\right)\right] h^{(6)}(x)dx \tag{2.46}$$

Inserting (2.45) and (2.46) in (2.44), and also using condition (2.40), we obtain

$$\begin{aligned}\int_I \left(f'''(x)\right)^2 h(x)dx \le & \left[\|f\|_{L^\infty} \int_I F(x) - \frac{9}{2}\sup_{x\in I}|f(x)| \int_I F(x)\right. \\ & \left. + \, 6\|f\|_{L^\infty} \int_I F(x) + \frac{5}{4}\int_I \left(f^2(x)\right)\right] h^{(4)}(x)dx \\ \le & \int_I \left(\frac{5(F(x))^2}{4} + \frac{5}{2}\|F\|_{L^\infty}\right) h^{(6)}(x).\end{aligned}$$

$\square$

The get the following weighted energy inequality for the smooth $6-$convex function f can be obtained simply by taking $F = f$ in the previous estimation, where $f \in C^6[a,b]$ and f and h satisfies the conditions of the last theorem.

$$\int_I |f'''(x)|^2 h(x)dx \le \int_I \left(\frac{5(f(x))^2}{4} + \frac{5}{2}\|f\|_{L^\infty} f(x)\right) h^{(6)}(x)dx.$$

The next result describes the energy estimate for the difference of two 6-convex functions.

Corollary 2.4 *Let $f_1,\ f_2 \in C^6[a,b]$ be the functions which are convex, $4-$convex and $6-$convex. Let $h : I \to \mathbb{R}$ be the weight function satisfying the conditions of the Theorem* 2.5. *Then the following energy estimate is valid*

$$\begin{aligned}\int_I |f_2'''(x) - f_1'''(x)|^2 h(x)dx \le & \left(\frac{5}{4}\|f_2 - f_1\|_{L^\infty}^2 + \frac{5}{2}\|f_2 - f_1\|_{L^\infty}\right. \\ & \left. \times \left(\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}\right)\right) \|h^{(6)}(x)\|_{L^1}.\end{aligned} \tag{2.47}$$

Proof. Take $f = f_2 - f_1$ and $F = f_1 + f_2$ in Theorem 2.5 to get

$$\int_I |f_2'''(x) - f_1'''(x)|^2 h(x)dx \leq \int_I \left[\frac{5}{4}\|f_2(x) - f_1(x)\|_{L^\infty}^2 + \frac{5}{2}\|f_2 - f_1\|_{L^\infty} \times \left(\|f_2\|_{L^\infty} + \|f_1\|_{L^\infty}\right)\right] h^{(6)}(x)dx.$$

□

We conclude the section with the following remark.

Remark 2.4 *Let f_1, f_2 and h be the same as in the last theorem. Then using the Hölder inequality, we have*

$$\int_I |f_2'''(x) - f_1'''(x)|^2 h(x)dx \leq \| \widetilde{f} \|_{L^p} \|h^{(6)}(x)\|_{L^q}.$$

Where $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$\widetilde{f}(x) = \frac{5\left(f_2(x) - f_1(x)\right)^2}{4} + \frac{5}{2}\|f_2 - f_1\|_{L^\infty}\left(f_1(x) + f_2(x)\right).$$

2.3.2 The case of an arbitrary 6-convex function

In this section we can use the mollification technique to prove previous results for arbitrary convex, $4-$convex and 6-convex function.

We use the mollification of arbitrary 6-convex function in $[a,b]$.

Let f be an arbitrary convex, 4-convex and $6-$convex function. Then by Theorem 1.2 $f \in C^3[a,b]$. Let $\theta_\varepsilon \in C^\infty(\mathbb{R})$ be mollifier,

$$\theta_\varepsilon(x) = \begin{cases} C exp\left(\frac{1}{x^2 - \varepsilon^2}\right), & x \leq \varepsilon, \\ 0, & x > \varepsilon, \end{cases}$$

where C is a constant such that

$$\int_{\mathbb{R}} \theta_\varepsilon(x)dx = 1.$$

Now using θ_ε as a kernal, we define the convolution of f and θ_ε as

$$f_\varepsilon(x) = \int_{\mathbb{R}} f(x-y)\theta_\varepsilon(y)dy = \int_{\mathbb{R}} f(y)\theta_\varepsilon(x-y)dy$$

Since $\theta_\varepsilon \in C^\infty(\mathbb{R})$, we have also $f_\varepsilon \in C^\infty(\mathbb{R})$.
If f is continous, then f_ε converges uniformly to f in any compact subset $K \subseteq I$,

$$\sup_{x \in K} |f_\varepsilon(x) - f(x)| \xrightarrow{\varepsilon \to 0} 0.$$

For $m \in \mathbb{N}$, let f_m denotes $\varepsilon = \frac{1}{m}-$ mollification of f. Then, specially,

$$\sup_{x \in K} |f_m(x) - f(x)| \xrightarrow{m \to \infty} 0.$$

Since f is convex, 4-convex and 6−convex function, we can, similar to Theorem 1.5 , show that its mollification f_ε (specially f_m) has the same properties.

Theorem 2.6 *Let f be convex, 4-convex and 6-convex function on an interval I. Let h_k be such that*

$$h_k^{(4)}(x) \geq 0, \ \ h_k''(x) \leq 0, \qquad x \in I_k$$

and

$$h_k'(x) = h_k''(x) = h_k'''(x) = h_k^{(4)}(x) = h_k^{(5)}(x) = 0, \qquad x \in \partial I,$$

where $\lim\limits_{k \to \infty} h_k(x) = h(x)$. *Then the following hold:*

$$\int_I \left|f'''(x)\right|^2 h(x)dx < \infty.$$

Proof. Writing the inequality (2.47) for the intervals $I_{k+l} \subset I$ and for the functions $f_1 = 0$ and $f_2 = f_m$ where f_m is the approximation of f, we get

$$\int_{I_{k+l}} \left|f_m'''(x)\right|^2 h_{k+l}(x)dx \ \ \leq \frac{15}{4} \|f_m\|^2_{L^\infty(I_{k+l})} \times \int_I \left|h_{k+l}^{(6)}(x)\right| dx \tag{2.48}$$

Taking limit $m \to \infty$ in (4.40), we have

$$\int_{I_{k+l}} \left|f'''(x)\right|^2 h_{k+l}(x)dx \leq \frac{15}{4} \|f\|^2_{L^\infty(I_{k+l})} \int_I \left|h_{k+l}^{(6)}(x)\right| dx.$$

Since $I_k \subset I_{k+l}$, writing left hand integral for smaller interval I_k, we get

$$\int_{I_k} \left|f'''(x)\right|^2 h_{k+l}(x)dx \leq \frac{15}{4} \|f\|^2_{L^\infty(I_{k+l})} \int_I \left|h_{k+l}^{(6)}(x)\right| dx.$$

If now let $l \to \infty$, we get

$$\int_{I_k} \left|f'''(x)\right|^2 h(x)dx \leq \frac{15}{4} |f(x)|^2_{L^\infty_I} \int_I \left|h^{(6)}(x)\right| dx.$$

Since left hand side is increasing, for $k \in \mathbb{N}$, and bounded, by dominated convergence, we have finally

$$\int_I \left|f'''(x)\right|^2 h(x)dx < \infty.$$

□

Theorem 2.7 *Let f_i $i=1,2$ be two convex, 4-convex and 6-convex functions over the interval I. Then the following holds*

$$\int_I \left|f_2'''(x)-f_1'''(x)\right|^2 h(x)dx \le \left[\frac{5}{4}\|f_2-f_1\|^2_{L^\infty}+\frac{5}{2}\|f_2-f_1\|_{L^\infty} \times\left(\|f_1\|_{L^\infty}+\|f_2\|_{L^\infty}\right)\right]\times\int_I\left|h^{(6)}(x)\right|dx, \tag{2.49}$$

where h is non-negative weight function satisfied the conditions in (2.39)

Proof. For 6-convex functions $f_i,\ i=1,2$, consider the smooth approximations $f_{m,i},\ i=1,2$.
For the interval I_{k+l} there exist an integer m_{k+l} such that $f_{m,i}(x)$ converges uniformly to $f_i(x)$ $i=1,2$. and also $f_{m,i}(x)$ is smooth for $m\ge m_{k+l}$.
Now writing the inequality (2.47) for the function $f_{m,1}$ and $f_{m,2}$ over interval I_{k+l}, we get

$$\int_{I_{k+l}} \left|f_{m,2}'''(x)-f_{m,1}'''(x)\right|^2 h(x)dx \le c_{k+l}\left[\frac{5}{4}\|f_{m,2}-f_{m,1}\|^2_{L^\infty} +\frac{5}{2}\|f_{m,2}-f_{m,1}\|_{L^\infty}\left(\|f_{m,1}\|_{L^\infty}+\|f_{m,2}\|_{L^\infty}\right)\right], \tag{2.50}$$

where $c_{k+l}=\int_{I_{k+l}}\left|h^{(6)}(x)\right|dx.$
Now taking limit $m\to\infty$ we obtain,

$$\int_{I_{k+l}} \left|f_2'''(x)-f_1'''(x)\right|^2 h(x)dx \le c_{k+l}\left[\frac{5}{4}\|f_2-f_1\|^2_{L^\infty(I_{k+l})}+\frac{5}{2}\|f_2-f_1\|_{L^\infty(I_{k+l})} \times\left(\|f_1\|_{L^\infty(I_{k+l})}+\|f_2\|_{L^\infty(I_{k+l})}\right)\right]. \tag{2.51}$$

Now writing left hand side integral for the smaller interval $I_k\subset I_{k+l}$ and taking limit as $l\to\infty$, we obtain

$$\int_{I_k} \left|f_2'''(x)-f_1'''(x)\right|^2 h(x)dx \le c_\infty\left[\frac{5}{4}\|f_2-f_1\|^2_{L^\infty(I)} +\frac{5}{2}\|f_2-f_1\|_{L^\infty(I)}\left(\|f_1\|_{L^\infty(I)}+\|f_2\|_{L^\infty(I)}\right)\right]. \tag{2.52}$$

Since we have

$$\int_I \left|f_i'''(x)\right|^2 h(x)dx<\infty,\ \ i=1,2.$$

Taking limit as $k\to\infty$ we obtain the required result. □

2.4 The weighted energy estimates for the (2,2)-convex function

The natural generalization of convex functions are sub-harmonic function and similarly the generalization of 4-convex function is the sub-solution of the fourth order Laplace equation i.e,

$$\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_2^4} + \dots + \frac{\partial^4 u}{\partial x_n^4} \geq 0.$$

The definition of divided and finite differences can be used in the definition for convex functions of several variables. This type of definition was firstly introduced by Popoviciu in [57].

Theorem 2.8 *Let* $f : I_1 \times I_2 \to \mathbb{R}$ *be* $(2,2)-$*convex such that* $f_{xx}(x_0,y_0) \geq 0,\ f_{yy}(x_0,y_0) \geq 0,$ *and* $f_{xy}(x_0,y_0) \geq 0$ *for every* $(x_0,y_0) \in I_1 \times I_2$.

Let $h : I_1 \times I_2 \to \mathbb{R}$ *be a non-negative weight function such that*

$$h(x_0,y_0) = h_x(x_0,y_0) = h_y(x_0,y_0) = h_{xx}(x_0,y_0) = h_{yy}(x_0,y_0) = h_{xy}(x_0,y_0) = 0, \tag{2.53}$$

for every $(x_0,y_0) \in \partial(I_1 \times I_2)$.
Then the following holds

$$\int\limits_{I_1\times I_2} (f_{xy})^2 h dxdy \leq \int\limits_{I_1\times I_2} \left(3\|f\|_{L^\infty} f + \frac{1}{2} f^2\right) h_{xxyy} dxdy. \tag{2.54}$$

Proof. First,

$$I = \int\limits_{I_1\times I_2} (f_{xy})^2 h dxdy = \int\limits_{I_1\times I_2} f_{xy}[f_{xy}h] dxdy,$$

and then using integration by parts formula with respect to y, we have

$$\begin{aligned} I &= -\int\limits_{I_1\times I_2} f_x (f_{xy}h)_y dxdy = -\int\limits_{I_1\times I_2} f_x f_{xyy} h dxdy - \int\limits_{I_1\times I_2} f_x f_{xy} h_y dxdy \\ &= -\int\limits_{I_1\times I_2} f_x f_{xyy} h dxdy - \frac{1}{2}\int\limits_{I_1\times I_2} [(f_x)^2]_y h_y dxdy. \end{aligned}$$

Using integration by parts on first integral with respect to x, we have

$$\begin{aligned} &\int\limits_{I_1\times I_2} f\left(f_{xyy}h\right)_x dxdy - \frac{1}{2}\int\limits_{I_1\times I_2} (f_x^2)_y h_y dxdy \\ &= \int\limits_{I_1\times I_2} f f_{xyy} h_x dxdy + \int\limits_{I_1\times I_2} f f_{xxyy} h dxdy - \frac{1}{2}\int\limits_{I_1\times I_2} [(f_x)^2]_y h_y dxdy. \end{aligned} \tag{2.55}$$

Taking first integral of (2.55) and using integrating by parts formula with respect to y, we have

$$\begin{aligned}\int\limits_{I_1\times I_2}(fh_x)f_{xyy}dxdy &= -\int\limits_{I_1\times I_2}f_{xy}[fh_x]_y dxdy = -\int\limits_{I_1\times I_2}f_{xy}f_yh_xdxdy - \int\limits_{I_1\times I_2}f_{xy}fh_{xy}dxdy\\ &= -\int\limits_{I_1\times I_2}f_yf_{xy}h_xdxdy - \int\limits_{I_1\times I_2}ff_{xy}h_{xy}dxdy\\ &= -\frac{1}{2}\int\limits_{I_1\times I_2}[f_y^2]_xh_xdxdy - \int\limits_{I_1\times I_2}ff_{xy}h_{xy}dxdy \end{aligned}\tag{2.56}$$

Using (2.56) in (2.55), we get

$$\begin{aligned}\int\limits_{I_1\times I_2}(f_{xy})^2hdxdy &= -\frac{1}{2}\int\limits_{I_1\times I_2}((f_y)^2)_xh_xdxdy - \int\limits_{I_1\times I_2}ff_{xy}h_{xy}dxdy\\ &\quad+\int\limits_{I_1\times I_2}ff_{xxyy}hdxdy - \frac{1}{2}\int\limits_{I_1\times I_2}((f_x)^2)_yh_ydxdy. \end{aligned}\tag{2.57}$$

Now we take the first integral of above and apply integration by parts formula,

$$-\frac{1}{2}\int\limits_{I_1\times I_2}((f_y)^2)_xh_xdxdy = \frac{1}{2}\int\limits_{I_1\times I_2}(f_y)^2h_{xx}dxdy = \frac{1}{2}\int\limits_{I_1\times I_2}f_y[f_yh_{xx}]dxdy. \tag{2.58}$$

Integrating by parts with respect to y, we have

$$\begin{aligned}-\frac{1}{2}\int\limits_{I_1\times I_2}((f_y)^2)_xh_xdxdy &= -\frac{1}{2}\int\limits_{I_1\times I_2}f[f_yh_{xx}]_ydxdy = -\frac{1}{2}\int\limits_{I_1\times I_2}f[f_{yy}h_{xx}+f_yh_{xxy}]dxdy\\ &= -\frac{1}{2}\int\limits_{I_1\times I_2}ff_{yy}h_{xx}dxdy - \frac{1}{4}\int\limits_{I_1\times I_2}[f^2]_yh_{xxy}dxdy\\ &= -\frac{1}{2}\int\limits_{I_1\times I_2}ff_{yy}h_{xx}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2}f^2h_{xxyy}dxdy. \end{aligned}\tag{2.59}$$

Now if we take the 4th integral of (2.57) we get similarly the result as in (2.59)

$$-\frac{1}{2}\int\limits_{I_1\times I_2}(f_x{}^2)_yh_ydxdy = -\frac{1}{2}\int\limits_{I_1\times I_2}ff_{xx}h_{yy}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2}f^2h_{xxyy}dxdy. \tag{2.60}$$

Now using (2.59) and (2.60) in (2.57), we obtain

$$\begin{aligned}\int\limits_{I_1\times I_2}(f_{xy})^2hdxdy = &-\frac{1}{2}\int\limits_{I_1\times I_2}ff_{yy}h_{xx}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2}f^2h_{xxyy}dxdy - \int\limits_{I_1\times I_2}ff_{xy}h_{xy}dxdy\\ &+\int\limits_{I_1\times I_2}ff_{xxyy}hdxdy - \frac{1}{2}\int\limits_{I_1\times I_2}ff_{xx}h_{yy}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2}f^2h_{xxyy}dxdy. \end{aligned}$$

Therefore

$$
\begin{aligned}
\int\limits_{I_1\times I_2} (f_{xy})^2 h dxdy \le{}& \frac{1}{2}\int\limits_{I_1\times I_2} |f||f_{yy}|h_{xx}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2} f^2 h_{xxyy}dxdy \\
&+ \int\limits_{I_1\times I_2} |f||f_{xy}|h_{xy}dxdy + \int\limits_{I_1\times I_2} |f||f_{xxyy}|hdxdy \\
&+ \frac{1}{2}\int\limits_{I_1\times I_2} |f||f_{xx}|h_{yy}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2} f^2 h_{xxyy}dxdy \\
\le{}& \frac{1}{2}\|f\|_{L^\infty}\int\limits_{I_1\times I_2} f h_{xxyy}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2} f^2 h_{xxyy}dxdy \\
&+ \|f\|_{L^\infty}\int\limits_{I_1\times I_2} f h_{xxyy}dxdy + \|f\|_{L^\infty}\int\limits_{I_1\times I_2} f h_{xxyy}dxdy \\
&+ \frac{1}{2}\|f\|_{L^\infty}\int\limits_{I_1\times I_2} f h_{xxyy}dxdy + \frac{1}{4}\int\limits_{I_1\times I_2} f^2 h_{xxyy}dxdy \\
={}& \int\limits_{I_1\times I_2} \left(3\|f\|_{L^\infty} f + \frac{1}{2}f^2\right) h_{xxyy}dxdy.
\end{aligned}
$$

□

The next result will give the similar inequality for the difference of $(2,2)-$convex functions.

Corollary 2.5 *Let* $f_i : I_1 \times I_2 \to \mathbb{R}$, $i = 1,2$, *be* $(2,2)-$*convex functions and let*

$$
\frac{\partial^2 f_i}{\partial x^2}(x_0,y_0) \ge 0, \frac{\partial^2 f_i}{\partial y^2}(x_0,y_0) \ge 0, \frac{\partial^2 f_i}{\partial x \partial y}(x_0,y_0) \ge 0,
$$

for every $(x_0,y_0) \in I_1 \times I_2$. *If* $h : I_1 \times I_2 \to \mathbb{R}$ *is non-negative weight function that satisfies* (2.53), *then the following energy estimate is valid*

$$
\int\limits_{I_1\times I_2} \left((f_2 - f_1)_{xy}\right)^2 hdxdy \le \int\limits_{I_1\times I_2} \left(3\|f_2 - f_1\|_{L^\infty}(f_1 + f_2) + \frac{(f_1 - f_2)^2}{2}\right) h_{xxyy}dxdy.
$$

Chapter 3

The weighted energy estimates for the vector valued functions

3.1 The weighted reverse Poincaré-type eEstimate for the difference of two convex vector functions

In order to understand the use of inequalities in optimization and uniform approximations, we refer [11] and [4]. Usually payoff function of the various options (for example, European and American options) in mathematical finance is convex and this property leads the corresponding value function to be convex with respect to the underlying stock price (see for detail El Karoui et al. [36] and Hobson [28]). Traders and practitioners dealing with real-world financial markets use value function to construct optimal hedging process of the options. When the value function is unknown, they use the above property to construct uniform approximations the unknown optimal hedging process. In this construction one has to pass some weighted integrals involve weak partial derivative of the value function. For this purpose, K. Shashiashvili and M. Shashiashvili [50] introduced a very particular weighted integral inequality for the derivative of bounded from below convex functions with a very particular weight function, with this they opened a new direction in the field of weighted inequalities. Hussain et al. [32, 33] extended this work to a variety of convex functions and subsequently applied to the hedging problems of financial mathematics.

Saleem et al. [61] studied the weighted reverse Poincaré type inequalities for the difference of two weak sub-solutions.

3.1.1 The reverse Poincaré inequalities for smooth vectors and approximation of arbitrary convex vectors by smooth ones

Let $h : [a,b] \to \mathbb{R}$ be the weight function which is non-negative and twice continuously differentiable and satisfying

$$h(a) = h(b) = 0,\ h'(a) = h'(b) = 0. \tag{3.1}$$

We state the following result of Hussain, Pečarić, and Shashiashvili (see [32]).

Lemma 3.1 *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be a smooth convex functions, and $h : I \to \mathbb{R}$, a non-negative weight function, which satisfies* (3.1). *Then*

$$\int_I \big(f'(x) - g'(x)\big)^2 h(x)dx \le \int_I \left[\left(\frac{f(x)-g(x)}{2}\right)^2 + \sup_{x\in I}|f(x)-g(x)| \right.$$
$$\left. \times\ \big(f(x)+g(x)\big)\right] |h''(x)|dx.$$

The latter result gives the following estimate for n-dimensional convex vectors.

Lemma 3.2 *Let $F : I \to \mathbb{R}^n$ and $G : I \to \mathbb{R}^n$ are two $n-$dimensional convex vectors on interval I and $h : I \to \mathbb{R}$ is smooth non-negative weight function satisfying* (3.1). *Then the following energy estimate is valid*

$$\int_I \left|F'(x) - G'(x)\right|^2 h(x)dx \le \sum_{i=1}^n \int_I \left[\frac{(f_i(x)-g_i(x))^2}{2} + \|f_i - g_i\|_{L^\infty} \right.$$
$$\left. \times\ \big(f_i(x)+g_i(x)\big)\right] |h''(x)|dx. \tag{3.2}$$

If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are concave functions on interval I then $-f($ and $-g$ become convex. Hence, we get the following result.

Corollary 3.1 *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be any two smooth concave functions on interval I and $h : I \to \mathbb{R}$ be a non-negative weight function as satisfies* (3.1)*. Then the following estimate does hold*

$$\int_I \big(f'(x) - g'(x)\big)^2 h(x)dx \le \int_I \left[\left(\frac{f(x)-g(x)}{2}\right)^2 - \|f-g\|_{L^\infty}\big(f(x)+g(x)\big)\right] |h''(x)|dx.$$

Taking supremum on both sides of (3.2), we find

Corollary 3.2 *Let $F : I \to \mathbb{R}^n$ and $G : I \to \mathbb{R}^n$ be two smooth convex vectors and non-negative weight function $h : I \to \mathbb{R}$ which satisfies* (3.1). *Then the following estimation is valid*

$$\int_I \left|F'(x) - G'(x)\right|^2 h(x)dx \le \sum_{i=1}^n \left[\frac{1}{2}\|f_i - g_i\|_{L^\infty}^2 + \|f_i(x) - g_i(x)\|_{L^\infty} \times \left(\|f_i\|_{L^\infty} + \|g_i\|_{L^\infty}\right)\right] \int_I |h''(x)|dx. \tag{3.3}$$

For concave n-dimensional vectors we have the following estimate.

Corollary 3.3 *Let $F : I \to \mathbb{R}^n$ and $G : I \to \mathbb{R}^n$ are n-dimensional concave vectors on I and the non-negative weight function $h : I \to \mathbb{R}$ satisfies* (3.1) *then*

$$\int_I \left|F'(x) - G'(x)\right|^2 h(x)dx \le \sum_{i=1}^n \int_I \left[\frac{(f_i(x) - g_i(x))^2}{2} + \|f_i - g_i\|_{L^\infty} \times \left(f_i(x) + g_i(x)\right)\right] |h''(x)|dx. \tag{3.4}$$

The next theorem gives the reversed Poincaré inequality for the difference of vectors that belong to $\chi_{[1,j]}^{[j+1,n]}[a,b]$.

Theorem 3.1 *Let $F : I \to \mathbb{R}^n$ and $G : I \to \mathbb{R}^n$ belongs to $\chi_{[1,j]}^{[j+1,n]}[a,b]$ and $h : I \to \mathbb{R}$ be a nonnegative weight function satisfying* (4.28)*. Then the following inequality is valid*

$$\begin{aligned}\int_I \left|F'(x) - G'(x)\right|^2 h(x)dx \le{}& \sum_{i=1}^n \int_I \frac{\left(f_i(x) - g_i(x)\right)^2}{2} h''(x)dx \\ &+ \left[\sum_{i=1}^j \sup_{x\in I}\left|f_i(x) - g_i(x)\right|\left[f_i(x) + g_i(x)\right]\right. \\ &- \sum_{i=j+1}^n \sup_{x\in I}\left|f_i(x) - g_i(x)\right| \\ &\left.\times \left[f_i(x) + g_i(x)\right]\right] \int_I |h''(x)|dx. \end{aligned} \tag{3.5}$$

Proof. Firstly

$$\begin{aligned}\int_I |F'(x) - G'(x)|^2 h(x)dx &= \sum_{i=1}^n \int_I \left(f_i'(x) - g_i'(x)\right)^2 h(x)dx \\ &= \sum_{i=1}^j \int_I \left(f_i'(x) - g_i'(x)\right)^2 h(x)dx\end{aligned}$$

$$+\sum_{i=j+1}^{n}\int_I \left(f_i'(x)-g_i'(x)\right)^2 h(x)dx.$$

Using Lemma 2.1 in the first sum on the right side of the latter expression, we obtain

$$\begin{aligned}\int_I (f_i'(x)-g_i'(x)\Big)^2 h(x)dx \le& \int_I \frac{\left(f_i(x)-g_i(x)\right)^2}{2}|h''(x)|dx+\sup_{x\in I}\left|f_i(x)-g_i(x)\right| \\ &\times \int_I \left(f_i(x)-g_i(x)\right)|h''(x)|dx, \end{aligned} \tag{3.6}$$

and Corollary 3.1 in the second sum, we get

$$\begin{aligned}\int_I \left(f_i'(x)-g_i'(x)\right)^2 h(x)dx \le& \int_I \Bigg[\frac{\left(f_i(x)-g_i(x)\right)^2}{2} - \sup_{x\in I}\left|f_i(x)-g_i(x)\right| \\ &\times \int_I \left(f_i(x)-g_i(x)\right)\Bigg]|h''(x)|dx \end{aligned} \tag{3.7}$$

Combining the inequality (3.6) and (3.7) we have the required inequality (3.5) □

Corollary 3.4 *Taking supremum of* (3.5)*, we obtain the following inequality*

$$\begin{aligned}\int_I \left|F'(x)-G'(x)\right|^2 h(x)dx \le& \sum_{i=1}^{n}\Bigg[\frac{1}{2}\|f_i-g_i\|^2_{L^\infty} \\ &+\Bigg[\sum_{i=1}^{n}\|f_i-g_i\|_{L^\infty}\left(\|f_i\|_{L^\infty}+\|g_i\|_{L^\infty}\right)\Bigg] \\ &\times \int_I |h''(x)|dx. \end{aligned} \tag{3.8}$$

Corollary 3.5 *Let $F:[a,b]\to\mathbb{R}^n$ and $G:[a,b]\to\mathbb{R}^n$ be two twice continuously diffier-entiable n-dimensional convex vectors, and weight function $h:[a,b]\to\mathbb{R}$*

$$h(x)=(x-a)^2(b-x)^2,\ a\le x\le b.$$

Then we have the estimate

$$\begin{aligned}\int_I \left|F'(x)-G'(x)\right|^2 h(x)dx \le& \frac{4\sqrt{3}}{9}\sum_{i=1}^{n}\Bigg[\frac{1}{2}\|f_i-g_i\|^2_{L^\infty}+\|f_i-g_i\|_{L^\infty} \\ &\times \left(\|f_i\|_{L^\infty}+\|g_i\|_{L^\infty}\right)\Bigg]\times(b-a)^3. \end{aligned} \tag{3.9}$$

Proof. Note

$$\int_I |h''(x)|\,dx = \frac{4\sqrt{3}}{9}(b-a)^3,$$

and then using the latter value in (3.5), we obtain the desired estimate. □

We define the vector convolution for $F(x) \in \chi_{[1,j]}^{[j+1,n]}[a,b]$ in the following way: Assume that

$$F(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)$$

is an n-dimensional vector and $\varepsilon = \big(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\big) \in \mathbb{R}^n$.
By $\varepsilon \to 0$ we mean $\max\{\varepsilon_1, \varepsilon_1, \dots, \varepsilon_n\} \to 0$. Now, we define

$$\theta_\varepsilon(x) = (\theta_{\varepsilon_1}(x), \theta_{\varepsilon_2}(x), \dots, \theta_{\varepsilon_n}(x))$$

where

$$\theta_{\varepsilon_i}(x) = \begin{cases} c_i exp\frac{1}{|x|^2-\varepsilon_i^2} & \text{if} \quad |x| < \varepsilon_i \\ 0 & \text{if} \quad |x| > \varepsilon_i, \end{cases},$$

$i = 1, 2, \dots, n$, where c_i are the constants such that

$$\int_I \theta_{\varepsilon_i}(x)dx = 1, \;\; i = 1, 2, \dots n.$$

Now we define the convolution of F and θ_ε as

$$F_\varepsilon(x) = F * \theta_\varepsilon(x) = \big(f_1 * \theta_{\varepsilon_1}, f_2 * \theta_{\varepsilon_2}, \dots.., f_n * \theta_{\varepsilon_n}\big)$$

where $f_i * \theta_{\varepsilon_i}$ is defined as

$$f_{\varepsilon_i} = \int_{\mathbb{R}} f_i(x-y)\theta_{\varepsilon_i}(y)dy.$$

The vector F_ε is called mollification of the vector F.
If f_i is continuous then f_{ε_i} converges uniformly to f_i on any compact subset $K \subseteq I$ i.e.

$$\sup_{x \in K} |f_{\varepsilon_i}(x) - f_i(x)| \xrightarrow{\varepsilon_i \to 0} 0,$$

which implies that

$$|F_\varepsilon - F|^2 = \sum_{i=1}^n |f_{\varepsilon_i} - f_i|^2 \xrightarrow{\varepsilon \to 0} 0,$$

uniformly on K.
Now it easy to see, with similar reasoning, that $F_\varepsilon \in \chi_{[1,j]}^{[j+1,n]}[a,b]$.

3.1.2 Existence of weak derivative and reverse Poincaré type inequality for arbitrary convex vectors

Through out this section we use $I_k = I(x_0, r_k)$ where radius r_k is defined as

$$r_k = \frac{k+1}{k+2} r.$$

It is trivial that $I_k \subset I_{k+1}$ and $\bigcup_{k=1}^{\infty} I_k = I$.
We come to the following result.

Theorem 3.2 *Let $F : I \to \mathbb{R}^n$ be an arbitrary convex vector. Then it possesses weak derivative F' in the interval $I = I(x_0, r)$ and satisfies*

$$\int_I \left|F'(x)\right|^2 h(x)dx < \infty,$$

where h is the weight function satisfying (3.1)

Proof. Let us consider F_ε, the mollification of the vector F.
Since F is continuous on interval I, we have

$$\sup_{x\in I_k} |f_{\varepsilon,i}(x) - f_i(x)| \xrightarrow{\varepsilon\to 0} 0,$$

for any closed interval $I_k \subset I$.
If $\varepsilon = \frac{1}{m}$, $m \in \mathbb{N}$, the the above convergence can be written as

$$\sup_{x\in I_k} |f_{m,i}(x) - f_i(x)| \xrightarrow{m\to\infty} 0.$$

Since $I_k \subset I$ for $p, m \in \mathbb{N}$, we write inequality (3.3) for the vectors F_p and F_m

$$\begin{aligned}\int_{I_{k+l}} \left|F'_p(x) - F'_m(x)\right|^2 h_{k+l}(x)dx \;&\le \sum_{i=1}^{n} \frac{1}{2} \|f_{p,i}(x) - f_{m,i}(x)\|^2_{L^\infty} + \sum_{i=1}^{n} \left\|f_{p,i}(x) - f_{m,i}(x)\right\|_{L^\infty} \\ &\times \left(\left\|f_{p,i}(x)\right\|_{L^\infty} + \|f_{m,i}(x)\| \right) \int_I |h''_{k+l}(x)|dx. \qquad (3.10)\end{aligned}$$

If we denote

$$c_{k+l} = \int_I |h''_{k+l}(x)|dx$$

and

$$\widehat{c}_{k+l} = \min_{x\in I} |h_{k+l}(x)|,$$

then we have

$$\widehat{c}_{k+l} \int_{I_k} \left|F'_p(x) - F'_m(x)\right|^2 dx \;\le c_{k+l} \sum_{i=1}^{n} \frac{1}{2} \|f_{p,i}(x) - f_{m,i}(x)\|^2_{L^\infty} + c_{k+l} \sum_{i=1}^{n} \left\|f_{p,i}(x) - f_{m,i}(x)\right\|_{L^\infty}$$

$$\times\left(\left\|f_{p,i}(x)\right\|_{L^\infty}+\|f_{m,i}(x)\|\right). \tag{3.11}$$

Since

$$\left\|f_{p,i}-f_{m,i}\right\|_{L^\infty_{I_{k+l}}}\to 0,\ m,p\to\infty,$$

we have

$$\lim_{m,p\to\infty}\int_{I_k}\left|F_p'(x)-F_m'(x)\right|^2dx=\lim_{m,p\to\infty}\sum_{i=1}^{n}\int_{I_k}\left(f_{p,i}'(x)-f_{m,i}'(x)\right)^2dx=0.$$

By the completeness of the space $L_\infty(I_k)$, there exist an n-dimensional measurable vector

$$g_k=(g_{k,1},g_{k,2},\ldots,g_{k,n})$$

such that

$$\lim_{m\to\ 0}\sum_{i=1}^{n}\int_{I_k}\left(f_{m,i}'(x)-g_{k,i}(x)\right)^2dx=0.$$

Let us extend g_k, trivially outside the Interval I_k by 0, and define

$$g(x)=\lim_{k\to\infty}\sup g_{k,i}(x).$$

It is obvious that $g(x)=g_k(x)$ on interval I_k.
We claim that

$$g(x)=\big(g_1(x),g_2(x),\ldots,g_n(x)\big)$$

is the weak derivative of

$$F(x)=\big(f_1(x),f_2(x),\ldots,f_n(x)\big).$$

To show this it is enough to prove that g_i is the weak partial derivative of f_i, for all $i=1,2,\ldots,n$.
To do this, let us take $\phi\in C_0^\infty(I)$. Then $\mathrm{supp}\phi\subset I_k$ for some k.
Hence

$$\int_{I_k}f_{m,i}'(x)\phi(x)=-\int_{I_k}f_{m,i}(x)\phi'(x)dx$$

Since

$$|f_{m,i}(x)-f_i(x)|_{I_k}\xrightarrow{m\to\infty}0$$

and

$$\left\|f_{m,i}'-g_i\right\|_{L^2(I_k)}\xrightarrow{m\to\infty}0$$

which implies

$$\int_{I_k}g_i(x)\phi(x)dx=-\int_{I_k}f_i(x)\phi'(x)dx.$$

Thus, g_i is the weak derivative of f_i for $i = 1, 2, \ldots, n$
Again, writing the inequality (3.3) for $F = F_m$ and $G = 0$, we have

$$\int_{I_{k+l}} \left|F_m'(x)\right|^2 h_{k+l} dx \leq \sum_{i=1}^{n} \left[\|f_m\|^2_{L^\infty_{I_{k+l}}} + \frac{1}{2} \|f_m\|^2_{L^\infty_{I_{k+l}}} \right] \int_{I_{k+l}} \left|h''(x) dx\right|.$$

Hence, we have

$$\int_{I_{k+l}} \left|F_m'(x)\right|^2 h_{k+l} dx \leq c_{k+l} \sum_{i=1}^{n} \left[\|f_m\|^2_{L^\infty_{I_{k+l}}} + \frac{1}{2} \|f_m\|^2_{L^\infty_{I_{k+l}}} \right].$$

Taking limit as $m \to \infty$, we get

$$\int_{I_{k+l}} \left|F'(x)\right|^2 h_{k+l} dx \leq c_{k+l} \|f\|^2_{L^\infty_{I_{k+l}}}$$

Since $I_k \subseteq I_{k+l}$ therefore

$$\int_{I_k} \left|F'(x)\right| h_{k+l}(x) dx \leq c_{k+l} \|F\|^2_{L^\infty_{I_{k+l}}}.$$

In the latter integral, letting $l \to \infty$ we find

$$\int_{I_k} \left|F'(x)\right|^2 h(x) dx \leq c_\infty \|F\|^2_{L^\infty_{I_k}} < \infty$$

Since above integral is bounded for each k, so we have

$$\int_{I} \left|F'(x)\right|^2 h(x) dx < \infty.$$

$\square$

Theorem 3.3 *Let $F : I \to \mathbb{R}^n$ and $G : I \to \mathbb{R}^n$ be two arbitrary convex vectors that belongs to $\chi^{[i,j]}_{[j+1,n]}[a,b]$ and let $h : I \to \mathbb{R}$ be the nonnegative weight function satisfying* (3.1) *on interval I, then the following estimate holds*

$$\int_{I} \left|F'(x) - G'(x)\right|^2 h(x) dx \leq \sum_{i=1}^{n} \frac{1}{2} \|f_i - g_i\|^2_{L^\infty} + \|f_i - g_i\|_{L^\infty} \times \left(\|f_i\|_{L^\infty} + \|g_i\|_{L^\infty}\right) \int_{I} |h''(x)| dx. \tag{3.12}$$

Proof. For arbitrary convex vectors F and G which are continuous, we take smooth approximations $F_{m,i}$ and $G_{m,i}$ $m,i \in \mathbb{N}$. Then, there exists integer m_{k+l} such that $F_{m,i}$ is smooth over the interval I_{k+l} and $F_{m,i}(x)$ converges uniformly to F for $m \geq m_{k+l}$.
Let us write the inequality (3.3) for the functions $F_{m,1}$ and $F_{m,2}$ on the interval I_{k+l} as

$$\begin{aligned}\int\limits_{I_{k+l}} \left|F'_{m,i}(x) - G'_{m,i}(x)\right|^2 h_{k+l}(x)dx \leq & \int\limits_{I_{k+l}} \left[\frac{1}{2}\sum_{i=1}^{n} \|f_{m,i} - g_{m,i}\|^2_{L^\infty}\right. \\ & + \sum_{i=1}^{n} \|f_{m,i} - g_{m,i}\|_{L^\infty} \\ & \left.\times \left(\|f_{m,i}\|_{L^\infty} + \|g_{m,i}\|_{L^\infty}\right)\right] |h''(x)|dx.\end{aligned}$$

Taking limit $m \to \infty$, we get

$$\begin{aligned}\int\limits_{I_{k+l}} \left|F'_i(x) - G'_i(x)\right|^2 h_{k+l}(x)dx \leq & \int\limits_{I_{k+l}} \left[\frac{1}{2}\sum_{i=1}^{n} \|f_i - g_i\|^2_{L^\infty}\right. \\ & + \sum_{i=1}^{n} \|f_i - g_i\|_{L^\infty} \\ & \left.\times \left(\|f_i\|_{L^\infty} + \|g_i\|_{L^\infty}\right)\right] |h''(x)|dx.\end{aligned}$$

As $I_k \subset I_{k+l}$, taking limit $l \to \infty$, we obtain

$$\begin{aligned}\int\limits_{I_k} \left|F'(x) - G'(x)\right|^2 h_k(x)dx \leq & \int\limits_{I} \left[\frac{1}{2}\sum_{i=1}^{n} \|f_i - g_i\|^2_{L^\infty}\right. \\ & + \sum_{i=1}^{n} \|f_i - g_i\|_{L^\infty} \\ & \left.\times \left(\|f_i\|_{L^\infty} + \|g_i\|_{L^\infty}\right)\right] |h''(x)|dx\end{aligned}$$

Using Theorem 3.2, we get

$$\int\limits_{I_k} \left|F'(x) - G'(x)\right|^2 h(x)dx < \infty.$$

By letting $k \to \infty$, we have complete proof of the theorem. □

3.2 Weighted energy estimates for second derivative of 4-convex vector

Similarly as in Section 2.1, the n-dimensional vector

$$F(x) = \big(f_1(x), f_2(x), \ldots.., f_n(x)\big) \tag{3.13}$$

is called smooth $4-$convex vector if

$$\frac{d^4}{dx^4} f_i(x) \geq 0, \; i = 1, 2, \ldots, n. \tag{3.14}$$

Let $\Upsilon_{[1,j]}^{[j+1,n]}[a,b]$ be the class of vectors having 4-convex function on its first j components and remaining components are 4-concave functions at interval $[a,b]$ and $\Upsilon_{[j+1,n]}^{[1,j]}[a,b]$ be the class of vectors having 4-concave functions on its first j components and remaining are 4-convex at the interval $[a,b]$. It is trivial that if $F \in \Upsilon_{[1,j]}^{[j+1,n]}[a,b]$ then $-F \in \Upsilon_{[j+1,n]}^{[1,j]}[a,b]$. Let h be the weight function which is non-negative 2-concave function in $C^4[a,b]$ satisfy

$$h(a) = h(b) = 0, h'(a) = h'(b) = 0, \; h''(a) = h''(b) = 0, \; h'''(a) = h'''(b) = 0, \tag{3.15}$$

for $a \leq x \leq b$.

We will start by the following theorem

Corollary 3.6 *Let $F : I \to \mathbb{R}^n$ and $G : I \to \mathbb{R}^n$ be the two smooth convex and $4-$convex vectors. Let $h : I \to \mathbb{R}$ be the smooth non negative weight function which satisfies* (3.15). *Then the following energy estimate is valid*

$$\begin{aligned} \int_I \left|F''(x) - G''(x)\right|^2 h(x)dx \; \leq \; & \sum_{i=1}^{n} \int_I \Bigg[\frac{(f_i(x) - g_i(x))^2}{2} - \sup_{x \in I} |f_i(x) - g_i(x)| \\ & \times \big(f_i(x) + g_i(x)\big) \Bigg] h^{(4)}(x)dx. \end{aligned} \tag{3.16}$$

Proof. On

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx = \sum_{i=1}^{n} \int_I \Big(f_i''(x) - g_i''(x)\Big)^2 h(x)dx,$$

we apply Lemma 3.1. □

Remark 3.1 *Taking the supremum of* (3.16)*, we obtain*

$$\int_I \left|F''(x)-G''(x)\right|^2 h(x)dx \leq \sum_{i=1}^{n}\left[\frac{1}{2}\|f_i-g_i\|_{L^\infty}^2+\|f_i-g_i\|_{L^\infty}\right.$$
$$\left.\times\ \left(\|f_i\|_{L^\infty}+\|g_i\|_{L^\infty}\right)\right]\int_I |h^{(4)}(x)|dx. \tag{3.17}$$

Remark 3.2 *If F and G are* 4*-concave vectors, then using Lemma 3.1, we have*

$$\int_I \left|F''(x)-G''(x)\right|^2 h(x)dx \leq \sum_{i=1}^{n}\int_I\left[\frac{(f_i(x)-g_i(x))^2}{2}+\sup_{x\in I}|f_i(x)-g_i(x)|\right.$$
$$\left.\times\ (f_i(x)+g_i(x))\right]h^{(4)}(x)dx. \tag{3.18}$$

Theorem 3.4 *Let $F:I\to\mathbb{R}^n$ and $G:I\to\mathbb{R}^n$ be the two vectors that belongs to $\Upsilon_{[1,j]}^{[j+1,n]}[a,b]$ and to $\chi_{[1,j]}^{[j+1,n]}[a,b]$, respectively. Let $h:I\to\mathbb{R}$ be the nonnegative weight function satisfying* (3.15)*. Then the following inequality is valid*

$$\int_I \left|F''(x)-G''(x)\right|^2 h(x)dx \leq \int_I\left[\sum_{i=1}^{n}\frac{\left(f_i(x)-g_i(x)\right)^2}{2}+\sum_{i=1}^{j}\sup_{x\in I}\left|f_i(x)-g_i(x)\right|\left[f_i(x)+g_i(x)\right]\right.$$
$$\left.-\sum_{i=j+1}^{n}\sup_{x\in I}\left|f_i(x)-g_i(x)\right|\times\left[f_i(x)+g_i(x)\right]\right]h^{(4)}(x)dx. \tag{3.19}$$

Proof. We have

$$\int_I |F''(x)-G''(x)|^2 h(x)dx=\sum_{i=1}^{n}\int_I\left(f_i''(x)-g_i''(x)\right)^2 h(x)dx \tag{3.20}$$
$$=\sum_{i=1}^{j}\int_I\left(f_i''(x)-g_i''(x)\right)^2 h(x)dx+\sum_{i=j+1}^{n}\int_I\left(f_i''(x)-g_i''(x)\right)^2 h(x)dx. \tag{3.21}$$

Using Lemma 3.1 on the first integral, we obtain

$$\int_I\left(f_i''(x)-g_i''(x)\right)^2 h(x)dx \leq \int_I\left[\sum_{i=1}^{n}\frac{\left(f_i(x)-g_i(x)\right)^2}{2}-\sup_{x\in I}\left|f_i(x)-g_i(x)\right|\right.$$
$$\left.\times\ \left(f_i(x)-g_i(x)\right)\right]h^{(4)}(x)dx. \tag{3.22}$$

Similarly using the Corollary 3.1 on the second integral, we have the following inequality (3.19)

$$\int_I\left(f_i''(x)-g_i''(x)\right)^2 h(x)dx \leq \int_I\left[\sum_{i=1}^{n}\frac{\left(f_i(x)-g_i(x)\right)^2}{2}+\sup_{x\in I}\left|f_i(x)-g_i(x)\right|\right.$$

$$\times \left(f_i(x) - g_i(x) \right) \Big] h^{(4)}(x)dx, \tag{3.23}$$

If we combine inequalities (3.22) and (3.23), we get the required inequality (3.19). □

3.2.1 The case of an arbitrary 4-convex vector

We will use the I_k for the interval $I(x_0, r_k)$, x_0 is the center and radius r_k is defined as where

$$r_k = \frac{k+1}{k+2} r.$$

It is trivial that $I_k \subset I_{k+1}$ and $\bigcup_{k=1}^{\infty} I_k = I$

Theorem 3.5 *Let $F : I \to \mathbb{R}^n$ be a continuous $4-$convex vector and let $h : I \to \mathbb{R}$ be the nonnegative weight function satisfying* (3.15). *Then*

$$\int_I \left| F''(x) \right|^2 h(x)dx < \infty.$$

Proof. Let F_m be the $\frac{1}{m}-$mollification of F. If we write the inequality (3.17) for $F = F_m$ and $G = 0$ and for intervals $I_{k+l} \subset I$, we have

$$\int_{I_{k+l}} \left| F_m''(x) \right|^2 h_{k+l} dx \leq \frac{3}{2} \sum_{i=1}^{n} \left[\| f_{m,i} \|^2_{L^\infty_{I_{k+l}}} \right] \int_{I_{k+l}} \left| h^{(4)}_{k+l}(x) \right| dx.$$

If we denote

$$c_{k+l} = \int_{I_{k+l}} \left| h^{(4)}_{k+l}(x) \right| dx,$$

we have

$$\int_{I_{k+l}} \left| F_m''(x) \right|^2 h_{k+l} dx \leq \frac{3c_{k+l}}{2} \sum_{i=1}^{n} \| f_{m,i} \|^2_{L^\infty_{I_{k+l}}}.$$

Applying limit as $m \to \infty$, we have

$$\int_{I_{k+l}} \left| F''(x) \right|^2 h_{k+l}(x) dx \leq \frac{3c_{k+l}}{2} \sum_{i=1}^{n} \| f_i \|^2_{L^\infty_{I_{k+l}}}.$$

Since $I_k \subseteq I_{k+l}$ so

$$\int_{I_k} \left| F''(x) \right|^2 h_{k+l}(x) dx \leq \frac{3c_{k+l}}{2} \| F \|^2_{L^\infty_{I_{k+l}}}.$$

In the above integral make $l \to \infty$, we have

$$\int_I \left|F''(x)\right|^2 h(x)dx \leq \frac{3c_\infty}{2} \|F(x)\|^2_{L^\infty_I} < \infty.$$

□

Theorem 3.6 *Let $F : I \to \mathbb{R}^n$ and $G : I \to \mathbb{R}^n$ be the two arbitrary 4−convex vectors that belongs to $\Upsilon^{[1,j]}_{[j+1,n]}[a,b]$ and also belongs to $\chi^{[1,j]}_{[j+1,n]}[a,b]$. Then the following inequality is valid*

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx \leq \sum_{i=1}^{n} \left[\frac{1}{2}\|f_i - g_i\|^2_{L^\infty} + \|f_i - g_i\|_{L^\infty} \times (\|f_i\|_{L^\infty} + \|g_i\|_{L^\infty})\right] \int_I |h^{(4)}(x)|dx. \tag{3.24}$$

Proof. For arbitrary continuous 4−convex vectors F and G respectively, take smooth approximation F_m and G_m. There exist integer m_{k+l} such that F_m and G_m is smooth over the interval I_{k+l} and F_m and G_m converges uniformly to F and G respectively for $m \geq m_{k+l}$. Let us write the inequality (3.17) for the functions F_m and G_m on the interval I_{k+l}

$$\int_{I_{k+l}} \left|F''_m(x) - G''_m(x)\right|^2 h_{k+l}dx \leq \sum_{i=1}^{n} \left[\frac{1}{2}\|f_{m,i} - g_{m,i}\|^2_{L^\infty_{I_{k+l}}} + \|f_{m,i} - g_{m,i}\|_{L^\infty_{I_{k+l}}} \times \left(\|f_{m,i}\|_{L^\infty_{I_{k+l}}} + \|g_{m,i}\|_{L^\infty_{I_{k+l}}}\right)\right] \int_{I_{k+l}} |h^{(4)}_{k+l}(x)|dx. \tag{3.25}$$

Applying limit $m \to \infty$, we get

$$\int_{I_{k+l}} \left|F''(x) - G''(x)\right|^2 h_{k+l}dx \leq \sum_{i=1}^{n} \left[\frac{1}{2}\|f_i - g_i\|^2_{L^\infty_{I_{k+l}}} + \|f_i - g_i\|_{L^\infty_{I_{k+l}}} \times \left(\|f_i\|_{L^\infty_{I_{k+l}}} + \|g_i\|_{L^\infty_{I_{k+l}}}\right)\right] \int_{I_{k+l}} \left|h^{(4)}_{k+l}(x)\right| dx. \tag{3.26}$$

Writing the left integral for smaller interval $I_k \subset I_{k+l}$ and taking limit as $l \to \infty$, we obtain

$$\int_{I_k} \left|F''(x) - G''(x)\right|^2 h(x)dx \leq \left[\frac{1}{2}\|f_i - g_i\|^2_{L^\infty_{I_{k+l}}} + \|f_i - g_i\|_{L^\infty_I} \left(\|f_i\|_{L^\infty_I} + \|g_i\|_{L^\infty_I}\right)\right] \int_I |h^{(4)}(x)|dx. \tag{3.27}$$

Since, by the last theorem, we have

$$\int_I \left|F''(x) - G''(x)\right|^2 h(x)dx < \infty,$$

then using dominated convergence theorem, after we take the limit, $k \to \infty$, we obtain desired result (3.24).

$\square$

Chapter 4

The weighted energy inequalities for subsolution of 2nd order partial differential equations

4.1 Reverse Poincaré-type inequalities for the difference of superharmonic functions

Throughout this chapter we assume that domain $D,\ D \subset R^n$ is bounded set having smooth boundary.

A function u is said to be smooth super-harmonic if $u \in C^2(B)$, and

$$\Delta u(x) \leq 0, \quad x \in \overline{D}. \tag{4.1}$$

A bounded measurable function u defined on ball B is said to be weak super-harmonic if for all non negative function $\phi \in C_0^2(B)$ the following holds

$$\int_B u(x)\Delta\phi(x)dx \leq 0. \tag{4.2}$$

We will consider the arbitrary smooth weight function satisfying the following:

$$\left.\begin{aligned} h(x) \geq 0 \ \text{ if } x \in D \\ h(x) = \frac{\partial h(x)}{\partial x_i} = 0, i = 1, 2, \ldots,\ n,\ x \in \partial D. \end{aligned}\right\} \tag{4.3}$$

We will also take particular form of weight function h for the ball $B(x_0, r)$

$$h(x) = (r^2 - \mid x - x_0 \mid^2)^2. \tag{4.4}$$

We will find

$$\frac{\partial h(x)}{\partial x_i} = 4(x_i - x_i^0)(r^2 - \mid x - x_0 \mid^2), \quad i = 1, 2, \ldots, n. \tag{4.5}$$

It is clear by definition of weight function h that $h(x) = \dfrac{\partial h(x)}{\partial x_i} = 0,\ i = 1, 2, \ldots, n$, for x on the boundary of the ball $B(x_0, r)$.

4.1.1 The case of smooth superharmonic functions and mollification of weak superharmonic functions

Our starting point will be the following theorem.

Theorem 4.1 *Let $u_i, i = 1, 2$, be two smooth superharmonic functions over domain $D \subset \mathbb{R}^n$ and h is the weight function defined in* (4.3)*. Then the following holds*

$$\int_D |\mathrm{grad}\, \mathrm{u}_2(\mathrm{x}) - \mathrm{grad}\, \mathrm{u}_1(\mathrm{x})|^2 \mathrm{h}(\mathrm{x}) \mathrm{dx} \leq \| \tilde{\mathrm{u}} \|_{\mathrm{L_p(D)}} \| \Delta \mathrm{h} \|_{\mathrm{L_q(D)}}, \tag{4.6}$$

where p and q are conjugates and

$$\tilde{u}(x) = \frac{1}{2}(u_2(x) - u_1(x))^2 - \| u_2 - u_1 \|_{L^\infty} (u_2(x) + u_1(x)).$$

Proof. Let us denote $u(x) = u_2(x) - u_1(x)$. Then

$$\int_D |\mathrm{grad}\, \mathrm{u}(\mathrm{x})|^2 \mathrm{h}(\mathrm{x}) \mathrm{dx} = \int_D \left(\frac{\partial u}{\partial x_1}\right)^2 h(x)dx + \int_D \left(\frac{\partial u}{\partial x_2}\right)^2 h(x)dx + \cdots + \int_D \left(\frac{\partial u}{\partial x_n}\right)^2 h(x)dx. \tag{4.7}$$

Consider the first integral on right hand side

$$\int_D \left(\frac{\partial u}{\partial x_1}\right)^2 h(x)dx = \int_D \frac{\partial u}{\partial x_1} \left(\frac{\partial u}{\partial x_1} h(x)\right) dx.$$

Using integration by parts and the fact that weight function vanishes on the boundary of the domain, we get

$$\begin{aligned}\int_D \left(\frac{\partial u}{\partial x_1}\right)^2 h(x)dx &= -\int_D u(x)\frac{\partial^2 u}{\partial x_1^2}h(x)dx - \int_D u(x)\frac{\partial u}{\partial x_1}\frac{\partial h(x)}{\partial x_1} \\ &= -\int_D u(x)\frac{\partial^2 u}{\partial x_1^2}h(x)dx - \frac{1}{2}\int_D \frac{\partial u^2(x)}{\partial x_1}\frac{\partial h(x)}{\partial x_1}dx\end{aligned}$$

Again using integration by parts formula on second integral of and also definition of weight function, we obtain

$$\int_D \left(\frac{\partial u}{\partial x_1}\right)^2 h(x)dx = -\int_D u(x)\frac{\partial^2 u}{\partial x_1^2}h(x)dx + \frac{1}{2}\int_D u^2(x)\frac{\partial^2 h(x)}{\partial x_1^2}dx$$

Solving all integrals of in the similar way, (4.7) becomes

$$\begin{aligned}\int_D |\mathrm{grad}\,\mathrm{u}(\mathrm{x})|^2\mathrm{h}(\mathrm{x})\mathrm{dx} &= \frac{1}{2}\int_D u^2(x)\Delta h(x)dx - \int_D u(x)\Delta u(x)h(x)dx \\ &\leq \frac{1}{2}\int_D u^2(x)\Delta h(x)dx + \sup_{x\in D}|\, u(x)\,|\int_D |\,\Delta u(x)\,|\, h(x)dx\end{aligned}$$

It is clear that $|\,\Delta u(x)\,|\leq|\,\Delta u_2(x)\,|+|\,\Delta u_1(x)\,|$, and since u_1 and u_2 are subharmonic, we have $|\,\Delta u_2(x)\,| = -\Delta u_2(x),\ |\,\Delta u_1(x)\,| = -\Delta u_1(x)$.
Now

$$\begin{aligned}\int_D |\mathrm{grad}\,\mathrm{u}(\mathrm{x})|^2\mathrm{h}(\mathrm{x})\mathrm{dx} \ \leq\ & \frac{1}{2}\int_D u^2(x)\Delta h(x)dx \\ & - \sup_{x\in D}|\, u(x)\,|\int_D (\Delta(u_2(x)+u_1(x)))h(x)dx.\end{aligned}$$

Using Green-Gauss theorem and the definition of weight function, we have

$$\begin{aligned}\int_D |\mathrm{grad}\,\mathrm{u}(\mathrm{x})|^2\mathrm{h}(\mathrm{x})\mathrm{dx} &\leq \frac{1}{2}\int_D u^2(x)\Delta h(x)dx - \sup_{x\in D}|\, u(x)\,|\int_D (u_2(x)+u_1(x))\Delta h(x)dx \qquad (4.8)\\ &\leq \int_D \left[\frac{u^2(x)}{2} - \|\, u(x)\,\|_{L^\infty}\,(u_2(x)+u_1(x))\right]\Delta h(x)dx.\end{aligned}$$

Hence,

$$\int_D |\mathrm{grad}\,\mathrm{u}(\mathrm{x})|^2\mathrm{h}(\mathrm{x})\mathrm{dx} \leq \int_D \left|\frac{\mathrm{u}^2(\mathrm{x})}{2} - \|\,\mathrm{u}\,\|_{\mathrm{L}^\infty}\,(\mathrm{u}_2(\mathrm{x})+\mathrm{u}_1(\mathrm{x}))\right| |\Delta\mathrm{h}(\mathrm{x})|\mathrm{dx}.$$

Finally using Hölder inequality we get the required result. □

Remark 4.1 *Using the definition of modulus on* (4.8) *we obtain the following inequality:*

$$\int_D |\mathrm{grad}\,\mathrm{u}(\mathrm{x})|^2 \mathrm{h}(\mathrm{x})\mathrm{dx} \leq \frac{1}{2} \parallel u_2 - u_1 \parallel^2_{L^\infty(D)} + \parallel u_2 - u_1 \parallel_{L^\infty(D)} \times (\parallel u_2 \parallel_{L^\infty(D)} + \parallel u_1 \parallel_{L^\infty(D)}) \int_D |\Delta h(x)|\, dx$$

Writing the above remark for arbitrary ball $B, B = B(x_0, r) \subset R^n$, we get

Remark 4.2

$$\int_B |\mathrm{grad}\,\mathrm{u}(\mathrm{x})|^2 \mathrm{h}(\mathrm{x})\mathrm{dx} \leq \frac{1}{2} \parallel u_2 - u_1 \parallel^2_{L^\infty(B)} + \parallel u_2 - u_1 \parallel_{L^\infty(B)} \times (\parallel u_2 \parallel_{L^\infty(B)} + \parallel u_1 \parallel_{L^\infty(B)}) \int_B |\Delta h(x)|\, dx. \tag{4.9}$$

Now we approximate the weak superharmonic function u by the smooth ones. For this we will again use the mollification technique.
Define

$$\eta(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1 \\ 0, \ |x| \geq 1 \end{cases} \tag{4.10}$$

where $x \in \mathbb{R}^n$, and $c > 0$ is constant such that

$$\int_{\mathbb{R}^n} \eta(x)dx = 1.$$

Let us define the mollification of bounded measurable function $u(x)$ on ball B

$$u_\varepsilon(x) = \int_B \eta\left(\frac{x-y}{\varepsilon}\right) u(y)dy. \tag{4.11}$$

If we denote $\eta_\varepsilon(x-y) = \eta\left(\frac{x-y}{\varepsilon}\right)$, then it is clear that

$$\frac{\partial^2}{\partial x_i^2}\eta_\varepsilon(x-y) = \frac{\partial^2}{\partial y_i^2}\eta_\varepsilon(x-y), \ i = 1, 2, \ldots, n. \tag{4.12}$$

Using (4.12) in (4.11), we have

$$\Delta_x u_\varepsilon(x) = \int_B u(y)\Delta_y \eta_\varepsilon(x-y)dy, \tag{4.13}$$

where Δ_x and Δ_y are the Laplace operator with respect to x and y respectively. Also define the balls B_k in the following way

$$B_k = B(x_0, r_k)$$

where $r_k = \frac{k+1}{k+2}r$.
The following theorem states that the functions u_ε are smooth superharmonic functions on B_k for sufficiently small ε.

Theorem 4.2 *Let u be the weak superharmonic function on ball $B = B(x_0, r)$. Then for any $k \in \mathbb{N}$ there exist $\hat{\varepsilon} > 0$ such that for each ε, $0 < \varepsilon < \hat{\varepsilon}$, each function $u_\varepsilon(x)$ is smooth superharmonic on ball B_k that is*

$$\Delta u_\varepsilon(x) \leq 0 \ \text{if } x \in B_k. \tag{4.14}$$

Proof. Take $\hat{\varepsilon} = \dfrac{r}{2(k+2)}$. By definition it is trivial that $u_\varepsilon(x), \varepsilon > 0$ is infinitely differentiable w.r.t x. Now we will see that for arbitrary $x \in B_k$ the function $\eta_\varepsilon(x-y)$ has compact support on B as a function of y.
Take the ball $\hat{B}_k$ in the following way

$$\hat{B}_k = B\left(x_0, \frac{2k+3}{2k+4} r\right)$$

Take $y \in \hat{B}_k$, then

$$|\, y - x \,| > \frac{1}{2(k+2)} r > \varepsilon.$$

Hence we have $\eta_\varepsilon(x-y) = 0$. Therefore the non negative function $\eta_\varepsilon(x-y)$ has a compact support in B as a function of y. So by the definition of weak super harmonic function $u(x)$, we have

$$\int_B u(y) \Delta_y \eta_\varepsilon(x-y) \leq 0$$

From (4.12) we get $\Delta u_\varepsilon(x) \leq 0$, if $x \in B_k$ and $\varepsilon < \hat{\varepsilon}$. □

4.1.2 Existence of Sobolev gradient

Let us introduce the weight function h_k corresponding to the balls B_k

$$h_k(x) = (r_k^2 - |\, x - x_0 \,|^2)^2, \ x \in \overline{B}_k, \ k \in \mathbb{N}. \tag{4.15}$$

The following theorem will show, the existence of the weak derivative, and square integrability with respect to weight function h.

Theorem 4.3 *Let u be a continuous weak superharmonic function, then it has weak partial derivatives $\frac{\partial u(x)}{\partial x_i}$, $i = 1, 2, \ldots, n$ in the ball $B(x_0, r)$ and they are square integrable with respect to the weight function h, i.e.*

$$\int_{B(x_0,r)} |\mathrm{grad\,u(x)}|^2 \mathrm{h(x)dx} < \infty. \tag{4.16}$$

Proof. If u is continuous in the ball B then on any compact set K, $K \subset B$, we have the uniform convergence(see, for example, Evans [14])

$$\sup_{x \in K} |\, u_\varepsilon(x) - u(x) \,| \xrightarrow{\varepsilon \to 0} 0,$$

where u_ε is the mollification of weak super harmonic function u.
Taking $\varepsilon = \frac{1}{m}$, $m = 1, 2, \cdots$, the latter convergence take the form

$$\sup_{x\in K} \mid u_m(x) - u(x) \mid \xrightarrow{m\to\infty} 0.$$

Since by definition it is clear that $B_k \subset B$ (compactly embedded), we have from Theorem 2.3, for any $k \in \mathbb{N}$, that there exists $m_k \in \mathbb{N}$, such that u_m is smooth subharmonic in the ball B_k for every $m \geq m_k$.
Now writing the inequality (4.9) for the ball B_{k+l} and for the functions

$$u_1(x) = u_m(x),\ u_2(x) = u_p(x),\ m, p \geq m_{k+l},$$

we get

$$\begin{aligned}\int\limits_{B_{k+l}} |\mathrm{grad\,u_p(x) - grad u_m(x)}|^2 \mathrm{h_{k+1}(x)dx} \leq \frac{1}{2} \parallel u_p - u_m \parallel^2_{L^\infty B_{k+l}} \\ + \parallel u_p - u_m \parallel_{L^\infty(B_{k+l})} (\parallel u_p \parallel_{L^\infty(B_{k+l})} \\ + \parallel u_m \parallel_{L^\infty(B_{k+l})}) \int_{B_{k+l}} \mid \Delta h(x) \mid dx. \qquad (4.17)\end{aligned}$$

Let us denote

$$\alpha_{k+l} = \int\limits_{B_{k+l}} \mid \Delta h_{k+l} \mid dx,\ \ \widehat{\alpha} = \inf_{x\in B_k} h_{k+l}(x).$$

Since, $B_k \subset B_{k+l}$, from (5.110), we have

$$\widehat{\alpha} \int\limits_{B_k} |\mathrm{grad\,u_p(x) - grad u_m(x)}|^2 \mathrm{dx} \leq \qquad (4.18)$$

$$\leq \alpha_{k+l} \left(\frac{1}{2} \parallel u_p - u_m \parallel^2_{L^\infty(B_{k+l})} + \parallel u_p - u_m \parallel_{L^\infty(B_{k+l})} (\parallel u_p \parallel_{L^\infty(B_{k+l})} + \parallel u_m \parallel_{L^\infty(B_{k+l})}) \right).$$

Since

$$\parallel u_p - u_m \parallel \xrightarrow{m,p\to\infty} 0,$$

from (4.18), we get

$$\lim_{m,p\to\infty} \sum_{i=1}^{n} \int\limits_{B_k} \left(\frac{\partial u_p(x)}{\partial x_i} - \frac{\partial u_m(x)}{\partial x_i} \right)^2 dx = 0$$

The space $L^2(B_k)$ is complete, so there exist family of measurable functions $v_{k,i} \in L^2(B_k)$, $i \in \mathbb{N}$ such that

$$\lim_{m\to\infty} \sum_{i=1}^{n} \int\limits_{B_k} \left(\frac{\partial u_m}{\partial x_i} - v_{k,i}(x) \right)^2 dx = 0.$$

Let us define $\widetilde{v}_{k,i}(x)$ in the the following way

$$\widetilde{v}_{k,i}(x) = \begin{cases} v_{k,i}(x), \, x \in B_k, \\ 0, \quad x \in B - B_k, \end{cases} \tag{4.19}$$

and then we define

$$v_i(x) = \limsup_{k \to \infty} \widetilde{v}_{k,i}(x),$$

$i = 1, 2, \ldots, n$.
By definition it is clear that

$$v_i(x) = v_{k,i}(x) \;\; x \in B_k.$$

Thus the functions v_i are locally square integrable on the ball B.
We claim that v_i, $i = 1, 2, \ldots, n$, is Sobolov weak derivatives of function u. To prove this, take an arbitrary

$$\phi \in C_0^\infty(B).$$

Then $\mathrm{supp}(\phi) \subseteq B_k$, for some $k \in \mathbb{N}$, and

$$\int_{B_k} \frac{\partial u_m}{\partial x_i} \phi(x) dx = -\int_{B_k} u_m \frac{\partial \phi_m}{\partial x_i} dx,$$

for any $m \geq m(k)$.
Hence, after we pass with limit $m \to \infty$, we have

$$\int_{B_k} v_i(x) \phi(x) dx = -\int_{B_k} u(x) \frac{\partial \phi}{\partial x_i} dx$$

This shows $v_i, \; i = 1, 2, \ldots, n$ is $i-$th partial derivative of u.
Again writing the inequality (4.9) for the ball $B_{k+1} \subseteq B$ and for $u_1 = 0$ $, u_2 = u_m$, we get

$$\int_{B_{k+l}} \mid \mathrm{grad}\, \mathrm{u_m(x)} \mid^2 \mathrm{h_{k+1}(x) dx} \leq \frac{3}{2} \parallel \mathrm{u_m} \parallel^2_{\mathrm{L^\infty(B_{k+1})}} \int_{\mathrm{B_{k+1}}} \mid \Delta \mathrm{h_{k+1}(x)} \mid \mathrm{dx}.$$

Passing with limit as $m \to \infty$, we get

$$\int_{B_{k+l}} \mid \mathrm{grad}\, \mathrm{u(x)} \mid^2 \mathrm{h_{k+1}(x) dx} \leq \frac{3}{2} \parallel \mathrm{u} \parallel^2_{\mathrm{L^\infty(B_{k+1})}} \int_{\mathrm{B_{k+1}}} \mid \Delta \mathrm{h_{k+1}(x)} \mid \mathrm{dx}.$$

Since $B_k + 1 \subseteq B$, taking left hand integral on the smaller ball, we have

$$\int_{B_k} \mid \mathrm{grad}\, \mathrm{u(x)} \mid^2 \mathrm{h_{k+1}(x) dx} \leq \frac{3}{2} \parallel \mathrm{u} \parallel^2_{\mathrm{L^\infty(B_{k+1})}} \int_{\mathrm{B_{k+1}}} \mid \Delta \mathrm{h_{k+1}(x)} \mid \mathrm{dx}.$$

Now, we let $l \to \infty$,

$$\int_{B_k} \mid \mathrm{grad}\, \mathrm{u(x)} \mid^2 \mathrm{h(x) dx} \leq \frac{3}{2} \parallel \mathrm{u} \parallel_{\mathrm{L^\infty(B)}} \int_{\mathrm{B}} \mid \Delta \mathrm{h(x)} \mid \mathrm{dx} < \infty.$$

The left hand side is increasing and bounded, then by dominated convergence theorem,

$$\int_B |\operatorname{grad} u(x)|^2 h(x)dx < \frac{3}{2} \| u \|_{L^\infty(B)} \int_B |\Delta h(x)| dx < \infty.$$

□

Theorem 4.4 *Let $u_i,\ i = 1,2$ be two continuous weak superharmonic functions on the ball $B = B(x_0, r)$, then the following energy estimate holds*

$$\begin{aligned}\int_B |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 h(x)dx \le{} & [\frac{1}{2} \| u_2 - u_1 \|^2_{L_\infty(B)} \\ & + \| u_2 - u_1 \|_{L_\infty(B)} (\| u_1 \|_{L_\infty(B)} \\ & + \| u_2 \|_{L_\infty(B)})] \int_B |\Delta h(x)| dx. \qquad (4.20)\end{aligned}$$

where h is the weight function defined in (4.4).

Proof. Let $u_{m,i},\ i = 1,2$, be two the mollifications of weak super-harmonic functions $u_i,\ i = 1,\ 2$.
Then we have that for a ball B_{k+l}, there exist integer m_{k+l} such that each function $u_{m,i}, i = 1,2$ is smooth superharmonic function on the ball B_{k+l} if $m \ge m_{k+l}$.
Also we have the following convergence

$$\| u_{m,i} - u_i \| \xrightarrow{m\to\infty} 0,\ i = 1,2.$$

Now we apply the inequality (2.4) for the functions $u_{m,1}$ and $u_{m,2}$ on the ball B_{k+l}. We have

$$\int_{B_{k+l}} |\operatorname{grad} u_{m,2}(x) - \operatorname{grad} u_{m,1}(x)|^2 h_{k+1}(x)dx \le \qquad (4.21)$$

$$\le \alpha_{k+l}[\frac{1}{2} \| u_{m,2} - u_{m,1} \|^2_{L_\infty(B_{k+l})} + (\| u_{m,2} - u_{m,1} \|_{L_\infty(B_{k+l})})(\| u_{m,1} \|_{L_\infty(B_{k+l})} + \| u_{m,2} \|_{L_\infty(B_{k+l})})]. \qquad (4.22)$$

Passing to the limit as $m \to \infty$, we obtain

$$\begin{aligned}\int_{B_{k+l}} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 h_{k+1}(x)dx \\ \le \alpha_{k+l}[\frac{1}{2} \| u_2 - u_1 \|^2_{L_\infty(B_{k+l})} + (\| u_2 - u_1 \|_{L_\infty(B_{k+l})}) \\ \times (\| u_1 \|_{L_\infty(B_{k+l})} + \| u_2 \|_{L_\infty(B_{k+l})})]. \qquad (4.23)\end{aligned}$$

Since $B_k \subseteq B_{k+l}$, so writing the left hand side for the smaller ball and passing to the limit $l \to \infty$, the above becomes

$$\int_{B_k} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 h(x)dx \le c_\infty[\frac{1}{2} \| u_2 - u_1 \|^2_{L_\infty(B)} \qquad (4.24)$$

$$+ \parallel u_2 - u_1 \parallel_{L_\infty(B)} (\parallel u_1 \parallel_{L_\infty(B)} + \parallel u_2 \parallel_{L_\infty(B)})].$$

By the Theorem 4.3, we have

$$\int_B |\,\mathrm{grad}\, \mathrm{u_i}(\mathrm{x})\,|^2\, \mathrm{h}(\mathrm{x})\mathrm{dx} < \infty, \ \mathrm{i} = 1,\ 2.$$

Passing to the limit as $k \to \infty$, we obtain the required result. □

For a continuous function in a closed ball $\overline{B}$, Wilson and Zwick [70] described best continuous subharmonic approximation. He found that the best subharmonic approximation of a continuous function f is just the greatest subharmonic minorant of the function. But in case of superharmonic approximation it will be smallest super-harmonic majorant. The details are given below.

In the problem when the analytic unknown exact solution must be super harmonic in the ball B, it makes interest find numerical approximation ε_0 that are super harmonic themselves. One expects that they will be better approximations to the unknown solution u(x) than the ones somehow constructed through the uniform approximation u_h.

Suppose u_h is the uniform approximation to the unknown superharmonic function u in $\overline{B}$. Then $-u$ will be the subharmonic function and $-u_h$ will approximate of $-u$.

$$-v_h(x) = \sup\{-g(x)| - g(x) \text{ is subharmonic and } - g(x) \le -u_h(x)\}$$
$$-v_h(x) = \sup\{-g(x)| - g(x) \text{ is subharmonic and } g(x) \ge u_h(x)\}$$
$$v_h(x) = \inf\{g(x)|g(x) \text{ is superharmonic in } B \text{ and } g(x) \ge u_h(x)\}$$

Denote

$$\delta = \|u - u_h\|_{L^\infty}.$$

Then

$$|u(x) - u_h(x)| \le \delta \iff \quad -\delta \le u(x) - u_h(x) \le \delta.$$

Thus $v_h(x) \ge u_h(x)$, and then $v_h(x) + \delta \ge u_h(x) + \delta \ge u(x)$, concluding

$$v_h(x) - u(x) \ge -\delta \tag{4.25}$$

Similarly,

$$v_h(x) - u(x) \le \delta \tag{4.26}$$

From (4.25) and (4.26)

$$\| v_h - u\|_{L^\infty} \le \| u_h - u\|_{L^\infty}.$$

Both v_h and u are superharmonic in B, and we also assume that they are continuous and bounded. By the use of inequality (1.4), we obtain the following important estimate

$$\begin{aligned}\int\limits_{B(x_0,r)} |\,\mathrm{grad}\ \mathrm{v_h}(\mathrm{x}) - \mathrm{grad}\,\mathrm{u}(\mathrm{x})\,|^2\, \mathrm{h}(\mathrm{x})\mathrm{dx} \le\ & \frac{1}{2} \parallel u_h - u \parallel^2_{L_\infty(B(x_0,r))} \\ & + (\parallel u_h - u \parallel_{L_\infty(B(x_0,r))})(\parallel u \parallel_{L_\infty(B(x_0,r))} \\ & + \parallel u_h \parallel_{L_\infty(B(x_0,r))}) \int\limits_{B(x_0,r)} |\,\Delta h(x)\,|\, dx. \end{aligned} \tag{4.27}$$

4.2 Reverse Poincaré-type Inequalities for the difference of superharmonic functions

In this section we develop the reverse Poincaré type estimate for the weak subsolution of heat equation. For this first we develop it for smooth ones and then using standard mollification technique for weak subsolution.
The heat equation is

$$\Delta u - u_t = 0 \tag{4.28}$$

where Δ is the classical Laplace operator. The heat equation appear in study of Brownian motion as well as the evolution in time of density for some quantity.
The physical interpretation and the derivation of the fundamental solution of heat equation is very well explained see e.g [14]
The fundamental solution of equation (4.28) is the function

$$\Psi(z,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4t}}, & z \in \mathbb{R}^n,\ t > 0; \\ 0, & z \in \mathbb{R}^n,\ t < 0 \end{cases} \tag{4.29}$$

For heat equation strong maximum principle is given as:
Assume that $v \in C_2^1(S) \cap C(\overline{S})$ solves the heat equation in S. Then
(i) $\max\limits_{\overline{S}} v = \max\limits_{\partial S} v$
(ii) Further if there exists a point $(z_0, t_0) \in S$ such that

$$v(z_0, t_0) = \max_{(z,t) \in \overline{S}} v(z,t)$$

then v is constant in $\overline{S_{t_0}}$.

Remark 4.3 *The strong maximum principle tells us that at any interior point if v attains its maximum then at all earlier times v will be a constant.*

Proof. We define for particular $z \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$.

$$E(z,t) = \left\{ (z,t) \in \mathbb{R}^{n+1} \ :\ s \leq t,\ and\ \ \psi(z-y, t-s) \geq \frac{1}{r^n} \right\}$$

This is called heat with center (z,t) of the top. The above region lies in space time whose boundary is level set of $\Psi(z-y, t-s)$.
Suppose there exist a point $(z_0, t_0) \in S_T$ with

$$v(z_0, t_0) = M = \max_{(z,t) \in \overline{S}_T} v(z,t).$$

Now for sufficiently small $r > 0$, $E(z_0,t_0;r) \subset S_T$. Now by mean value property we have

$$M = v(z_0,t_0) = \frac{1}{4r^n} \int \int\limits_{E(z_0,t_0;r)} v(y,s) \frac{|z_0 - y|^2}{(t_0 - s)^2} dyds \leq M,$$

Since

$$1 = \frac{1}{4r^n} \int \int\limits_{E(z_0,t_0;r)} \frac{|z_0 - y|^2}{(t_0 - s)^2} dyds$$

if v is identically equal to M within $E(x_0,t_0;r)$ then above equality holds for a line segment $L \in S_T$ joining (z_0,t_0) to some other point (y_0,s_0) in S_T, with $s_0 < t_0$
Consider

$$r_0 := \min\{s \geq s_0 \mid v(z,t) = M,\ (z,t) \in L,\ s \leq t \leq t_0\}.$$

Due to the continuity of v it will surely attain its minimum.
If $r_0 > s_0$. Then $v(x,r_0) = M$, $(x_0,r_0) \in L \cap S_T$, so $v \equiv M$, on $E(x_0,r_0;r)$. We obtain a contradiction because $E(z_0,r_0;r)$ contains $L \cap \{r_0 - \sigma \leq t \leq r_0\}$ for some small $\sigma > 0$, so $r_0 = s_0$ and $v \equiv M$ on L.
For a fixed $z \in S$ in time $0 \leq t < t_0$, there are exit points $\{z_0, z_1, \dots, z_m = z\}$ such that consecutive line segment connecting z_i $i = 1, 2, \dots, m$ lie in S.
For times $t_0 > t_1 > t_2 > \cdots > t_m = t$, the line segments in $\mathbb{R}^{n+1}$ connecting (z_{i-1}, t_{i-1}) to (z_i, t_i) lie in S_T.
According to previous $v \equiv M$ on each such segment and so $v(z,t) = M$. □

4.3 The energy estimates for smooth subsolution and approximation of weak subsolution

Let us define the parabolic cylinder $S(r,s)$,

$$S(r,s) = B(z_0,r) \times (s, T - s),$$

where $0 < s < \frac{T}{2}$ and $(0,T)$ is the basic time interval. For simplicity we denote

$$S = B(z_0,r) \times (0,T).$$

We organize the section in the the following way. Firstly we will develop the estimate for the smooth subsolution of the heat equation and also we will approximate the weak subsolution by smooth ones. Secondly, we will prove that the continuous weak subsolution possesses first order weak partial derivative and, finally, we will develop the reverse

Poincaré inequality for weak subsolutions.
Throughout this section we use the particular weight function

$$w(z,t) = [r^2 - (z - z_0)^2]t^2(t - T)^2. \tag{4.30}$$

It is obvious $w(z,t) = \frac{\partial w(z,t)}{\partial z_i} = \frac{\partial w}{\partial t} = 0, \quad z \in \partial S.$

Theorem 4.5 *Let* $v_i \in C^{2,1}S(r,s), \ i = 1,2$ *be a two arbitrary smooth subsolutions of* (4.28)*. Let* w *be the smooth weight function defined with* (4.30)*. Then following estimate is valid*

$$\int_S |\mathrm{grad}\,(v_2(z,t)) - \mathrm{grad}\,(v_1(z,t))|^2 w(z,t)dxdt \leq \tag{4.31}$$

$$\leq \sup|v(z,t)| \int_S (v_2(z,t) + v_1(z,t))\,\widetilde{\Delta^*} w(z,t)dzdt + \frac{1}{2}\int_S v^2(z,t)\,\widetilde{\Delta^*} w(z,t)dzdt. \tag{4.32}$$

Proof. Take $v \equiv v_2 - v_1$. Then

$$\begin{aligned}
\int_S |\mathrm{grad}\,(v(z,t))|^2 w(z,t)dzdt &= \int_S \left[\left(\frac{\partial v}{\partial z_1}\right)^2 + \left(\frac{\partial v}{\partial z_2}\right)^2 + \ldots + \left(\frac{\partial v}{\partial z_n}\right)^2\right] w(z,t)dzdt \\
&= \int_S \left(\frac{\partial v}{\partial x_1}\right)^2 w(z,t)dzdt + \int_S \left(\frac{\partial v}{\partial x_2}\right)^2 w(z,t)dzdt + \ldots \\
&+ \int_S \left(\frac{\partial v}{\partial z_n}\right)^2 w(z,t)dzdt.
\end{aligned} \tag{4.33}$$

Using integration by parts

$$\begin{aligned}
\int_S \left(\frac{\partial v}{\partial z_1}\right)^2 w(z,t)dzdt &= \int_S \frac{\partial v}{\partial z_1}\left[(\frac{\partial v}{\partial z_1})w(z,t)\right] dzdt \\
&= -\int_S v(z,t)\frac{\partial}{\partial z_1}\left(\frac{\partial v}{\partial x_1}w(z,t)\right) dzdt \\
&= -\int_S v(z,t)\left(\frac{\partial^2 v}{\partial z_1^2}w(z,t) + \frac{\partial v}{\partial z_1}\frac{\partial}{\partial z_1}w(z,t)\right) dzdt \\
&= -\int_S v(z,t)\frac{\partial^2 v}{\partial z_1^2}w(z,t)dzdt - \int_S v(z,t)\frac{\partial u}{\partial z_1}\frac{\partial}{\partial z_1}w(z,t)dzdt \\
&= -\int_S v(z,t)\frac{\partial^2 v}{\partial z_1^2}w(z,t)dzdt - \frac{1}{2}\int_S \frac{\partial}{\partial z_1}(v(z,t))^2\frac{\partial}{\partial z_1}w(z,t)dzdt.
\end{aligned}$$

Using again integration by parts on the second term in the last line we get

$$-\int_S v(z,t)\frac{\partial^2 v}{\partial z_1^2}w(z,t)dzdt+\frac{1}{2}\int_S v^2(z,t)\frac{\partial^2}{\partial z_1^2}w(z,t)dzdt$$

Making similar calculations on all other integral of (4.33), we obtain the following

$$\begin{aligned}
I=\int_S |\mathrm{grad}\,(v(z,t))|^2\,w(z,t)dzdt = & -\int_S v(z,t)\frac{\partial^2 v}{\partial z_1^2}w(z,t)dzdt-\int_S v(z,t)\frac{\partial^2 v}{\partial z_2^2}w(z,t)dzdt-\ldots \\
& -\int_S v(z,t)\frac{\partial^2 v}{\partial z_n^2}w(z,t)dzdt+\frac{1}{2}\int_S v^2(z,t)\frac{\partial^2}{\partial x_1^2}w(z,t)dzdt \\
& +\frac{1}{2}\int_S v^2(z,t)\frac{\partial^2}{\partial z_2^2}w(z,t)dzdt+\ldots+\frac{1}{2}\int_S v^2(z,t)\frac{\partial^2}{\partial x_n^2}w(z,t)dzdt \\
& =-\int_S v(z,t)\left(\frac{\partial^2 v}{\partial z_1^2}+\frac{\partial^2 v}{\partial z_2^2}+\ldots+\frac{\partial^2 v}{\partial z_n^2}\right)w(z,t)dzdt \\
& +\frac{1}{2}\int_S v^2(z,t)\left(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\ldots+\frac{\partial^2}{\partial z_n^2}\right)w(z,t)dzdt \\
& =-\int_S v(z,t)\left(\frac{\partial^2 v}{\partial z_1^2}+\frac{\partial^2 v}{\partial z_2^2}+\ldots+\frac{\partial^2 v}{\partial z_n^2}-\frac{\partial v}{\partial t}+\frac{\partial v}{\partial t}\right)w(z,t)dzdt \\
& +\frac{1}{2}\int_S v^2(z,t)\left(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\ldots+\frac{\partial^2}{\partial z_n^2}\right)w(z,t)dzdt \\
& =-\int_S v(z,t)\left[\widetilde{\Delta}v+\frac{\partial v}{\partial t}\right]w(z,t)dzdt+\frac{1}{2}\int_S v^2(z,t)\,\Delta w(z,t)dzdt \\
& =-\int_S v(z,t)\,\widetilde{\Delta}v\,w(z,t)dzdt-\int_S v(z,t)\frac{\partial v}{\partial t}\,w(z,t)dzdt \\
& +\frac{1}{2}\int_S u^2(z,t)\,\Delta w(z,t)dzdt \\
& =-\int_S v(z,t)\,\widetilde{\Delta}v\,w(z,t)dzdt-\frac{1}{2}\int_S \frac{\partial v^2(z,t)}{\partial t}\,w(z,t)dzdt \\
& +\frac{1}{2}\int_S v^2(z,t)\,\Delta w(z,t)dzdt.
\end{aligned}$$

Using integration by parts on the middle term in the last line

$$I=-\int_S v(z,t)\,\widetilde{\Delta}v\,w(z,t)dzdt+\frac{1}{2}\int_S v^2(z,t)\frac{\partial w(z,t)}{\partial t}\,dzdt+\frac{1}{2}\int_S v^2(z,t)\,\Delta w(z,t)dzdt$$

$$= -\int_S v(z,t)\,\widetilde{\Delta} v\, w(z,t)dzdt + \frac{1}{2}\int_S v^2(z,t)\,\left(\Delta + \frac{\partial}{\partial t}\right) w(z,t)dzdt$$
$$= -\int_S v(z,t)\,\widetilde{\Delta} v\, w(z,t)dzdt + \frac{1}{2}\int_S v^2(z,t)\,\widetilde{\Delta^*} w(z,t)dzdt,$$

where

$$\widetilde{\Delta} = \left(\frac{\partial^2 v}{\partial z_1^2} + \frac{\partial^2 v}{\partial z_2^2} + \ldots + \frac{\partial^2 v}{\partial z_n^2} - \frac{\partial v}{\partial t}\right)$$

and

$$\widetilde{\Delta^*} = \left(\frac{\partial^2 v}{\partial z_1^2} + \frac{\partial^2 v}{\partial z_2^2} + \ldots + \frac{\partial^2 v}{\partial z_n^2} + \frac{\partial v}{\partial t}\right).$$

Now taking the modulus value and the fact that $w(z,t) \geq 0$ we get

$$I \leq \sup|v(z,t)| \int_S \left|\widetilde{\Delta} v(z,t)\right|\, w(z,t)dzdt + \frac{1}{2}\int_S v^2(z,t)\,\widetilde{\Delta^*} w(z,t)dzdt,$$

and, further, using $v(z,t) = v_2(z,t) - v_1(z,t)$ we have

$$I \leq \sup|v(z,t)| \int_S \left(\left|\widetilde{\Delta} v_2(z,t)\right| + \left|\widetilde{\Delta} v_1(z,t)\right|\right) w(z,t)dzdt + \frac{1}{2}\int_S v^2(z,t)\,\widetilde{\Delta^*} w(z,t)dzdt$$

By using Gauss-Green theorem we get the inequality (4.31). □

Remark 4.4 *Rewriting the inequality* (4.31) *and using Hölder inequality on* (4.31) *we obtain*

$$\int_S |\mathrm{grad}\,(\mathrm{v_2(z,t)}) - \mathrm{grad}\,(\mathrm{v_1(z,t)})|^2\, w(z,t)dzdt \leq \|\hat{v}\|_{L^p} \cdot \left\|\widetilde{\Delta^*} w(z,t)\right\|_{L^q} \tag{4.34}$$

where

$$\hat{v}(z,t) = \|v_2 - v_1\|_{L^\infty} (v_2 - v_1) + \frac{(v_2 - v_1)^2}{2}.$$

Remark 4.5 *If we apply L^∞ norm on* (4.31) *we obtain the following*

$$\int_S |\mathrm{grad}\,(\mathrm{v_2(z,t)}) - \mathrm{grad}\,(\mathrm{v_1(z,t)})|^2\, w(z,t)dzdt$$
$$\leq \left[\|v_2 - v_1\|_{L^\infty} (\|v_2\|_{L^\infty} + \|v_1\|_{L^\infty}) + \frac{1}{2}\|v_2 - v_1\|_{L^\infty}^2\right] \int_S \left|\widetilde{\Delta^*} w(z,t)\right| dzdt. \tag{4.35}$$

A bounded measurable function $v(z,t)$ defined in the cylinder $S = B(z_0, r) \times (0,T)$ is called the weak subsolution of heat equation

$$\widetilde{\Delta^*} v(z,t) = 0$$

in the cylinder S if for every non-negative function $\phi(z,t) \in C_0^{2,1}(S)$ the following holds

$$\int_0^T \int_B v(z,t)\widetilde{\Delta^*}\phi(z,t)dzdt \geq 0.$$

Now we will approximate weak sub-solution of heat equation by the smooth ones. We will do it with the help of mollification.
Define

$$\eta_n(y) = \begin{cases} C\,exp\left(\frac{1}{|y|^2-1}\right), & |y|<1; \\ 0, & |y| \geq 1, \end{cases} \tag{4.36}$$

where $y \in \mathbb{R}^n,\ n \in \mathbb{N}$, and C is a positive constant that satisfies

$$\int_{\mathbb{R}^n} \eta_n(y)dy = 1.$$

For the bounded measurable function $v(z,t)$ defined on the cylinder S, we define its mollification

$$v_\varepsilon(z,t) = \varepsilon^{-(n+t)} \int_0^T \int_B \eta_n\left(\frac{z-y}{\varepsilon}\right)\eta_1\left(\frac{t-s}{\varepsilon}\right)v(y,s)$$

for arbitrary $\varepsilon > 0$.
If we denote

$$\eta_\varepsilon(z-y,t-s) = \varepsilon^{-(n+1)}\eta_n\left(\frac{z-y}{\varepsilon}\right)\eta_1\left(\frac{t-s}{\varepsilon}\right),$$

then it is trivial that

$$\frac{\partial^2}{\partial {z_i}^2}\eta_\varepsilon(z-y,t-s) = \frac{\partial^2}{\partial {y_i}^2}\eta_\varepsilon(z-y,t-s)$$

$$\frac{\partial}{\partial t}\eta_\varepsilon(z-y,t-s) = -\frac{\partial}{\partial s}\eta_\varepsilon(z-y,t-s),$$

and from above we conclude that

$$\widetilde{\Delta^*_{z,t}}\eta_\varepsilon(z-y,t-s) = \widetilde{\Delta^*_{y,s}}\eta_\varepsilon(z-y,t-s), \tag{4.37}$$

where $\widetilde{\Delta^*_{z,t}}$ and $\widetilde{\Delta^*_{y,s}}$ are heat operators with argument (z,t) and its adjoint operator with respect to argument (y,s) respectively.
Let us define the cylinder S_k in the following way

$$S_k = S\left(r_k, \frac{T}{k+2}\right) = B(z_0,r_k) \times \left(\frac{T}{k+2}, \frac{k+1}{k+2}T\right),$$

where

$$r_k = \frac{k+1}{k+2}r, \quad k \in \mathbb{N}.$$

The following theorem tells that the function $v_\varepsilon(x,t)$ is smooth subsolution of heat equation in the cylinder S_k for sufficiently small ε.

Theorem 4.6 *Consider the weak subsolution* $v(z,t)$ *of* (4.28) *in the cylinder* $S = B(z_0, r) \times (0,T)$ *then for every* $k \in \mathbb{N}$ *there exist* $\widetilde{\varepsilon} > 0$, *such that for every* ε, $0 < \varepsilon < \widetilde{\varepsilon}$ *each function* $v_\varepsilon(z,t)$ *is the smooth parabolic subsolution in the cylinder* S_k, *that is* $\widetilde{\Delta} v_\varepsilon(z,t) \geq 0$, *if* $(z,t) \in S_k$.

Proof. It is obvious that for arbitrary ε, $v_\varepsilon(z,t)$ is infinitely continuously differentiable function with respect to its argument in $\mathbb{R}^{n+1}$. Now we check that for arbitrary$(z,t) \in S_k$ the $\eta_\varepsilon(z-y, t-z)$ has compact support in the cylinder as a function of (y,s).
Let us fix $k = 1, 2, 3, \ldots, n$

$$\widehat{\varepsilon} = \min\left(\frac{r}{2(k+2)}, \frac{T}{2(k+2)}\right)$$

Define the cylinder $\widehat{S_k}$ in the following way

$$\widehat{S_k} = B\left(z_0, \frac{2k+3}{2k+4} r\right) \times \left(\frac{T}{2k+2,}, \frac{2k+3}{2k+4} T\right)$$

If $(y,s) \notin \widehat{S_k}$ then either $y \notin B(z_0, \frac{2k+2}{2k+4} r)$ *or* $S \notin (\frac{T}{2k+4}, \frac{2k+3}{2k+4} T)$.
For the first case

$$|y-z| > \left(\frac{2k+3}{2k+4} - \frac{2k+2}{2k+4}\right) r = \frac{1}{2(k+2)} r > \varepsilon,$$

and for second one

$$|t-z| > \left(\frac{2}{2k+4} - \frac{1}{2k+4}\right) T > \varepsilon.$$

Hence, in both cases we have $\eta_\varepsilon(z-y, t-s) = 0$, so the non-negative smooth function $\eta_\varepsilon(z-y, t-s)$ has compact support in cylinder S as a function of (y,s) if $\varepsilon < \hat{\varepsilon}$. By the definition of weak subsolution of heat equation we have

$$\int_0^T \int_B v(y,s) \widetilde{\Delta^*_{y,s}} \eta_\varepsilon(z-y, t-s) dy ds \geq 0$$

which proves the required result. □

4.4 The case of weak subsoution of wave equation

Now we prove that Sobolev gradient of weak subsolution of wave exit equation is also weighted square integrable.
Now define the smooth weight function for corresponding cylinder

$$w_k(z,t) = (r_k^2 - |z-z_0|^2)\left(t - \frac{T}{k+2}\right)\left(\frac{k+1}{k+2} T - t\right)$$

$(z,t) \in \overline{S_k},\ k \in \mathbb{N}$.

Our smooth basic weight function $w(z,t)$ is $\left[(r^2 - |z - z_0|^2)t(T-t)\right]^2,\ (z,t) \in \overline{S}$.

The next theorem shows us that the continuous weak subsolution posses first order weak partial derivative and also it is square integrable with respect to weight function $w(z,t)$.

Theorem 4.7 *Let $v(z,t)$ be the continuous weak subsolution of heat equation* (4.28). *Then it has weak partial derivative $\frac{\partial v}{\partial z_i}$, $i = 1,2,\ldots,n$ and the following holds*

$$\int_S |\mathrm{grad}\,(\mathrm{u(z,t)})|^2 w(z,t)dzdt < \infty.$$

Proof. Take $v_\varepsilon(z,t)$, the mollification of weak subsolution $v(z,t)$ is continuous in the cylinder S so by Evans [14] it is well known over any compact subset $C \subset S$ we have the convergence

$$\sup_{(z,t)\in C} |v_\varepsilon(z,t) - v(z,t)| \xrightarrow{\varepsilon \to 0} 0.$$

If we will change $\varepsilon = \frac{1}{m}$, $m \in \mathbb{N}$, the above convergence will become

$$\sup_{(z,t)\in C} |v_m(z,t) - v(z,t)| \xrightarrow{m \to \infty} 0.$$

As the cylinder S_k are completely imbedded in the cylinder we have that for any $k \in \mathbb{N}$ there exist such m_k that each $v_m(z,t)$ is smooth subsolution of heat equation in the cylinder S_k if $m \geq m_k$.

Now rewrite the inequality (4.35) for the cylinder S_k and for the functions

$$v_1(z,t) = v_m(z,t),\ \ v_2(z,t) = v_p(z,t),\ \ m,p \geq m_{k+l}$$

$$\int_{S_{k+l}} \left|\mathrm{grad}\,(\mathrm{v_p(z,t)}) - \mathrm{grad}\,(\mathrm{v_m(z,t)})\right| w_{k+l}(z,t)dzdt \leq \left[\left\|v_p - u_m\right\|_{L^\infty}\left(\left\|v_p\right\|_{L^\infty} + \|v_m\|_{L^\infty}\right)\right.$$
$$\left. + \frac{1}{2}\left\|v_p - v_m\right\|^2\right] \int_{S_{k+l}} \left|\widetilde{\Delta^*} w_{k+l}(z,t)\right| dzdt \tag{4.38}$$

Let us denote

$$\widehat{\alpha}_{k+l} = \inf_{(z,t)\in S_k} w_{k+l}$$

and

$$\alpha_{k+l} = \int_{S_{k+l}} \left|\widetilde{\Delta^*} w_{k+l}(z,t)\right| dzdt$$

If we restrict the left hand side of the integral over smaller cylinder S_k, we obtain

$$\widehat{\alpha}_{k+l} \int_{S_k} \left|\mathrm{grad}\,(\mathrm{v_p(z,t)}) - \mathrm{grad}\,(\mathrm{v_m(z,t)})\right|^2 dzdt \tag{4.39}$$

$$\leq \alpha_{k+l}\left[\left\|v_p - v_m\right\|_{L^\infty}\left(\left\|v_p\right\|_{L^\infty} + \|v_m\|_{L^\infty}\right) + \frac{1}{2}\left\|v_p - v_m\right\|^2\right]$$

Since we have $\left\|v_p - v_m\right\|_{L^\infty}(S_{k+l}) \to 0,\ m,p \to \infty$, letting the limit in the inequality (4.39) we obtain

$$\lim_{m,p\to\infty}\sum_{i=1}^{n}\int_{S_k}\left(\frac{\partial v_p(z,t)}{\partial z_i} - \frac{\partial v_m(z,t)}{\partial z_i}\right)^2 = 0. \tag{4.40}$$

Since $L^2(S_k)$ is complete space so the above sequence will converge, i.e. there exist a collection of measurable functions $u_{k,i}(z,t) \in L^2(S_k)$ such that

$$\lim_{m\to\infty}\sum_{i=1}^{n}\int_{S_k}\left(\frac{\partial v_m(z,t)}{\partial z_i} - u_{k,i}(z,t)\right)^2 dzdt = 0, \qquad k \in \mathbb{N}. \tag{4.41}$$

Let us extend the function $u_{k,i}(z,t)$ outside S_k trivially by zero in this way

$$u_i(z,t) = \limsup_{k\to\infty} u_{k,i}(z,t), \quad i = 1,2,3,\ldots,n. \tag{4.42}$$

It is clear by definition that the function $u_{k+l,i}(z,t),\ l \in \mathbb{N}$, have the same value on cylinder S_k and therefore $u_i(z,t) = u_{k,i}(z,t)$ $a.e$ on cylinder S_k
Now we will check that $u_i(z,t)$ is partial derivative of function $v(z,t)$
To prove this take arbitrary function $\phi(z,t)$ which is infinitely differentiable and having compact support in S. The support $\phi(z,t)$ is contained in S_k, for some k, so we have

$$\int_{S_k}\frac{\partial v_m(z,t)}{\partial z_i}\phi(z,t)dzdt = -\int_{S_k} v_m(z,t)\frac{\partial \phi_m(z,t)}{\partial z_i}dxdt \tag{4.43}$$

for any $m \geq m_k$.
But

$$\sup_{(z,t)\in S_k}|v_m(z,t) - v(z,t)| \xrightarrow{m\to\infty} 0. \tag{4.44}$$

and $\frac{\partial v_m}{\partial z_i}$ converge to $u_i(z,t)$ in $L^2(S_k)$.
So applying limit $m \to \infty$, we obtain

$$\int_{S_k} u_i(z,t)\phi(z,t)dzdt = -\int_{S_k} v(z,t)\frac{\partial \phi(z,t)}{\partial x_i}dzdt. \tag{4.45}$$

This shows that $u_i(z,t)$ represents the weak partial derivative of the functions $v(z,t)$.
Writing the equality $v_1(z,t) = 0$ and $v_2(z,t) = v_m(z,t) = 0$ for $m \geq m_{k+l}$ and the cylinder, we have,

$$\int_{S_{k+l}} |\mathrm{grad}\,(\mathrm{v_m(z,t)})|^2 v_{k+l}(z,t)dzdt \leq 3\widehat{\alpha}_{k+l}\|v_m(z,t)\|^2_{L^\infty(S_{k+l})}.$$

If we let $m \to \infty$, we obtain the following

$$\int_{S_{k+l}} |\mathrm{grad}\,(\mathrm{v(z,t)})|^2 w_{k+l}(z,t)dxdt \leq 3\widehat{\alpha}_{k+l} \|v(z,t)\|^2_{L^\infty(S_{k+l})}.$$

Considering the left hand integral for the smaller cylinder S_k and letting the limit $l \to \infty$, we get,

$$\int_{S_k} |\mathrm{grad}\,(\mathrm{v(z,t)})|^2 w(z,t)dzdt \leq 3C_\infty \|v(z,t)\|^2_{L^\infty}(S) < \infty, \tag{4.46}$$

where

$$C_\infty = \int_S \left|\widetilde{\Delta^*} w_{k+l}(z,t)\right| dzdt.$$

The left hand side of (4.46) is bounded, with the respect to k, is bounded and increasing, so the limit is finite by dominated convergence theorem. Hence,

$$\int_S |\mathrm{grad}\,(\mathrm{v(z,t)})|^2 w(z,t)dzdt < \infty.$$

□

4.5 The weighted energy estimates for the difference of weak subsolutions of wave equation

The wave equation is simplified model for a vibrating string ($n = 1$), membrane ($n = 2$) or elastic solid ($n = 3$). In these physical interpretations $u(x,t)$ represents the displacement in some direction of point x at time $t \geq 0$. The reverse Poincaré inequality will be helpful for the study of qualitative properties of solution of wave equation.

Let $B = B(x_0, r)$ be the ball having center x_0 and radius r and $Q(r,s)$ is the cylinder defined as

$$Q(r,s) = B(x_0, r) \times (s, T - s).$$

where $0 < s < \frac{T}{2}$ and $(0,T)$ is the time interval. For consistency we denote $Q = B(x_0, r) \times (0,T)$.

Let $C_c^{2,2}(\overline{Q(r,s)})$ be the space of twice continuous differentiable functions with respect to $x = (x_1, x_2, \ldots, x_n)$ and t, on the closure $\overline{Q(r,s)}$.
We consider the n-dimensional wave equation

$$L(u(x,t)) = \Delta u(x,t) - \frac{\partial^2 u}{\partial t^2} = 0, \tag{4.47}$$

where Δ is the classical n-dimensional Laplace operator

$$\Delta u(x,t) = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}.$$

We define

$$\operatorname{grad} u(x,t) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right),$$

and also the extended gradient as

$$\widetilde{\operatorname{grad}} u(x,t) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t} \right).$$

The function $u(x,t) \in C^{2,2}(Q(r,s))$ is said to be smooth subsolution of wave equation if

$$L(u(x,t)) \geq 0. \tag{4.48}$$

The function $h(x,t)$ denotes the weight function

$$h(x,t) = [r^2 - (x - x_0)^2] t^2 (T - t)^2. \tag{4.49}$$

The bounded measurable function $u(x,t)$ is said to be weak subsolution of (4.47) if for all non-negative functions $\phi(x,t) \in C_c^{2,2}(Q(r,s))$, we have

$$\int_Q u(x,t) L(\phi(x,t)) dx dt \geq 0. \tag{4.50}$$

4.5.1 Approximation of weak subsolution

Now we approximate the weak subsolution $u(x,t)$ of (4.47) using mollification technique. Denote

$$u_\varepsilon(x,t) = \varepsilon^{-(n+1)} \int_Q \eta_n \left(\frac{x-y}{\varepsilon} \right) \eta_1 \left(\frac{t-s}{\varepsilon} \right) u(y,s) dy ds, \tag{4.51}$$

for arbitrary $\varepsilon > 0$, where η_n is defined in (4.36).
Denote further

$$\eta_\varepsilon(x-y, t-s) = \varepsilon^{-(n+1)} \eta_n \left(\frac{x-y}{\varepsilon} \right) \eta_1 \left(\frac{t-s}{\varepsilon} \right).$$

The following is obvious

$$\frac{\partial^2}{\partial x_i^2}\eta_\varepsilon(x-y,t-s) = \frac{\partial^2}{\partial y_i^2}\eta_\varepsilon(x-y,t-s),$$

and

$$\frac{\partial^2}{\partial t^2}\eta_\varepsilon(x-y,t-s) = \frac{\partial^2}{\partial s^2}\eta_\varepsilon(x-y,t-s).$$

Then, we check the following

$$L_{x,t}\eta_\varepsilon(x-y,t-s) = L_{y,s}\eta_\varepsilon(x-y,t-s),$$

$$L_{x,t}\eta_\varepsilon(x-y,t-s) = L^*_{y,s}\eta_\varepsilon(x-y,t-s) \tag{4.52}$$

where $L_{x,t}$ and $L^*_{y,s}$ are operators using arguments (x,t) and (y,s), respectively. From (4.52), we have

$$L_{x,t}u_\varepsilon(x,t) = \int_Q u(y,s)L^*_{y,s}\eta_\varepsilon(x-y,t-s)\,dyds \text{ for arbitrary } \varepsilon > 0. \tag{4.53}$$

Also, define the cylinder Q_k

$$Q_k = Q\left(r_k, \frac{T}{k+2}\right) = B(x_0,r_k)\times\left(\frac{T-1}{k+2},\frac{k+1}{k+2}\right),$$

where

$$r_k = \frac{k+1}{k+2}r, \quad k\in\mathbb{N}.$$

Now $u_\varepsilon(x,t)$ is infinitely differentiable with respect to its arguments on $\mathbb{R}^{n+1}$.

4.5.2 Reverse Poincaré type estimate for weak subsolution of wave equation

We start with the following theorem

Theorem 4.8 *If $u_1(x,t)$ and $u_2(x,t)$ are the smooth subsolutions of the wave equation* (4.47) *and let $h(x,t)$ be the weight function defined by* (4.49). *Then the following energy estimate is valid*

$$\int\Big[E(u(x,t))\Big]h(x,t)dxdt \le \int_Q\left(\sup|u(x,t)|\,(u_2+u_1)+\frac{1}{2}(u(x,t))^2\right)L(h(x,t))dxdt,$$

where

$$E(u(x,t)) = |\operatorname{grad} u(x,t)|^2 - \left(\frac{\partial u}{\partial t}\right)^2,$$

and

$$u(x,t) = u_2(x,t) - u_1(x,t).$$

Proof. Take

$$\int_Q \left(|\mathrm{grad}\, u(x,t)|^2 - \left(\frac{\partial u}{\partial t}\right)^2 \right) h(x,t)dxdt = \int_Q |\mathrm{grad}\, u(x,t)|^2 h(x,t)dxdt - \int_Q \left(\frac{\partial u}{\partial t}\right)^2 h(x,t)dxdt$$
$$= \int_Q \left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)^2 \right) h(x,t)dxdt - \int_Q \left(\frac{\partial u}{\partial t}\right)^2 h(x,t)dxdt$$
$$= \int_Q \left(\frac{\partial u}{\partial x_1}\right)^2 h(x,t)dxdt + \int_Q \left(\frac{\partial u}{\partial x_2}\right)^2 h(x,t)dxdt + \ldots$$
$$+ \int_Q \left(\frac{\partial u}{\partial x_n}\right)^2 h(x,t)dxdt - \int_Q \left(\frac{\partial u}{\partial t}\right)^2 h(x,t)dxdt. \tag{4.54}$$

Applying integration by parts on

$$\int_Q \left(\frac{\partial u}{\partial x_1}\right)^2 h(x,t)dxdt = \int_Q \left(\frac{\partial u}{\partial x_1}\right) \left(\frac{\partial u}{\partial x_1} h(x,t)\right) dxdt$$
$$= -\int_Q u(x,t) \left(\frac{\partial^2 u}{\partial x_1^2}\right) h(x,t)dxdt - \int_Q u(x,t) \left(\frac{\partial u}{\partial x_1}\right) \left(\frac{\partial}{\partial x_1} h(x,t)\right) dxdt$$
$$= -\int_Q u(x,t) \left(\frac{\partial^2 u}{\partial x_1^2}\right) h(x,t)dxdt - \frac{1}{2}\int_Q \left(\frac{\partial (u(x,t))^2}{\partial x_1}\right) \left(\frac{\partial}{\partial x_1} h(x,t)\right) dxdt. \tag{4.55}$$

Again using integration by parts on second integral of (4.55), we have

$$-\int_Q u(x,t) \left(\frac{\partial^2 u}{\partial x_1^2}\right) h(x,t)dxdt + \frac{1}{2}\int_Q u^2(x,t) \frac{\partial^2 h(x,t)}{\partial x_1^2} dxdt.$$

Similarly solving the other integrals of (4.54), we get

$$\begin{aligned}
\int_Q \left(\frac{\partial u}{\partial x_1}\right)^2 h(x,t)dxdt &= -\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}\right) h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t) \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right) h(x,t)dxdt \\
&+ \int_Q u(x,t) \left(\frac{\partial^2 u}{\partial t^2}\right) h(x,t)dxdt - \frac{1}{2}\int_Q u^2(x,t) \frac{\partial^2 h(x,t)}{\partial t^2} dxdt \\
&= -\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial^2 u}{\partial t^2}\right) h(x,t)dxdt
\end{aligned}$$

$$+\frac{1}{2}\int_Q u^2(x,t)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}-\frac{\partial^2}{\partial t^2}\right)h(x,t)dxdt$$

$$=-\int_Q u(x,t)\left(\Delta u(x,t)-\frac{\partial^2 u}{\partial t^2}\right)h(x,t)dxdt$$

$$+\frac{1}{2}\int_Q u^2(x,t)\left(\Delta h(x,t)-\frac{\partial^2 h(x,t)}{\partial t^2}\right)dxdt. \tag{4.56}$$

Now using (4.47), we have

$$\int_Q\left(\frac{\partial u}{\partial x_1}\right)^2 h(x,t)dxdt = -\int_Q u(x,t)L(u(x,t))h(x,t)dxdt + \frac{1}{2}\int_Q (u(x,t))^2 L(h(x,t))dxdt.$$

$$\leq \int_Q \sup|u(x,t)|\,|L(u(x,t))|\,|h(x,t)|\,dxdt+\frac{1}{2}\int_Q u^2(x,t)\,|L(h(x,t))|\,dxdt. \tag{4.57}$$

Since $L_i(u(x,t))\geq 0,\ i=1,2$ and also $|L(u(x,t))|=|L(u_2)-L(u_1)|\leq|L(u_2)|+|L(u_1)|=L(u_2)+L(u_1)$, so (4.57) becomes

$$\int_Q\left[|\mathrm{grad}\,u(x,t)|^2-\left(\frac{\partial u}{\partial t}\right)^2\right]h(x,t)dxdt$$

$$\leq \int_Q \sup|u(x,t)|\,(L(u_2)+L(u_1))h(x,t)dxdt+\frac{1}{2}\int_Q u^2(x,t)L(h(x,t))dxdt$$

$$\leq \int_Q \sup|u(x,t)|\,L(u_2+u_1)h(x,t)dxdt+\frac{1}{2}\int_Q u^2(x,t)L(h(x,t))dxdt.$$

Using Gauss-Green theorem, we get

$$\int_Q \sup|u(x,t)|\,(u_2+u_1)L(h(x,t))dxdt+\frac{1}{2}\int_Q u^2(x,t)L(h(x,t))dxdt$$

$$\leq\left(\int_Q \sup|u(x,t)|\,(u_2+u_1)+\frac{1}{2}u^2(x,t)\right)L(h(x,t))dxdt.$$

□

Theorem 4.9 *Let $u_1(x,t)$ and $u_2(x,t)$ be the two smooth subsolutions of* (4.47) *and $h(x,t)$ be the weight function defined by* (4.49). *Suppose furhter that $\frac{\partial^2 u_i}{\partial t^2}\geq 0,\ i=1,2$. Then the* *following estimate holds*

$$\int\limits_Q \left|\widetilde{\operatorname{grad}} u(x,t)\right|^2 h(x,t)dxdt \le \int\limits_Q \left(|u(x,t)|\,(u_1+u_2)+\frac{1}{2}u^2(x,t)\right) \times \left(L(h(x,t))+2\frac{\partial^2 h(x,t)}{\partial t^2}\right)dxdt.$$

Proof.

$$\begin{aligned}\int\limits_Q \left|\widetilde{\operatorname{grad}} u(x,t)\right|^2 h(x,t)dxdt &= \int\limits_Q \left(\left(\frac{\partial u}{\partial x_1}\right)^2+\ldots+\left(\frac{\partial u}{\partial x_n}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right)h(x,t)dxdt\\ &= \int\limits_Q \left(\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\ldots+\left(\frac{\partial u}{\partial x_n}\right)^2\right.\\ &\quad \left.-\left(\frac{\partial u}{\partial t}\right)^2\right)h(x,t)dxdt+2\int\limits_Q \left(\frac{\partial u}{\partial t}\right)^2 h(x,t)dxdt.\end{aligned}$$

Now using (4.57) on first integral and integration by parts formula on second integral, we have

$$\begin{aligned}&-\int\limits_Q u(x,t)L(u(x,t))h(x,t)dxdt+\frac{1}{2}\int\limits_Q u^2(x,t)L(h(x,t))dxdt\\ &-2\int\limits_Q u(x,t)\frac{\partial^2 u}{\partial t^2}h(x,t)dxdt+\int\limits_Q u^2(x,t)\frac{\partial^2(h(x,t))}{\partial t^2}dxdt\\ &=-\int\limits_Q u(x,t)L(u(x,t))h(x,t)dxdt-2\int\limits_Q u(x,t)\frac{\partial^2 u}{\partial t^2}h(x,t)dxdt\\ &+\frac{1}{2}\int\limits_Q u^2(x,t)L(h(x,t))dxdt+\int\limits_Q u^2(x,t)\frac{\partial^2(h(x,t))}{\partial t^2}dxdt\\ &\le |u(x,t)|\int\limits_Q |L(u(x,t)|\,h(x,t)dxdt+2\,|u(x,t)|\int\limits_Q \left|\frac{\partial^2 u}{\partial t^2}\right| h(x,t)dxdt\\ &+\frac{1}{2}\int\limits_Q u^2(x,t)L(h(x,t))dxdt+\int\limits_Q u^2(x,t)\frac{\partial^2 h(x,t))}{\partial t^2}dxdt. \end{aligned} \tag{4.58}$$

Since

$$|L(u(x,t))| = |L(u_2(x,t))-L(u_1(x,t))| \le |L(u_2)|+|L(u_1)| = L(u_1)+L(u_2)$$

$L(u_i(x,t)) \ge 0,\ i=1,2$, and $\frac{\partial^2 u_i}{\partial t^2} > 0$, we have

$$\left|\frac{\partial^2 u}{\partial t^2}\right| = \left|\frac{\partial^2 u_2}{\partial t^2}-\frac{\partial^2 u_1}{\partial t^2}\right| \le \left|\frac{\partial^2 u_2}{\partial t^2}\right|+\left|\frac{\partial^2 u_1}{\partial t^2}\right|,$$

Then inequality (4.58) becomes

$$|u(x,t)| \int_Q (L(u_1)+L(u_2))h(x,t)dxdt + 2\,|u(x,t)| \int_Q (\frac{\partial^2 u_2}{\partial t^2} + \frac{\partial^2 u_1}{\partial t^2})h(x,t)$$
$$+\frac{1}{2}\int_Q u^2(x,t)L(h(x,t))dxdt + \int_Q u^2(x,t)\frac{\partial^2(h(x,t))}{\partial t^2}dxdt.$$

Since L is self adjoint operator, so by using Gauss-Green theorem, we obtain further sequence of inequalities

$$\begin{aligned}
&|u(x,t)| \int_Q (u_1+u_2)L(h(x,t))dxdt + 2\,|u(x,t)| \int_Q (u_1+u_2)\frac{\partial^2 h(x,t)}{\partial t^2}dxdt \\
&+\frac{1}{2}\int_Q u^2(x,t)L(h(x,t))dxdt + \int_Q u^2(x,t)\frac{\partial^2(h(x,t))}{\partial t^2}dxdt. \\
&\leq |u(x,t)| \int_Q (u_1+u_2)\left(L(h(x,t)) + 2\frac{\partial^2 h(x,t)}{\partial t^2}\right)dxdt \\
&+\frac{1}{2}\int_Q u^2(x,t)\left(L(h(x,t)) + 2\frac{\partial^2 h(x,t)}{\partial t^2}\right)dxdt \\
&\leq \int_Q \left(|u(x,t)|\,(u_1+u_2) + \frac{1}{2}u^2(x,t)\right)\left(L(h(x,t)) + 2\frac{\partial^2 h(x,t)}{\partial t^2}\right)dxdt. \qquad (4.59)
\end{aligned}$$

□

Remark 4.6 *Taking supremum norm in* (4.59)*, we obtain the following,*

$$\int_Q \left|\widetilde{\text{grad}}\,(u_2(x,t)-u_1(x,t))\right|^2 h(x,t)dxdt \leq \qquad (4.60)$$
$$\leq \left[\|u_2-u_1\|_{L^\infty}\left(\|u_1\|_{L^\infty}\|u_2\|_{L^\infty}\right) + \frac{1}{2}\|u_2-u_1\|^2_{L^\infty}\right]\int_Q \left|L(h(x,t)) + 2\frac{\partial^2 h(x,t)}{\partial t^2}\right|dxdt.$$

With same arguments as in Theorem 4.6, we can now prove the next theorem.

Theorem 4.10 *Consider the weak subsolution $u(x,t)$ of wave equation in the cylinder $Q, Q = B\times(0,T)$, then for any $k\in\mathbb{N}$ there exists $\hat{\varepsilon}>0$ such that for any $\varepsilon,\ 0<\varepsilon<\hat{\varepsilon}$, each function $u_\varepsilon(x,t)$ is the smooth subsolution of wave equation in the cylinder Q_k, that is*

$$Lu_\varepsilon(x,t) \geq 0,\ (x,t)\in Q_k. \qquad (4.61)$$

Also, using the same technique as in Theorem 4.7, we can prove the next theorem.

Theorem 4.11 *Any continuous weak subsolution $u(x,t)$ of wave equation has weak partial derivatives $\frac{\partial u(x,t)}{\partial x_i}$, $i = 1,\dots,n$, and $\frac{\partial u}{\partial t}$ also exist in the cylinder Q, $Q = B(x_0,R)\times(0,T)$, and they are square integrable with respect to the weight function $h(x,t)$, i.e.*

$$\int\limits_{Q} \left|\widetilde{\operatorname{grad}} u(x,t)\right|^2 h(x,t)dxdt < \infty. \tag{4.62}$$

Theorem 4.12 *Consider two arbitrary continuous weak subsolutions of wave equation $u_i(x,t), i = 1,2$, in the cylinder Q, $Q = B(x_0,R)\times(0,T)$. Then the following weighted reverse poincare type inequality holds for the difference $u_2(x,t)-u_1(x,t)$ of two weak subsolutions*

$$\int\limits_{Q} \left|\widetilde{\operatorname{grad}} u_2(x,t)-\widetilde{\operatorname{grad}} u_1(x,t)\right|^2 h(x,t)dxdt \leq \tag{4.63}$$

$$\left(\|u_2-u_1\|_{L^\infty(Q)}\left(\|u_1\|_{L^\infty(Q)}+\|u_2\|_{L^\infty(Q)}\right)+\frac{1}{2}\|u_2-u_1\|^2_{L^\infty(Q)}\right)\int\limits_{Q}\left|Lh(x,t))+2\frac{\partial^2 h(x,t)}{\partial t^2}\right|dxdt.$$

Proof. Consider mollifications $u_{m,i}(x,t)$, $i = 1,2$ of the continuous weak subsolutions $u_i(x,t), i = 1,2$. We already know that for a cylinder Q_{k+l} there exists integer m_{k+l} such that each function $u_{m,i}(x,t), i = 1,2$ is the smooth subsolution of wave equation in the cylinder Q_{k+l} if $m \geq m_{k+l}$.
We have the following uniform convergence

$$\|u_{m,i}-u_i\|_{L^\infty(Q_{k+l})} \xrightarrow{m\to\infty} 0,\ \ i = 1,2.$$

Let us apply the inequality (4.60) to the functions $u_{m,1}(x,t)$ and $u_{m,2}(x,t)$ and the cylinder Q_{k+l}. We have

$$\int\limits_{Q_{k+l}} \left|\widetilde{\operatorname{grad}} u_{m,2}(x,t)-\widetilde{\operatorname{grad}} u_{m,1}(x,t)\right|^2 h_{k+l}(x,t)dxdt \leq \tag{4.64}$$

$$\leq \left[\|u_{m,2}-u_{m,1}\|_{L^\infty(Q_{k+l})}\left(\|u_{m,1}\|_{L^\infty(Q_{k+l})}+\|u_{m,2}\|_{L^\infty(Q_{k+l})}\right)+\frac{1}{2}\|u_{m,2}-u_{m,1}\|^2_{L^\infty(Q_{k+l})}\right]$$

$$\times \int\limits_{Q_{k+l}}\left|L(h_{k+l}(x,t))+2\frac{\partial^2 h_{k+l}(x,t)}{\partial t^2}\right|dxdt.$$

Passing to the limit as $m\to\infty$ in the latter inequality, we get

$$\int\limits_{Q_{k+l}} \left|\widetilde{\operatorname{grad}} u_2(x,t)-\widetilde{\operatorname{grad}} u_1(x,t)\right|^2 h_{k+l}(x,t)dxdt \leq \tag{4.65}$$

$$\leq \left[\|u_2-u_1\|_{L^\infty(Q_{k+l})}\left(\|u_1\|_{L^\infty(Qk+l)}+\|u_2\|_{L^\infty(Qk+l)}\right)+\frac{1}{2}\|u_2-u_1\|^2_{L^\infty(Qk+l)}\right]$$

$$\times \int\limits_{Q_{k+l}}\left|L(h_{k+l}(x,t))+2\frac{\partial^2 h_{k+l}(x,t)}{\partial t^2}\right|dxdt.$$

Restricting the integral on the left-hand side of (4.65) over the cylinder Q_k and then passing to the limit as $l \to \infty$, we obtain

$$\int\limits_{Q_k} \left|\widetilde{\operatorname{grad}} u_2(x,t) - \widetilde{\operatorname{grad}} u_1(x,t)\right|^2 h(x,t)dxdt \tag{4.66}$$
$$\leq \left[\|u_2 - u_1\|_{L^\infty(Q)} (\|u_1\|_{L^\infty(Q)} + \|u_2\|_{L^\infty(Q)}) + \frac{1}{2}\|u_2 - u_1\|^2_{L^\infty(Q)} \right] \int\limits_{Q_k} \left| L(h(x,t)) + 2\frac{\partial^2 h(x,t)}{\partial t^2} \right| dxdt.$$

By Theorem 4.11, we have

$$\int\limits_{Q} \left|\widetilde{\operatorname{grad}} u_i(x,t)\right|^2 h(x,t)dxdt < \infty,\ i = 1,2, \tag{4.67}$$

passing now to the limit in the inequality (4.66) as $k \to \infty$, we obtain desired result. □

4.6 The weighted energy estimates for the difference of weak subsolutions of telegraph equation

Let $B = B(x_0, r)$ be the ball having center x_0 and radius r. Let $Q(r,s)$ is the cylinder defined as

$$Q(r,s) = B(x_0, r) \times (s, T - s).$$

$C\big(Q(r,s)\big)$ be the space of continuous functions on Q and $C^{2,2}\overline{(Q(r,s))}$ be the space of twice continuous differentiable functions with respect to argument $x = (x_1, x_2, \ldots, x_n)$ and t on the closure $\overline{(Q(r,s))}$. Let us suppose the n-dimensional telegraph equation

$$L(u(x,t)) = \Delta u(x,t) - \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t} = 0. \tag{4.68}$$

The function $u(x,t) \in C^{2,2}(Q(r,s))$ is said to be smooth sub-solution of telegraph equation if

$$L\big(u(x,t)\big) \geq 0.$$

The bounded measurable function $u(x,t)$ is said to be weak sub-solution of (4.68) if for all non-negative functions $\phi(x,t) \in C^{2,2}(Q(r,s))$, we have the following

$$\int_Q u(x,t)L(\phi(x,t))dxdt \geq 0.$$

Theorem 4.13 *Let $u_1(x,t)$ and $u_2(x,t)$ are the smooth subsolution of telegraph equation* (4.68)*. Let $h(x,t)$ be the our weight function having compact support. Then the following energy estimate is valid.*

$$\int_Q |E(u(x,t))|\, h(x,t)dxdt \leq \sup|u(x,t)| \int_Q \left[u_2(x,t)+u_1(x,t)+\frac{1}{2}(u(x,t))^2\right] L^*(h(x,t))dxdt$$

where $u = u_2 - u_1$ and

$$Eu(x,t) = |\mathrm{grad}\, u(x,t)|^2 - \left(\frac{\partial u}{\partial t}\right)^2$$

Proof.

$$\int_Q \left[|\mathrm{grad}\,(\mathrm{u(x,t)})|^2 - \left(\frac{\partial u}{\partial t}\right)^2\right] h(x,t)dxdt = \sum_{i=1}^{n} \int_Q \left(\frac{\partial u}{\partial x_n}\right)^2 h(x,t)dxdt - \int_Q \left(\frac{\partial u}{\partial t}\right)^2 h(x,t)dxdt.$$

Using integration by parts formula and the fact that $h(x,t)$ vanishes on the boundary, we have

$$\begin{aligned}
&\int_Q \left(\frac{\partial u}{\partial x_1}\right)^2 h(x,t)dxdt = \int_Q \left(\frac{\partial u}{\partial x_1}\right)\left[\left(\frac{\partial u}{\partial x_1}\right)h(x,t)\right]dxdt \\
&= -\int_Q u(x,t)\frac{\partial}{\partial x_1}\left((\frac{\partial u}{\partial x_1})h(x,t)\right)dxdt \\
&= -\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2}\right)h(x,t)dxdt - \int_Q u(x,t)\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial}{\partial x_1}h(x,t)\right)dxdt \\
&= -\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2}\right)h(x,t)dxdt - \frac{1}{2}\int_Q \left(\frac{\partial (u(x,t))^2}{\partial x_1}\right)\left(\frac{\partial}{\partial x_1}h(x,t)\right)dxdt.
\end{aligned} \tag{4.69}$$

Applying integration by parts on second integral of (4.69) we get

$$-\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2}\right)h(x,t)dxdt + \frac{1}{2}\int_Q u^2(x,t)\frac{\partial^2 h(x,t)}{\partial x_1^2}dxdt.$$

Now

$$\begin{aligned}
&\int_Q \left[|\mathrm{grad}\,(\mathrm{u(x,t)})|^2 - \left(\frac{\partial u}{\partial t}\right)^2\right] h(x,t)dxdt = \\
&-\int_Q u(x,t)\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}\right] h(x,t)dxdt \\
&+\frac{1}{2}\int_Q u^2(x,t)\left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)h(x,t)\right]dxdt \\
&+\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial t^2}\right)h(x,t)dxdt - \frac{1}{2}\int_Q u^2(x,t)\frac{\partial^2 h(x,t)}{\partial t^2}dxdt \\
&= -\int_Q u(x,t)\left[\Delta u(x,t) - \frac{\partial^2 u}{\partial t^2}\right]h(x,t)dxdt + \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) - \frac{\partial^2 h(x,t)}{\partial t^2}\right]dxdt
\end{aligned}$$

$$
\begin{aligned}
&= -\int_Q u(x,t)\left[\Delta u(x,t) - \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t}\right] h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) - \frac{\partial^2 h(x,t)}{\partial t^2}\right] dxdt.
\end{aligned}
$$

Using (4.68), we get further

$$
\begin{aligned}
&- \int_Q u(x,t)\left[L(u(x,t)) + \frac{\partial u}{\partial t}\right] h(x,t)dxdt + \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) - \frac{\partial^2 h(x,t)}{\partial t^2}\right] dxdt \\
&= -\int_Q u(x,t)L(u(x,t))h(x,t)dxdt - \int_Q u(x,t)\frac{\partial u}{\partial t} h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) - \frac{\partial^2 h(x,t)}{\partial t^2}\right] dxdt \qquad (4.70) \\
&= -\int_Q u(x,t)L(u(x,t))h(x,t)dxdt - \frac{1}{2}\int_Q \frac{\partial (u(x,t))^2}{\partial t} h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) - \frac{\partial^2 h(x,t)}{\partial t^2}\right] dxdt. \qquad (4.71)
\end{aligned}
$$

Again, using integration by parts on middle integral we have

$$
\begin{aligned}
&- \int_Q u(x,t)L(u(x,t))h(x,t)dxdt + \frac{1}{2}\int_Q u^2(x,t)\frac{\partial h}{\partial t} dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) - \frac{\partial^2 h(x,t)}{\partial t^2}\right] dxdt \\
&= -\int_Q u(x,t)L(u(x,t))h(x,t)dxdt + \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t)\right. \\
&\left. - \frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial h}{\partial t}\right] dxdt \\
&= -\int_Q u(x,t)L(u(x,t))h(x,t)dxdt + \frac{1}{2}\int_Q u^2(x,t)L^*(h(x,t))dxdt \\
&\le \int_Q \sup|u(x,t)|\,|L(u(x,t))|\,|h(x,t)|\,dxdt + \frac{1}{2}\int_Q u^2(x,t)\,|L^*(h(x,t))|\,dxdt, \qquad (4.72)
\end{aligned}
$$

where L^* is self-adjoint operator of L.
Since $Lu \ge 0$ and also $|L(u(x,t))| = |L(u_2) - L(u_1)| \le |L(u_2)| + |L(u_1)|$, (5.75) becomes

$$
\begin{aligned}
\int_Q \left[|\mathrm{grad}\, u(x,t)|^2 - \left(\frac{\partial u}{\partial t}\right)^2\right] h(x,t)dxdt \;&\le \int_Q \sup|u(x,t)|\Big(L(u_2) + L(u_1)\Big) h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)L^*(h(x,t))dxdt. \\
&\le \int_Q \sup|u(x,t)|\,L(u_2 + u_1)h(x,t)dxdt
\end{aligned}
$$

$$+\frac{1}{2}\int_Q u^2(x,t)L^*(h(x,t))dxdt. \tag{4.73}$$

Using Green-Gauss theorem, the right hand side of (4.73) is equal to

$$\int_Q \sup|u(x,t)|\,(u_2+u_1)L(h(x,t))dxdt+\frac{1}{2}\int_Q u^2(x,t)L^*(h(x,t))dxdt.$$
$$\leq \sup|u(x,t)|\int_Q\left[(u_2+u_1)+\frac{1}{2}(u^2(x,t))\right]L^*(h(x,t))dxdt.$$

□

Theorem 4.14 *Let $u_1(x,t)$ and $u_2(x,t)$ be the two smooth subsolutions of* (4.68) *and let $h(x,t)$ be the weight function, $h(x,t)=[r^2-(x-x_0)^2]t^2(T-t)^2$. Also suppose that $\frac{\partial^2 u_i}{\partial t^2}\geq 0,\ i=1,2$. Then the following estimate holds,*

$$\int_Q\left|\widetilde{\mathrm{grad}}\,u(x,t)\right|^2 h(x,t)dxdt \;\leq\; |u(x,t)|\int_Q(u_1+u_2)+\left[Lh(x,t)+2\frac{\partial^2 h}{\partial t^2}\right]dxdt$$
$$+\frac{1}{2}\int_Q u^2(x,t)\left[L^*h(x,t)+2\frac{\partial^2 h}{\partial t^2}\right]dxdt, \tag{4.74}$$

where $u=u_2-u_1$.

Proof.

$$\int_Q\left|\widetilde{\mathrm{grad}}\,u(x,t)\right|^2 h(x,t)dxdt=\int_Q\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\ldots+\left(\frac{\partial u}{\partial x_n}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]h(x,t)dxdt.$$
$$=\int_Q\left(\frac{\partial u}{\partial x_1}\right)^2 h(x,t)dxdt+\int_Q\left(\frac{\partial u}{\partial x_2}\right)^2 h(x,t)dxdt+\ldots$$
$$+\int_Q\left(\frac{\partial u}{\partial x_n}\right)^2 h(x,t)dxdt+\int_Q\left(\frac{\partial u}{\partial t}\right)^2 h(x,t)dxdt. \tag{4.75}$$

Using integration by parts formula and the fact that $h(x,t)$ vanishes on the boundary we calculate

$$\int_Q\left(\frac{\partial u}{\partial x_1}\right)^2 h(x,t)dxdt=\int_Q\left(\frac{\partial u}{\partial x_1}\right)\left[\left(\frac{\partial u}{\partial x_1}\right)h(x,t)\right]dxdt$$
$$=-\int_Q u(x,t)\frac{\partial}{\partial x_1}\left((\frac{\partial u}{\partial x_1})h(x,t)\right)dxdt$$
$$=-\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2}\right)h(x,t)dxdt-\int_Q u(x,t)\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial}{\partial x_1}h(x,t)\right)dxdt$$
$$=-\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2}\right)h(x,t)dxdt-\frac{1}{2}\int_Q\left(\frac{\partial(u^2(x,t))}{\partial x_1}\right)\left(\frac{\partial}{\partial x_1}h(x,t)\right)dxdt$$
$$=-\int_Q u(x,t)\left(\frac{\partial^2 u}{\partial x_1^2}\right)h(x,t)dxdt+\frac{1}{2}\int_Q u^2(x,t)\frac{\partial^2 h(x,t)}{\partial x_1^2}dxdt.$$

Similarly, solving the other integrals of (4.75), we get

$$
\begin{aligned}
\int_Q \left|\widetilde{\operatorname{grad}} u(x,t)\right|^2 h(x,t)dxdt &= -\int_Q u(x,t)\left[\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\ldots+\frac{\partial^2 u}{\partial x_n^2}\right]h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}\right)h(x,t)\right]dxdt \\
&- \int_Q u(x,t)\left(\frac{\partial^2 u}{\partial t^2}\right)h(x,t)dxdt+\frac{1}{2}\int_Q u^2(x,t)\frac{\partial^2 h(x,t)}{\partial t^2}dxdt \\
&= -\int_Q u(x,t)\left[\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\ldots+\frac{\partial^2 u}{\partial x_n^2}+\frac{\partial^2 u}{\partial t^2}\right]h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}+\frac{\partial^2}{\partial x_n^2}\right)h(x,t)\right]dxdt \\
&= -\int_Q u(x,t)\left[\Delta u(x,t)+\frac{\partial^2 u}{\partial t^2}\right]h(x,t)dxdt \\
&+\frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t)+\frac{\partial^2 h(x,t)}{\partial t^2}\right]dxdt. \\
&= -\int_Q u(x,t)\left[\Delta u(x,t)-\frac{\partial^2 u}{\partial t^2}+2\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-\frac{\partial u}{\partial t}\right]h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t)+\frac{\partial^2 h(x,t)}{\partial t^2}\right]dxdt \\
&= -\int_Q u(x,t)\left[Lu(x,t)+2\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right]h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t)+\frac{\partial^2 h(x,t)}{\partial t^2}\right]dxdt \\
= &- \int_Q u(x,t)Lu(x,t)h(x,t)dxdt-2\int_Q u(x,t)\frac{\partial^2 u}{\partial t^2}h(x,t)dxdt \\
&- \int_Q u(x,t)\frac{\partial u}{\partial t}h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t)+\frac{\partial^2 h(x,t)}{\partial t^2}\right]dxdt \\
= &- \int_Q u(x,t)Lu(x,t)h(x,t)dxdt-2\int_Q u(x,t)\frac{\partial^2 u}{\partial t^2}h(x,t)dxdt \\
&- \frac{1}{2}\int_Q \frac{\partial u^2}{\partial t}h(x,t)dxdt \\
&+ \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t)+\frac{\partial^2 h(x,t)}{\partial t^2}\right]dxdt \\
= &- \int_Q u(x,t)Lu(x,t)h(x,t)dxdt-2\int_Q u(x,t)\frac{\partial^2 u}{\partial t^2}h(x,t)dxdt
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{1}{2}\int_Q u^2 \frac{\partial h(x,t)}{\partial t} dxdt \\
& + \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) + \frac{\partial^2 h(x,t)}{\partial t^2}\right] dxdt \\
= & - \int_Q u(x,t)Lu(x,t)h(x,t)dxdt - 2\int_Q u(x,t)\frac{\partial^2 u}{\partial t^2}h(x,t)dxdt \\
& + \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) + \frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial h(x,t)}{\partial t}\right] dxdt \\
& \leq |u(x,t)|\int_Q |Lu(x,t)|\,h(x,t)dxdt + 2\,|u(x,t)|\int_Q \left|\frac{\partial^2 u}{\partial t^2}\right| h(x,t)dxdt \\
& + \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) + \frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial h(x,t)}{\partial t}\right] dxdt. \qquad (4.76)
\end{aligned}
$$

Since

$$
|L(u(x,t)| = |L(u_2(x,t)) - L(u_1(x,t))| \leq |L(u_2)| + |L(u_1)| = L(u_1) + L(u_2),
$$

$$
L(u_i(x,t)) \geq 0, \quad \frac{\partial^2 u_i}{\partial t^2} \geq 0,\ i = 1,2
$$

we have

$$
\left|\frac{\partial^2 u}{\partial t^2}\right| = \left|\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_1}{\partial t^2}\right| \leq \frac{\partial^2 u_2}{\partial t^2} + \frac{\partial^2 u_1}{\partial t^2}.
$$

Now we further estimate (4.76) by

$$
\begin{aligned}
\int_Q \left|\widetilde{\operatorname{grad}} u(x,t)\right|^2 h(x,t)dxdt \ \leq & \ |u(x,t)|\int_Q (u_1+u_2)Lh(x,t)dxdt + 2\,|u(x,t)|\int_Q (u_1+u_2)\frac{\partial^2 h(x,t)}{\partial t^2}dxdt \\
& + \frac{1}{2}\int_Q u^2(x,t)\frac{\partial h}{\partial t}\left[\Delta h(x,t) + \frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial h(x,t)}{\partial t}\right] dxdt
\end{aligned}
$$

$$
\begin{aligned}
& \leq |u(x,t)|\int_Q (u_1+u_2)Lh(x,t)dxdt + 2\,|u(x,t)|\int_Q (u_1+u_2)\frac{\partial^2 h(x,t)}{\partial t^2}dxdt \\
& + \frac{1}{2}\int_Q u^2(x,t)\left[\Delta h(x,t) - \frac{\partial^2 h(x,t)}{\partial t^2} + \frac{\partial h}{\partial t} + \frac{\partial^2 h(x,t)}{\partial t^2}\right] dxdt \\
& \leq |u(x,t)|\int_Q (u_1+u_2) + \left[Lh(x,t) + 2\frac{\partial^2 h}{\partial t^2}\right] dxdt + \frac{1}{2}\int_Q u^2(x,t)\left[L^* h(x,t) + 2\frac{\partial^2 h}{\partial t^2}\right] dxdt.
\end{aligned}
$$

□

In the sequel, we again use the mollification of bounded, measurable function $u(x,t)$ on the cylinder P :

$$
u_\varepsilon(x,t) = \varepsilon^{-(n+1)} \int_0^T \int_B \eta_n\left(\frac{x-y}{\varepsilon}\right)\eta_1\left(\frac{t-s}{\varepsilon}\right)u(y,s)dyds, \qquad (4.77)
$$

for arbitrary $\varepsilon > 0$.
Let us denote

$$\eta_\varepsilon(x-y,t-s) = \varepsilon^{-(n+1)}\eta_n\left(\frac{x-y}{\varepsilon}\right)\eta_1\left(\frac{t-s}{\varepsilon}\right),$$

then following will be trivial

$$\frac{\partial^2}{\partial x_i^2}\eta_\varepsilon(x-y,t-s) = \frac{\partial^2}{\partial y_i^2}\eta_\varepsilon(x-y,t-s),$$

and

$$\frac{\partial^2}{\partial t^2}\eta_\varepsilon(x-y,t-s) = \frac{\partial^2}{\partial s^2}\eta_\varepsilon(x-y,t-s).$$

$$\frac{\partial}{\partial t}\eta_\varepsilon(x-y,t-s) = \frac{\partial}{\partial s}\eta_\varepsilon(x-y,t-s).$$

From the equations, we can easily deduce

$$L_{x,t}\eta_\varepsilon(x-y,t-s) = L_{y,s}\eta_\varepsilon(x-y,t-s) = L^*_{y,s}\eta_\varepsilon(x-y,t-s)$$

where operators $L_{x,t}$ and $L_{y,s}$ act on arguments (x,t) and (y,s), respectively.
From the above (4.77) becomes

$$L_{x,t}u_\varepsilon(x,t) = \int_0^T\int_B u(y,s)L^*_{y,s}\eta_\varepsilon(x-y,t-s)\,dy\,ds$$

Also define the cylinders P_k

$$P_k = P\left(r_k, \frac{T}{k+2}\right) = B(x_0,r_k)\times\left(\frac{T-1}{k+2}, r_k\right)$$

$r_k = \frac{k+1}{k+2}r, \quad k\in\mathbb{N}$.
It is obvious, by construction, that $u(x,t)$ is infinitely differentiable with respect to its arguments on $\mathbb{R}^{n+1}$

With same arguments as in Theorem 4.6, we can now prove the next theorem.

Theorem 4.15 *Consider the weak subsolution $u(x,t)$ of telegraph equation in the cylinder $Q = B\times(0,T)$. Then for any $k\in\mathbb{N}$ there exists $\hat{\varepsilon} > 0$ such that for every ε, $0<\varepsilon<\hat{\varepsilon}$ each function $u_\varepsilon(x,t)$ is the smooth subsolution of telegraph equation in the cylinder Q_k, that is*

$$Lu_\varepsilon(x,t) \geq 0,\ (x,t)\in Q_k.$$

With same arguments as in Theorem 4.7, we can now prove the next theorem.

Theorem 4.16 *Any continuous weak subsolution $u(x,t)$ of telegraph equation has weak partial derivatives $\frac{\partial u(x,t)}{\partial x_i}$, $i=1,\dots,n$ in the cylinder $Q, Q = B(x_0,R)\times(0,T)$, $\frac{\partial u}{\partial t}$ also exist and they are square integrable with respect to the weight function $h(x,t)$, i.e.*

$$\int_Q \left|\widetilde{\operatorname{grad}}\,u(x,t)\right|^2 h(x,t)\,dx\,dt < \infty. \tag{4.78}$$

With same arguments as in Theorem 4.12, we can now prove the next theorem.

Theorem 4.17 *Consider two arbitrary continuous weak subsolution of telegraph equation $u_i(x,t), i = 1,2$ in the cylinder $Q, Q = B(x_0,R)\times(0,T)$. Then the following weighted reverse poincare type inequality holds for the difference $u_2(x,t)-u_1(x,t)$ of two weak subsolutions.*

$$\int_Q \left|\widetilde{\operatorname{grad}}\, u_2(x,t) - \widetilde{\operatorname{grad}}\, u_1(x,t)\right|^2 h(x,t)dxdt \le$$

$$\left[\|u_2-u_1\|_{L^\infty(Q)}\left(\|u_1\|_{L^\infty(Q)}+\|u_2\|_{L^\infty(Q)}\right)\int_Q\left[Lh(x,t)+2\frac{\partial^2 h}{\partial t^2}\right]dxdt+\frac{1}{2}\|u_2-u_1\|^2_{L^\infty(Q)}\right]$$

$$\times\int_Q\left|L^*h(x,t))+2\frac{\partial^2 h(x,t)}{\partial t^2}\right|dxdt.$$

4.7 The weighted reverse Poincaré type inequalities for elliptic subsolution

The reverse Poincaré (or the Caccioppoli) inequality represents an important tool in the study of qualitative properties of solutions of elliptic as well as parabolic partial differential equations (see, e.g. Giaquinta [24], Heinonen, Kilpelainen, Martio [29] , Perić, Žubrinić [54], Lieberman [44]).

Consider the second order uniformly elliptic partial differential operator Lu acting on real valued smooth functions u defined in an n-dimensional ball $B = B(x_0,R)$. The function u is called the classical sub-solution of the partial differential equation $Lu(x) = 0$ in B, if u is twice continuously differentiable and satisfy the differential inequality

$$Lu(x)\ge 0 \ \text{ in } B. \tag{4.79}$$

The classical Caccioppoli inequality bounds the subsolution's gradient norm $\|\operatorname{grad}\mathrm{u(x)}\|_{\mathrm{L^2(B)}}$ by the norm $\|u(x)\|_{L^2(\widehat{B})}$ of the subsolution itself, where $\widehat{B} = B(x_0,2R)$. Littman [46] gave a very fruitful generalization of the notion of the classical subsolution to the case of functions which need to be only locally integrable without any regularity requirements.

According to Littman [46] the locally integrable function u defined in the ball B is called a weak $L-$sub-solution if for all nonnegative functions v, which are twice continuously differentiable with compact support in B, the following inequality is valid

$$\int_B u(x)L^*v(x)\,dx\ge 0, \tag{4.80}$$

where $L^*v(x)$ denotes the adjoint operator to $Lv(x)$.
For the elliptic differential operator $Lu(x)$ with smooth coefficients and for arbitrary continuous weak $L-$subsolution u Littman [46] proves the fundamental approximation theorem, which states that there exist a sequence of smooth classical subsolutions u_m, $m \in \mathbb{N}$, in the ball B, such that on each compact subset $K \subset B$ we have the uniform convergence

$$\sup_{x\in K} |u_m(x) - u(x)| \xrightarrow{m\to\infty} 0. \tag{4.81}$$

Based on the latter approximation theorem of Littman we establish in the definition (4.80) of the weak $L-$subsolution which requires no a priori regularity in fact leads to the existence and the integrability of the Sobolev gradient $\mathrm{grad}\,\mathrm{u(x)}$ of the continuous weak $L-$subsolution $u(x)$.
This remarkable fact enabled us to establish a new type weighted reverse Poincaré inequality for a difference of two continuous weak $L-$subsolutions. We should note here that the difference of two $L-$subsolutions is neither $L-$subsolution, nor $L-$supersolution in general, and therefore this type of inequality can not be reduced to the classical one.

4.7.1 Subsolutions that are close in the uniform norm are close in the Sobolev norm as well

Consider two arbitrary finite convex functions f and φ on a closed interval $[a,b]$. The following energy inequality was established by K. Shashiashvili and M. Shashiashvili in [62, Theorem 2.1]

$$\begin{aligned}
&\int_a^b (x-a)^2(b-x)^2(f'(x-)-\varphi'(x-))^2\,dx \\
&\qquad \le \frac{8}{9}\sqrt{3}\sup_{x\in(a,b)}|f(x)-\varphi(x)|\sup_{x\in(a,b)}|f(x)+\varphi(x)|(b-a)^3 \\
&\qquad +\frac{4}{3}\left(\sup_{x\in(a,b)}|f(x)-\varphi(x)|\right)^2(b-a)^3.
\end{aligned} \tag{4.82}$$

This kind of estimate with weight functions on an infinite interval $[0,\infty)$ was subsequently applied to hedging problems of mathematical finance in S. Hussain and M. Shashiashvili [21] (see also S. Hussain, J. Pečarić and M. Shashiashvili [32]). The natural generalization of univariate convex functions to the case of several variables are subharmonic functions that share many convenient attributes of the former functions. An extensive study of the properties of subharmonic functions was carried out by L. Hörmander in his well-known book [19, Chapter 3].

A locally integrable function u in the ball B is said to be a weak Δ-subsolution of the Laplace equation

$$\Delta u(x) = 0 \text{ in the ball } B$$

if

$$\int_B u(x)\Delta v(x)\,dx \geq 0 \tag{4.83}$$

for all nonnegative v, such that $v \in C_0^2(B)$ (i.e. $\Delta u \geq 0$ in the sense of the distribution theory). Theorem 3.2.11 in [19] states the equivalence between the notion of a subharmonic function and the notion of a weak Δ-subsolution.

Consider a sequence of subharmonic functions u_m, $m \in \mathbb{N}$,, on the ball B, which converges to a subharmonic function u in $L^1_{loc}(B)$. Theorem 3.2.13 in L. Hörmander [19] asserts that weak partial derivatives $\frac{\partial u_m(x)}{\partial x_i}$, $i = 1,\dots,n$, tend to $\frac{\partial u(x)}{\partial x_i}$, $i = 1,\dots,n$, in $L^p_{loc}(B)$ for an exponent p with $1 \leq p < \frac{n}{n-1}$.
Proposition 3.4.19 in [19] considers a sequence of bounded nonpositive subharmonic functions u_m in the ball B, such that $u_m\big|_{\partial B} = 0$ and $\sup \Delta u_m$ is contained in a fixed compact set $K \subset B$. It is proved there that if

$$u_m(x) \downarrow u(x) \ \text{ when } \ m \to \infty,$$

then weak partial derivatives $\frac{\partial u_m(x)}{\partial x_i}$ converge to $\frac{\partial u(x)}{\partial x_i}$, $i = 1,\dots,n$, in $L^2(B)$.
It seems reasonable to ask whether the mapping $u(x) \to \operatorname{grad} u(x)$ possesses some Hölder continuity property when restricted to the class of subharmonic functions defined on the ball B. W. Littman [46] gave a very fruitful generalization of the notion of a subharmonic function to the case of general type (with variable coefficients) second order linear elliptic partial differential operators.
According to Littman [46], the locally integrable function u defined in the ball B is called a generalized subharmonic function if for all nonnegative functions $v \in C_0^2(B)$ the following inequality holds

$$\int_B u(x)L^*v(x)\,dx \geq 0 \tag{4.84}$$

(i.e. $Lu(x) \geq 0$ in the sense of the distribution theory), where $L^*v(x)$ is the adjoint operator to $Lv(x)$,

$$\begin{aligned} Lu(x) &= \sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\,\frac{\partial u(x)}{\partial x_i} + c(x)u(x), \\ L^*u(x) &= \sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^*(x)\,\frac{\partial u(x)}{\partial x_i} + c^*(x)u(x), \end{aligned} \tag{4.85}$$

where

$$\begin{aligned} b_i^*(x) &= -b_i(x) + 2\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}, \\ c^*(x) &= c(x) - \sum_{i=1}^{n} \frac{\partial b_i(x)}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} \end{aligned} \tag{4.86}$$

with $a_{ij}(x) = a_{ji}(x)$, $i,j = 1,\dots,n$. It is assumed that the operator L is uniformly elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x)y_i y_j \geq \alpha |y|^2, \ \ x \in B, \ \ y \in \mathbb{R}^n, \tag{4.87}$$

where $\alpha > 0$ is the elliptic constant and the coefficients satisfy the smoothness conditions

$$\begin{gathered} a_{ij}(x) \in C^{2+\gamma}(\overline{B}),\ \ b_i(x) \in C^{1+\gamma}(\overline{B}), \\ c(x) \in C^{\gamma}(\overline{B}),\ \ i,j = 1,\dots,n, \end{gathered} \tag{4.88}$$

with a Hölder exponent γ, $0 < \gamma \leq 1$.

Note that for the sake of simplicity we use the term a weak L-subsolution instead of the term Littman's generalized subharmonic function.

In this section we establish an estimate for a difference of two continuous weak L-subsolutions in an n-dimensional ball B, which is analogous to the one-dimensional estimate (2.37).

4.7.2 Preliminary material and the formulation of the basic result

Consider the twice continuously differentiable functions u and h in the ball $B = B(x_0, R)$. We start with the well-known Green's identity (see e.g. A. Friedman [18, chapter 6, section 4])

$$\begin{aligned} &h(x)Lu(x) - u(x)L^*h(x) = \\ &\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\left(h(x)a_{ij}(x)\,\frac{\partial u(x)}{\partial x_j} - u(x)a_{ij}(x)\,\frac{\partial h(x)}{\partial x_j} - u(x)h(x)\,\frac{\partial a_{ij}(x)}{\partial x_j}\right) + b_i(x)u(x)h(x)\right]. \end{aligned} \tag{4.89}$$

Suppose now that $u \in C^2(\overline{B})$, $h \in C^2(\overline{B})$ and integrate the identity (4.89) using the Gauss-Ostrogradski divergence theorem. We get

$$\begin{aligned} \int_B Lu(x)h(x)\,dx = &\int_B u(x)L^*h(x)\,dx + \\ &+ \int_{\partial B} \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(h(x)a_{ij}(x)\,\frac{\partial u(x)}{\partial x_j} - u(x)a_{ij}(x)\,\frac{\partial h(x)}{\partial x_j} - \right.\right. \\ &\left.\left. - u(x)h(x)\,\frac{\partial a_{ij}(x)}{\partial x_j}\right)n_i(x) + b_i(x)u(x)h(x)n_i(x)\right] d\sigma, \end{aligned} \tag{4.90}$$

where $n(x) = (n_i(x))_{i=1,\dots,n}$ is the outward pointing unit normal vector at $x \in \partial B$, and $d\sigma$ is an $(n-1)$-dimensional surface measure of the ball B.

We say that h, $h \in C(\overline{B})$, is a weight function if

$$h(x) > 0 \ \text{ in a ball } B \text{ and } \ h(x)\big|_{\partial B} = 0.$$

Let us consider a weight function $h \in C^2(\overline{B})$. Then from the equality (4.90) we get the Green's second formula

$$\int_B Lu(x)h(x)\,dx = \int_B u(x)L^*h(x)\,dx - \int_{\partial B} u(x)\big(\operatorname{grad} h(x), \gamma_a(x)\big)\,d\sigma, \tag{4.91}$$

where

$$\operatorname{grad} h(x) = \left(\frac{\partial h(x)}{\partial x_i}\right)_{i=1,\dots,n}, \quad \gamma_a(x) = \big(\gamma_{ai}(x)\big)_{i=1,\dots,n},$$

where

$$\gamma_{ai}(x) = \sum_{j=1}^{n} a_{ji}(x) n_j(x), \quad i = 1, \dots, n.$$

We have

$$\big(\gamma_a(x), n(x)\big) = \sum_{i,j=1}^{n} a_{ij}(x) n_i(x) n_j(x) \geq \alpha |n(x)|^2 = \alpha > 0$$

by the uniform ellipticity condition (4.87).

Hence for $x \in \partial B$

$$\big(\operatorname{grad} h(x), \gamma_a(x)\big) = \lim_{t \downarrow 0} \frac{h(x) - h(x - t\gamma_a(x))}{t} \leq 0.$$

Let us write the operator $Lu(x)$ in the variational form

$$Lu(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) - \sum_{i=1}^{n} b_i^*(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x)$$

and introduce the bilinear form $a(u,v)$ on the product space $C^1(\overline{B}) \times C^1(\overline{B})$

$$a(u,v) = \int_B \left[\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i} + \sum_{i=1}^{n} b_i^*(x)\frac{\partial u(x)}{\partial x_i} v(x) - c(x)u(x)v(x)\right] dx.$$

In the sequel we will need the Green's first formula (see e.g. C. Baiocchi and A. Capelo [3, Chapter 18])

$$a(u,v) = -\int_B Lu(x)v(x)\,dx + \int_{\partial B} v(x)\big(\operatorname{grad} u(x), \gamma_a(x)\big)\,d\sigma \tag{4.92}$$

for $u \in C^2(\overline{B})$ and $v \in C^1(\overline{B})$.

Consider now the linear space S of locally integrable functions u in the ball B, which have weak (Sobolev) derivatives $\frac{\partial u(x)}{\partial x_i}$, $i = 1, \dots, n$.

Define the weight functions

$$\begin{aligned} \widehat{h}(\beta) \equiv \widehat{h}(\beta; x) &= R^{2-\beta} \operatorname{dist}^{\beta}(x, \partial B), \quad \beta \geq 1, \\ \overline{h}(x) &= R^2 - |x - x_0|^2. \end{aligned} \tag{4.93}$$

Introduce a subspace $H^1(B; \widehat{h}(\beta))$ of the space S consisting of functions $u \in S$ for which the following integral is finite

$$\int_B u^2(x)\,dx + \sum_{i=1}^{n} \int_B \left(\frac{\partial u(x)}{\partial x_i}\right)^2 \widehat{h}(\beta; x)\,dx \equiv \|u\|^2_{H^1(B;\widehat{h}(\beta))}. \tag{4.94}$$

One can easily check that $H^1(B;\widehat{h}(\beta))$ is a complete linear space. We call it the weighted Sobolev space. The following inclusion is obvious

$$H^1(B) \subseteq H^1(B;\widehat{h}(\beta)) \subseteq H^1_{loc}(B), \tag{4.95}$$

where $H^1(B)$ and $H^1_{loc}(B)$ are respectively the first order Sobolev and the corresponding local Sobolev spaces.

Note that (4.95) asserts that if two bounded continuous weak L-subsolutions in a ball B are close in the uniform norm, then they remain close in the weighted Sobolev norm as well.

4.7.3 Auxiliary propositions and the proof of the basic result

Consider a weight function $h \in C^2(\overline{B})$ and two arbitrary smooth L-subsolutions $u_i \in C^2(\overline{B})$ in the ball $B = B(x_0, r)$, $r > 0$, i.e.

$$Lu_i(x) \geq 0 \ \text{ for all } \ x \in B, \ \ i = 1,2. \tag{4.96}$$

Proposition 4.1 *Suppose that the uniform ellipticity condition* (4.87) *is satisfied and the coefficients of the differential operator $Lu(x)$ are smooth, i.e.*

$$a_{ij} \in C^2(\overline{B}), \quad b_i \in C^1(\overline{B}), \quad c \in C(\overline{B}), \quad i,j = 1,\dots,n. \tag{4.97}$$

If u_1 and u_2 are smooth L-subsolutions satisfying the inequality (4.96) *then the following energy inequality is valid*

$$\int\limits_B \Big|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\Big|^2 h(x)\,dx \leq \frac{1}{\alpha}\int\limits_B \Big(|L^* h(x)| + |c(x)|h(x)\Big)\,dx \tag{4.98}$$
$$\times \Big[2\|u_2 - u_1\|_{L^\infty(B)}\big(\|u_1\|_{L^\infty(B)} + \|u_2\|_{L^\infty(B)}\big) + \|u_2 - u_1\|^2_{L^\infty(B)}\Big].$$

Proof. Denote $u(x) = u_2(x) - u_1(x)$, $x \in \overline{B}$. Taking u^2 instead of u in the Green's second formula (4.91), we have

$$\int\limits_B Lu^2(x)h(x)\,dx = \int\limits_B u^2(x)L^*h(x)\,dx - \int\limits_{\partial B} u^2(x)\big(\operatorname{grad} h(x), \gamma_a(x)\big)\,d\sigma.$$

It is easy to see then

$$Lu^2(x) = 2\sum_{i,j=1}^n a_{ij}(x)\,\frac{\partial u(x)}{\partial x_i}\,\frac{\partial u(x)}{\partial x_j} + 2u(x)Lu(x) - c(x)u^2(x).$$

Now

$$2\int\limits_B \sum_{i,j=1}^n a_{ij}(x)\,\frac{\partial u(x)}{\partial x_i}\,\frac{\partial u(x)}{\partial x_j} h(x)\,dx + 2\int\limits_B u(x)Lu(x)h(x)\,dx$$

$$= \int_B \big(L^*h(x) + c(x)h(x)\big)u^2(x)\,dx - \int_{\partial B} u^2(x)\big(\operatorname{grad} h(x), \gamma_a(x)\big)\,d\sigma. \quad (4.99)$$

From (4.99) we get

$$2\alpha \int_B |\operatorname{grad} u(x)|^2 h(x)\,dx \le 2\sup_B |u(x)| \int_B |Lu(x)|h(x)\,dx$$
$$+ \sup_B u^2(x) \int_B \Big(|L^*h(x)| + |c(x)|h(x)\Big)\,dx + \sup_{\partial B} u^2(x) \int_{\partial B} \Big|\big(\operatorname{grad} h(x), \gamma_a(x)\big)\Big|\,d\sigma. \quad (4.100)$$

Taking $u(x) = 1$ in the equality (4.99),

$$\int_{\partial B} \big(\operatorname{grad} h(x), \gamma_a(x)\big)\,d\sigma = \int_B \big(L^*h(x) - c(x)h(x)\big)\,dx. \quad (4.101)$$

Since

$$\big(\operatorname{grad} h(x), \gamma_a(x)\big) \le 0 \quad (4.102)$$

from the relation (4.100) we derive the estimate

$$\alpha \int_B |\operatorname{grad} u(x)|^2 h(x)\,dx \le \sup_B |u(x)| \int_B |Lu(x)|h(x)\,dx$$
$$+ \sup_B u^2(x) \int_B \Big(|L^*h(x)| + |c(x)|h(x)\Big)\,dx. \quad (4.103)$$

Futher, $|Lu(x)| = \big|Lu_2(x) - Lu_1(x)\big| \le L\big(u_1(x) + u_2(x)\big)$, hence

$$\int_B |Lu(x)|h(x)\,dx \le \int_B L\big(u_1(x) + u_2(x)\big)h(x)\,dx.$$

From the Green's second formula (4.91) we can write

$$\int_B L\big(u_1(x) + u_2(x)\big)h(x)\,dx = \int_B \big(u_1(x) + u_2(x)\big)L^*h(x)\,dx$$
$$+ \int_{\partial B} \big(u_1(x) + u_2(x)\big)\big(\operatorname{grad} h(x), -\gamma_a(x)\big)\,d\sigma. \quad (4.104)$$

Using (4.99)–(4.102) we know that

$$\big(\operatorname{grad} h(x), -\gamma_a(x)\big) \ge 0,$$
$$\int_{\partial B} \big(\operatorname{grad} h(x), -\gamma_a(x)\big)\,d\sigma = \int_B \big(-L^*h(x) + c(x)h(x)\big)\,dx, \quad (4.105)$$

therefore

$$\int_B |Lu(x)|h(x)\,dx \le 2\sup_B \big|u_1(x) + u_2(x)\big| \int_B \Big(|L^*h(x)| + |c(x)|h(x)\Big)\,dx. \quad (4.106)$$

Using estimates (4.103) and (4.106) we obtain the desired inequality (4.98). □

In order to extend the inequality (4.98) to the general case of weak L-subsolutions we need to approximate an arbitrary continuous weak L-subsolution by a sequence of smooth L-subsolutions. It turns out that in case of the variable coefficients of the differential operator $Lu(x)$ this is not a trivial task (since the standard mollification arguments work only for the case with constant coefficients). The technique of approximation for this kind of problem was developed by W. Littman in [46] and we make essential use of it.

For an arbitrary continuous weak L-subsolution u W. Littman constructed a monotonic nonincreasing sequence u_m, $m \in \mathbb{N}$ of functions in the ball B, such that on each compact subset $K \subset B$

$$\begin{gathered} u_m \in C^{2+\beta}(K), \quad Lu_m(x) \geq 0, \ \ x \in K, \\ \lim_{m\to\infty} \downarrow u_m(x) = u(x), \ \ x \in K \end{gathered} \tag{4.107}$$

for m sufficiently large (that depends on K).

Here we consider only the continuous weak L-subsolutions u in the ball B. By Dini's classical theorem the latter convergence is uniform

$$\sup_{x\in K} \left| u_m(x) - u(x) \right| \xrightarrow{m\to\infty} 0.$$

Let us consider the balls $B_k = B(x_0, r_k)$, $r_k = R\,\frac{k}{k+1}$, $k \in \mathbb{N}$, which are compactly imbedded in the original ball $B = B(x_0, R)$.
We also introduce the smooth weight functions

$$\begin{gathered} h_k(x) = r_k^2 - |x - x_0|^2, \ \ x \in \overline{B}_k, \ \ k \in \mathbb{N}, \\ h_\infty(x) = R^2 - |x - x_0|^2, \ \ x \in \overline{B}. \end{gathered} \tag{4.108}$$

Now we will show that any continuous weak L-subsolution $u(x)$ in the ball B has all first order weak (Sobolev) derivatives

$$\frac{\partial u(x)}{\partial x_i}, \ \ i = 1, \ldots, n.$$

Theorem 4.18 *Suppose that the conditions* (4.87) – (4.88) *are satisfied. Then any continuous weak L-subsolution u has weak partial derivatives* $\frac{\partial u(x)}{\partial x_i}$, $i = 1, \ldots, n$, *in the ball* $B = B(x_0, R)$.

Proof. Let us consider the sequence u_m approximating the function u. If we write the inequality (4.98) for

$$u_1 = u_m, \quad u_2 = u_l$$

and for the ball B_{k+1}, then we get

$$\int_{B_{k+1}} \left| \operatorname{grad} u_m(x) - \operatorname{grad} u_l(x) \right|^2 h_{k+1}(x)\, dx$$

$$\leq \frac{c_{k+1}}{\alpha}\Big[2\|u_m-u_l\|_{L^\infty(B_{k+1})}\big(\|u_m\|_{L^\infty(B_{k+1})}+\|u_l\|_{L^\infty(B_{k+1})}\big)+\|u_m-u_l\|^2_{L^\infty(B_{k+1})}\Big], \tag{4.109}$$

where

$$c_k=\int\limits_{B_k}\Big(|L^*h_k(x)|+|c(x)|h_k(x)\Big)\,dx. \tag{4.110}$$

Note that for $x\in B_k$ the following estimate is valid:

$$h_{k+1}(x)\geq \frac{R^2}{(k+1)(k+2)}. \tag{4.111}$$

Therefore if we restrict the integral on the left-hand side of (4.109) over the ball B_k, then we have

$$\begin{aligned}&\frac{R^2}{(k+1)(k+2)}\int\limits_{B_k}\big|\operatorname{grad}u_m(x)-\operatorname{grad}u_l(x)\big|^2\,dx \\ &\leq \frac{c_{k+1}}{\alpha}\Big[2\|u_m-u_l\|_{L^\infty(B_{k+1})}\big(\|u_m\|_{L^\infty(B_{k+1})}+\|u_l\|_{L^\infty(B_{k+1})}\big)+\|u_m-u_l\|^2_{L^\infty(B_{k+1})}\Big].\end{aligned} \tag{4.112}$$

Since the sequence u_m converges to u in the norm $L^\infty(B_{k+1})$, we can write

$$\|u_m-u_l\|_{L^\infty(B_{k+1})}\longrightarrow 0,\quad m,l\to\infty.$$

Passing to the limit in the inequality (4.112) as $m,l\to\infty$, we obtain

$$\lim_{m,l\to\infty}\sum_{i=1}^n\int\limits_{B_k}\Big(\frac{\partial u_m(x)}{\partial x_i}-\frac{\partial u_l(x)}{\partial x_i}\Big)^2\,dx=0. \tag{4.113}$$

By the completeness of the space $L^2(B_k)$, there exists a family of measurable functions $g_{k,i}(x)$, $i=1,\dots,n$, $k=1,2,\dots$, such that $g_{k,i}(x)\in L^2(B_k)$, $i=1,\dots,n$, and

$$\lim_{m\to\infty}\sum_{i=1}^n\int\limits_{B_k}\Big(\frac{\partial u_m(x)}{\partial x_i}-g_{k,i}(x)\Big)^2\,dx=0,\ \ k=1,2,\dots. \tag{4.114}$$

Let us extend the functions $g_{k,i}$ trivially outside B_k as follows

$$g_{k,i}(x)=\begin{cases}g_{k,i}(x) & \text{for }\ x\in B_k,\\ 0 & \text{for }\ x\in B\setminus B_k,\end{cases}$$

and define the functions g_i, $i=1,\dots,n$, on the ball B by

$$g_i(x)=\limsup_{k\to\infty}g_{k,i}(x),\ \ i=1,\dots,n. \tag{4.115}$$

It is obvious that the functions $g_{k+l,i}$, $l=0,1,2,\dots$, agree on the ball B_k and therefore

$$g_i(x)=g_{k,i}(x)\ \text{(a.e.) on a ball } B_k. \tag{4.116}$$

Thus the functions g_i, $i = 1, \ldots, n$, are locally square integrable on the ball B.

Let us check that g_i, $i = 1, \ldots, n$, represent the weak partial derivatives of the function u. Take any continuously differentiable function φ with compact support in B (i.e. $\varphi \in C_0^1(B)$). Then supp $\varphi \subset B_k$ for some k. We have

$$\int_{B_k} \frac{\partial u_m(x)}{\partial x_i} \varphi(x)\, dx = -\int_{B_k} u_m(x)\, \frac{\partial \varphi(x)}{\partial x_i}\, dx.$$

But u_m converges uniformly to u on B_k, and $\frac{\partial u_m}{\partial x_i}$ converges to g_i in $L^2(B_k)$. Hence, passing to the limit as $m \to \infty$ we obtain the equality

$$\int_{B_k} g_i(x)\varphi(x)\, dx = -\int_{B_k} u(x)\, \frac{\partial \varphi(x)}{\partial x_i}\, dx, \tag{4.117}$$

which means that g_i, $i = 1, \ldots, n$, are indeed the weak partial derivatives of the function u. $\square$

Theorem 4.19 *Assume the conditions* (4.87) – (4.88) *are satisfied. Then any continuous bounded weak L-subsolution u in the ball B belongs to the weighted Sobolev space* $H^1(B; \widehat{h}(\beta))$, $\beta \geq 1$.

Proof. We write the inequality (4.98) for the functions $u_1(x) = 0$ and $u_2(x) = u_m(x)$ and the ball B_{k+l}, where the sequence $u_m(x)$ converges to $u(x)$. We obtain

$$\int_{B_{k+l}} |\mathrm{grad}\, u_m(x)|^2 h_{k+l}(x)\, dx \leq \frac{c_{k+l}}{\alpha} 3\|u_m\|^2_{L^\infty(B_{k+l})}.$$

Next, passing to the limit as $m \to \infty$, we get

$$\int_{B_{k+l}} |\mathrm{grad}\, u(x)|^2 h_{k+l}(x)\, dx \leq \frac{c_{k+l}}{\alpha}\, 3\|u\|^2_{L^\infty(B_{k+l})}.$$

Restricting the integral on the left-hand side of this inequality over the ball B_k and making the integer l tend to infinity, we obtain

$$\int_{B_k} |\mathrm{grad}\, u(x)|^2 h_\infty(x)\, dx \leq \frac{c_\infty}{\alpha}\, 3\|u\|^2_{L^\infty(B)} < \infty. \tag{4.118}$$

Since the left-hand side of (4.118) is increasing with respect to k and bounded, using dominate convergence we have

$$\int_{B} |\mathrm{grad}\, u(x)|^2 h_\infty(x)\, dx \leq \frac{3c_\infty}{\alpha}\, \|u\|^2_{L^\infty(B)}.$$

But

$$h_\infty(x) = R^2 - |x - x_0|^2 \geq R^2 \left(\frac{\mathrm{dist}(x, \partial B)}{R}\right) \geq R^{2-\beta}\, \mathrm{dist}^\beta(x, \partial B),$$

for $\beta \geq 1$. Hence, we get the energy estimate

$$\int_B |\operatorname{grad} u(x)|^2 \widehat{h}(\beta;x)\,dx \leq \frac{3c_\infty}{\alpha}\,\|u\|^2_{L^\infty(B)} < \infty,$$

where

$$c_\infty = \int_B \Big(|L^* h_\infty(x)| + |c(x)| h_\infty(x)\Big)\,dx.$$

□

Theorem 4.20 (THE WEIGHTED REVERSE POINCARÉ INEQUALITY) *Assume that the conditions* (4.87) – (4.88) *are satisfied. Consider two weak L-subsolutions u_i, $i = 1,2$, in the ball B, such that*

$$u_i \in C(B) \cap L^\infty(B), \ \ i = 1,2. \tag{4.119}$$

Then the functions u_i belong to the weighted Sobolev space $H^1(B;\widehat{h}(\beta))$, $\beta \geq 1$, and the following reverse Poincaré type inequality holds for the difference $u_2 - u_1$ of two weak L-subsolutions

$$\begin{aligned} &\|u_2 - u_1\|^2_{H^1(B;\widehat{h}(\beta))} \leq \\ &\leq \Big(\frac{c}{\alpha} + \lambda(B)\Big) \times \Big[2\|u_2 - u_1\|_{L^\infty(B)}\big(\|u_1\|_{L^\infty(B)} + \|u_2\|_{L^\infty(B)}\big) + \|u_2 - u_1\|^2_{L^\infty(B)}\Big], \end{aligned} \tag{4.120}$$

where

$$c = \int_B \Big(|L^* \overline{h}(x)| + |c(x)| \overline{h}(x)\Big)\,dx,$$

and $\alpha > 0$ is the constant of the uniform ellipticity.

Proof. We consider the sequences of smooth L-subsolutions $u_{m,i}$, $i = 1,2$, $m = 1,2,\ldots$, converging on the balls B_{k+l} uniformly to weak L-subsolutions u_i, $i = 1,2$. By the assumption of the theorem the functions u_i, $i = 1,2$, are continuous and bounded on the ball B, i.e.

$$u_i \in C(B) \cap L^\infty(B), \ \ i = 1,2.$$

Let us apply the inequality (4.98) to the functions $u_{m,1}$ and $u_{m,2}$ and the balls B_{k+l}, $k,l \in \mathbb{N}$. We have

$$\begin{aligned} &\int_{B_{k+l}} \big|\operatorname{grad} u_{m,2}(x) - \operatorname{grad} u_{m,1}(x)\big|^2 h_{k+l}(x)\,dx \\ &\quad \leq \frac{c_{k+l}}{\alpha}\Big[2\big\|u_{m,2}(x) - u_{m,1}(x)\big\|_{L^\infty(B_{k+l})}\big(\|u_{m,2}\|_{L^\infty(B_{k+l})} + \|u_{m,1}\|_{L^\infty(B_{k+l})}\big) \\ &\quad + \|u_{m,2} - u_{m,1}\|^2_{L^\infty(B_{k+l})}\Big]. \end{aligned} \tag{4.121}$$

Passing to the limit as $m \to \infty$ in this inequality, by Proposition 4.18 we get

$$\int_{B_{k+l}} \big|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\big|^2 h_{k+l}(x)\,dx$$

$$\leq \frac{c_{k+l}}{\alpha}\Big[2\|u_2-u_1\|_{L^\infty(B_{k+l})}\big(\|u_2\|_{L^\infty(B_{k+l})}+\|u_1\|_{L^\infty(B_{k+l})}\big)+\|u_2-u_1\|^2_{L^\infty(B_{k+l})}\Big]. \tag{4.122}$$

Restricting the integral on the left-hand side of (4.122) over the ball B_k and then passing to the limit as $l\to\infty$, we obtain

$$\int_{B_k}\big|\operatorname{grad} u_2(x)-\operatorname{grad} u_1(x)\big|^2 h_\infty(x)\,dx$$

$$\leq \frac{c_\infty}{\alpha}\Big[2\|u_2-u_1\|_{L^\infty(B)}\big(\|u_2\|_{L^\infty(B)}+\|u_1\|_{L^\infty(B)}\big)+\|u_2-u_1\|^2_{L^\infty(B)}\Big], \tag{4.123}$$

where we have used the assumption on the boundedness of u_i, $i=1,2$, on the ball B.

By the energy estimates (4.7.3) and (4.7.3) we get that u_i, $i=1,2$, belong to the weighted Sobolev spaces $H^1(B;h_\infty)$ and $H^1(B;\widehat{h}(\beta))$, $\beta\geq 1$.

Passing to the limit in the inequality (4.123), as $k\to\infty$, we obtain

$$\int_{B}\big|\operatorname{grad} u_2(x)-\operatorname{grad} u_1(x)\big|^2 h_\infty(x)\,dx$$

$$\leq \frac{c_\infty}{\alpha}\Big[2\|u_2-u_1\|_{L^\infty(B)}\big(\|u_2\|_{L^\infty(B)}+\|u_1\|_{L^\infty(B)}\big)+\|u_2-u_1\|^2_{L^\infty(B)}\Big], \tag{4.124}$$

from which taking into account the inequality (4.7.3) the desired estimate (4.120) follows. □

It can be easily calculated that $\Delta\overline{h}(x)=-2n$ and therefore the constant c in (4.120) is equal to

$$c=2n\lambda(B). \tag{4.125}$$

Wilson and Zwick [70] studied the problem of best approximation in the norm of $L^\infty(B)$ of a given function f by subharmonic functions. For a continuous function in $\overline{B}$ they characterized best continuous subharmonic approximations. It turned out that the best subharmonic approximation of a continuous function f is just the greatest subharmonic minorant of f adjusted by a constant.

In problems for which it is known a priori that the analytically unknown continuous exact solution u must be subharmonic in the ball B it makes sense to seek for numerical approximations v_h (h is some small parameter) that are subharmonic themselves. One expects that they will better imitate the unknown solution u than the somehow constructed continuous uniform approximation u_h.

Suppose we are given some continuous uniform approximation u_h to the unknown subharmonic function u in the ball $\overline{B}$. The nice idea of Wilson and Zwick [70] consists in replacing u_h by its greatest subharmonic minorant v_h defined by

$$v_h(x)=\sup\Big\{g(x): g(x) \text{ is subharmonic in } B \text{ and } g(x)\leq u_h(x)\Big\}. \tag{4.126}$$

If we denote $\delta=\|u_h-u\|_{L^\infty(B)}$, then we obtain

$$u_h(x)-\delta\leq u(x),\quad u(x)-\delta\leq u_h(x).$$

Hence
$$v_h(x) - \delta \le u_h(x) - \delta \le u(x)$$
and as the subharmonic function $u(x) - \delta$ is the minorant of $u_h(x)$, we have
$$u(x) - \delta \le v_h(x).$$
Hence we get
$$\|v_h - u\|_{L^\infty(B)} \le \|u_h - u\|_{L^\infty(B)}. \tag{4.127}$$
So, both functions v_h and u are subharmonic in B (and we assume they are bounded and continuous), so that we can apply the energy inequality (4.124) and obtain the following important estimate
$$\left\|\operatorname{grad} v_h - \operatorname{grad} u\right\|^2_{L^2(B;\widehat{h}(\beta))} \le 2n\lambda(B)\left[4\|u_h - u\|_{L^\infty(B)}\|u\|_{L^\infty(B)} + 3\|u_h - u\|^2_{L^\infty(B)}\right]. \tag{4.128}$$
Thus, the subharmonic approximation v_h indeed better imitates the unknown exact solution u than the initial uniform approximation u_h.

4.8 The weighted reverse Poincaré inequality for bounded smooth domains

We generalize the estimates from the previous section established for a ball B, $B \subset \mathbb{R}^n$, to the case of arbitrary bounded smooth domains D, $D \subset \mathbb{R}^n$, such that $D \in C^{2+\gamma}, 0 < \gamma \le 1$.

The additional assumption on the operator L, see (4.85), which will be assumed throughout this section is the following
$$L^*1 = c^*(x) \le 0 \quad x \in D. \tag{4.129}$$
Let us consider the Dirichlet problem
$$\begin{cases} L^*h(x) = -1, & x \in D \\ h(x) = 0, & \text{on } x \in \partial D \end{cases} \tag{4.130}$$
Consider a sequence $D_k,\ k = 1, 2, \ldots$ of subdomains of D, such that $D_k \in C^{2+\gamma}$ and
$$\overline{D}_k \subset D_{k+1} \subset \overline{D}_{k+1} \subset D,\ D = \bigcup_{k=1}^{\infty} D_k. \tag{4.131}$$
Aside of (4.130), let us consider the Dirichlet problem for each subdomain $D_k,\ k \in \mathbb{N}$,
$$\begin{cases} L^*h_k(x) = -1 & \text{in } D_k \\ h_k(x) = 0 & \text{on } \partial D_k. \end{cases} \tag{4.132}$$

From the Theorem 6.14 on the global regularity in Gilbarg and Trudinger [23, Chapter 6] we have that the Dirichlet problems (4.130) and (4.132) have the unique solutions h and h_k, respectively, which are smooth up to the boundary, i.e.

$$\begin{cases} h \in C^{2+\gamma}(\overline{D}) \\ h_k \in C^{2+\gamma}(\overline{D}). \end{cases} \tag{4.133}$$

By the Hopf's maximum principle we obtain

$$\begin{cases} h(x) > 0, \;\; x \in D, \\ h_k(x) > 0, \;\; x \in D_k. \end{cases} \tag{4.134}$$

Hence h and h_k can be considered as the smooth weight functions in the corresponding domains.
Further we claim that

$$\lim_{k\to\infty} h_k(x) = h(x), \; x \in D. \tag{4.135}$$

In order to prove the claim let us define

$$\varepsilon_k = \sup_{x\in\partial D_k} h(x), \quad k \in \mathbb{N}. \tag{4.136}$$

Since $D_k \subset D_{k+1}$ and $D = \bigcup\limits_{k=1}^{\infty} D_k$, then for any $\gamma > 0$ there is a number $k(\gamma)$ such for $k \geq 4k(\gamma)$ the boundaries of D_k lie in the $\gamma-$neighbourhood of ∂D.
As h is uniformly continuous in $\overline{D}$ and $h(x) = 0$ for $x \in \partial D$, we get that $\varepsilon_k \to 0$ as $k \to \infty$.
From the definition of ε_k it follows that

$$L^*(h(x) - h_k(x) - \varepsilon_k) = -c^*(x)\varepsilon_k \geq 0, \quad x \in D_k \tag{4.137}$$

and

$$h(x) - h_k(x) - \varepsilon_k = h(x) - \varepsilon_k \leq 0, \quad x \in \partial D_k.$$

By the maximum principle we have

$$h(x) - h_k(x) - \varepsilon_k \leq 0, \; x \in D_k,$$

hence

$$h(x) - h_k(x) \leq \varepsilon_k \; in \; D_k,$$

and similarly

$$h_k(x) - h(x) \leq \varepsilon_k, \; x \in D_k.$$

Altogether,

$$|h_k(x) - h(x)| \leq \varepsilon_k, \; x \in D_k,$$

which gives

$$\lim_{k\to\infty} h_k(x) = h(x), \; x \in D.$$

4.8.1 The energy inequality for the smooth L-subsolutions

Our objective in this section is to establish the reverse Poincaré type inequality for smooth subsolutions in the smooth domains A, $A \subseteq D$.

Consider arbitrary two smooth L-subsolutions u_i, $i = 1,2$, in the domain A, that is, $u_i(x) \in C^2(\overline{A})$ and

$$Lu_i(x) \geq 0, \quad x \in A, \; i = 1,2. \tag{4.138}$$

Proposition 4.2 *Let A be a smooth subdomain of D and let h_A be a smooth weight function in the domain A with $h_A(x) > 0$, $x \in A$, and $h_A(x) = 0$, $x \in \partial A$. Suppose that the uniform ellipticity condition* (4.87) *is satisfied and the coefficients of the differential operator L are smooth, i.e.*

$$a_{ij}(x) \in C^2(\overline{A}), \quad b_i(x) \in C^1(\overline{A}), \quad c(x) \in C(\overline{A}), \; i,j = 1,\ldots,n. \tag{4.139}$$

Then the following energy inequality is valid

$$\int_A \left|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\right|^2 h_A(x)\, dx \leq \frac{1}{\alpha} \int_A \Big(|L^* h_A(x)| + |c(x)| h_A(x) \Big)\, dx$$
$$\times \left[2\|u_2 - u_1\|_{L^\infty(A)} \big(\|u_1\|_{L^\infty(A)} + \|u_2\|_{L^\infty(A)} \big) + \|u_2 - u_1\|^2_{L^\infty(A)} \right] \tag{4.140}$$

for the difference $u_2 - u_1$ of smooth L-subsolutions u_i, $i = 1,2$, in A.

Proof. Let us denote $u(x) = u_2(x) - u_1(x)$ If we take $y = \operatorname{grad} u(x)$ in (4.87), multiply by $h_A(x)$, and then integrate over the domain A, we have

$$\alpha \int_A |\mathrm{grad\, u(x))}|^2 h_A(x) dx \leq \sum_{i,j=1}^{n} \int_A a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} h_A(x) dx. \tag{4.141}$$

We start with the equality

$$\int_A v(x) Lu(x) dx = \int_A [\sum_{i,j=1}^{n} a_{i,j}(x) v(x) \frac{\partial^2 u(x)}{\partial x_i x_j} + \sum_{i=1}^{n} b_i(x) v(x) \frac{\partial u(x)}{\partial x_i} + c(x) v(x) u(x)] dx, \tag{4.142}$$

where v is an arbitrary smooth function in the domain A, i.e. $v \in C^2(\overline{A})$.
If we take $v(x) = u(x) h_A(x)$ in (4.142), we have

$$\int_A u(x) h_A(x) Lu(x) dx$$
$$= \int_A \left(\sum_{i,j=1}^{n} a_{i,j}(x) u(x) h_A(x) \frac{\partial^2 u(x)}{\partial x_i x_j} + \sum_{i=1}^{n} b_i(x) u(x) h_A(x) \frac{\partial u(x)}{\partial x_i} + c(x) u^2(x) h_A(x) \right) dx$$
$$= \sum_{i,j=1}^{n} \int_A a_{i,j}(x) u(x) h_A(x) \frac{\partial^2 u(x)}{\partial x_i x_j} dx + \sum_{i=1}^{n} \int_A b_i(x) u(x) h_A(x) \frac{\partial u(x)}{\partial x_i} dx + \int_A c(x) u^2(x) h_A(x) dx. \tag{4.143}$$

Using integration by parts formula in multidimensional domain A (see [14, Appendix C]) in the first and second integral of the latter expression, and then using the fact that h_A+ vanishes on the boundary, we have

$$\begin{aligned}
\int_A u(x)h_A(x)Lu(x)dx &= -\sum_{i,j=1}^{n}\int_A \frac{\partial}{\partial x_i}(a_{i,j}(x)u(x)h_A(x))\frac{\partial u(x)}{\partial x_j}dx \\
&\quad -\frac{1}{2}\sum_{i=1}^{n}\int_A \frac{\partial(b_i(x).h_A(x))}{\partial x_i}.u^2(x)dx+\int_A c(x)u^2(x)h_A(x)dx. \\
&= -\sum_{i,j=1}^{n}\int_A \frac{\partial(a_{i,j}(x)h_A(x))}{\partial x_i}u(x)\frac{\partial u(x)}{\partial x_j}dx \\
&\quad -\sum_{i,j=1}^{n}\int_A a_{i,j}(x)h_A(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}dx \\
&\quad -\frac{1}{2}\sum_{i=1}^{n}\int_A \frac{\partial(b_i(x).h_A(x))}{\partial x_i}.u^2(x)dx+\int_A c(x)u^2(x)h_A(x)dx.
\end{aligned}$$

The above implies that

$$\begin{aligned}
&\sum_{i,j=1}^{n}\int_A a_{i,j}(x)h_A(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}dx \\
&= -\int_A u(x)h_A(x)Lu(x)dx-\frac{1}{2}\sum_{i,j=1}^{n}\int_A \frac{\partial(a_i(x).h_A(x))}{\partial x_i}\frac{\partial u^2(x)}{\partial x_j}dx \\
&\quad -\frac{1}{2}\sum_{i=1}^{n}\int_A \frac{\partial(b_i(x).h_A(x))}{\partial x_i}.u^2(x)dx+\int_A c(x)u^2(x)h_A(x)dx. \qquad (4.144)
\end{aligned}$$

Using again integration by parts formula on second integral of right hand side of (4.144), we get

$$\begin{aligned}
&\sum_{i,j=1}^{n}\int_A a_{i,j}(x)h_A(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}dx \\
&= -\int_A u(x)h_A(x)Lu(x)dx+\frac{1}{2}\sum_{i,j=1}^{n}\int_A \frac{\partial^2(a_i(x)h_A(x))}{\partial x_i\partial x_j}u^2(x)dx \\
&\quad -\frac{1}{2}\sum_{i,j=1}^{n}\int_{\partial A} \frac{\partial(a_{ij}(x)h_A(x))}{\partial x_i}u^2(x)n_i d\sigma-\frac{1}{2}\sum_{i=1}^{n}\int_A \frac{\partial(b_i(x)h_A(x))}{\partial x_i}u^2(x)dx \\
&\quad +\int_A c(x)u^2(x)h_A(x)dx = -\int_A u(x)h_A(x)Lu(x)dx+\frac{1}{2}\int_A\Big[\sum_{i,j=1}^{n}\frac{\partial^2(a_i(x)h_A(x))}{\partial x_i\partial x_j}
\end{aligned}$$

$$-\sum_{i=1}^{n}\frac{\partial(b_i(x)h_A(x))}{\partial x_i}+c(x)h_A(x)\Big]u^2(x)dx-\frac{1}{2}\sum_{i,j=1}^{n}\int\limits_{\partial A}\frac{\partial(a_{ij}(x)h_A(x))}{\partial x_i}u^2(x)n_i d\sigma$$
$$+\frac{1}{2}\int\limits_A c(x)u^2(x)h_A(x)dx. \tag{4.145}$$

By the definition of adjoint operator, the equality (4.145) becomes

$$\begin{aligned}&\sum_{i,j=1}^{n}\int\limits_A a_{i,j}(x)h_A(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}dx\\ &=-\int\limits_A u(x)h_A(x)Lu(x)dx+\frac{1}{2}\int\limits_A(L^*h_A(x)+c(x)h_A(x))u^2(x)dx\\ &\quad-\frac{1}{2}\sum_{i,j=1}^{n}\int\limits_{\partial A}\frac{\partial(a_{ij}(x).h_A(x))}{\partial x_i}u^2(x)n_i d\sigma.\end{aligned} \tag{4.146}$$

Now we transform the surface integral in (4.146)

$$\begin{aligned}&\sum_{i,j=1}^{n}\int\limits_{\partial A}\frac{\partial(a_{ij}(x)h_A(x))}{\partial x_i}u^2(x)n_i d\sigma\\ &=\sum_{i,j=1}^{n}\int\limits_{\partial A}\frac{\partial a_{ij}(x)}{\partial x_i}h_A(x).u^2(x)n_i d\sigma+\sum_{i,j=1}^{n}\int\limits_{\partial A}a_{ij}(x)\frac{\partial h_A(x)}{\partial x_i}u^2(x)n_i d\sigma.\end{aligned} \tag{4.147}$$

The first integral vanishes in (4.147) due to the definition of weight function h_A, hence we get

$$\left|\sum_{i,j=1}^{n}\int\limits_{\partial A}a_{ij}(x)\frac{\partial h_A(x)}{\partial x_i}u^2(x)n_i d\sigma\right|\leq\sup_{x\in\partial A}u^2(x)\int\limits_{\partial A}|\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial h_A(x)}{\partial x_i}n_i|d\sigma \tag{4.148}$$

Let us consider the vector

$$\gamma_a(x)=\big(\gamma_{ai}(x)\big)_{i=1,\ldots,n},$$

where

$$\gamma_{ai}(x)=\sum_{j=1}^{n}a_{ji}(x)n_j(x),\ \ i=1,\ldots,n. \tag{4.149}$$

By the definition of directional derivative we have

$$\frac{\partial h_A(x)}{\partial\gamma_a}=\sum_{i=1}^{n}\frac{\partial h_A(x)}{\partial x_i}\gamma_{ai}(x)$$

hence from (4.149) we get

$$\frac{\partial h_A(x)}{\partial\gamma_a}=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial h_A(x)}{\partial x_i}n_i. \tag{4.150}$$

Now consider

$$\big(\gamma_a(x), n(x)\big) = \sum_{i,j=1}^{n} a_{ij}(x) n_i(x) n_j(x) \geq \alpha |n(x)|^2 = \alpha > 0. \tag{4.151}$$

Also if $x \in \partial A$, we have

$$\big(\operatorname{grad} h_A(x), \gamma_a(x)\big) = \lim_{t \downarrow 0} \frac{h_A(x) - h_A(x - t\gamma_a(x))}{t} \leq 0. \tag{4.152}$$

Using (4.141) and (4.151) in (4.148) we have

$$\begin{aligned} \alpha \int_A |\operatorname{grad} u(x)|^2 h_A(x)\, dx \leq \sup_A |u(x)| \int_A |Lu(x)| h_A(x)\, dx \\ + \frac{1}{2} \sup_{x\in A} u^2(x) \int_A \Big(|L^* h_A(x)| + |c(x)| h_A(x) \Big)\, dx \\ + \frac{1}{2} \sup_{x \in \partial A} u^2(x) \int_{\partial A} \big| \big(\operatorname{grad} h_A(x), \gamma_a(x)\big) \big|\, d\sigma. \end{aligned} \tag{4.153}$$

Taking $u(x) = 1$ in (4.146),we get

$$\int_{\partial A} \big(\operatorname{grad} h_A(x), \gamma_a(x)\big)\, d\sigma = \int_A \big(L^* h_A(x) - c(x) h_A(x)\big)\, dx. \tag{4.154}$$

Also we know from (4.152) that for any $x \in \partial A$

$$\big(\operatorname{grad} h_A(x), \gamma_a(x)\big) \leq 0. \tag{4.155}$$

Hence from the inequality (4.153) we derive the estimate

$$\begin{aligned} \alpha \int_A |\operatorname{grad} u(x)|^2 h_A(x)\, dx \leq \sup_{x\in A} |u(x)| \int_A |Lu(x)| h_A(x)\, dx \\ + \sup_{x\in A} u^2(x) \int_A \Big(|L^* h_A(x)| + |c(x)| h_A(x) \Big)\, dx. \end{aligned} \tag{4.156}$$

Now we bound the integral $\int_A |Lu(x)| h_A(x)\, dx$ from above.

We have

$$|Lu(x)| = \big|Lu_2(x) - Lu_1(x)\big| \leq L\big(u_1(x) + u_2(x)\big), \tag{4.157}$$

hence

$$\int_A |Lu(x)| h_A(x)\, dx \leq \int_A L\big(u_1(x) + u_2(x)\big) h_A(x)\, dx. \tag{4.158}$$

From Green's formula we can write

$$\int_A L\big(u_1(x)+u_2(x)\big)h_A(x)dx = \int_A \big(u_1(x)+u_2(x)\big)L^* h_A(x)dx$$
$$+\int_{\partial A} \big(u_1(x)+u_2(x)\big)\big(\operatorname{grad} h_A(x), -\gamma_a(x)\big)d\sigma. \qquad (4.159)$$

By (4.154) and (4.155) we know that

$$\big(\operatorname{grad} h_A(x), -\gamma_a(x)\big) \geq 0,$$
$$\int_{\partial A}\big(\operatorname{grad} h_A(x), -\gamma_a(x)\big)\, d\sigma = \int_A \big(-L^* h_A(x)+c(x)h_A(x)\big)\, dx, \qquad (4.160)$$

therefore

$$\int_A |Lu(x)|h_A(x)\,dx \leq 2\sup_{x\in A}\big|u_1(x)+u_2(x)\big| \int_A \Big(|L^* h_A(x)|+|c(x)|h_A(x)\Big)\,dx. \qquad (4.161)$$

From the estimates (4.156) and (4.161) we obtain the desired result (4.140). □

4.8.2 The existence and integrability of first order weak partial derivatives for continuous weak L-subsolutions and the weighted reverse Poincaré inequality

In order to extend the inequality (4.140) to the general case of weak $L-$subsolutions, we need the same technique of W. Littman [46] which was used in the previous section for approximating an arbitrary weak $L-$subsolution by a sequence of smooth $L-$subsolutions.

For an arbitrary continuous weak $L-$subsolution u, there exists a monotonic nonincreasing sequence $u_m,\ m\in\mathbb{N}$, of functions in the domain D such that on each compact subset $K\subset D$ for sufficiently large $m\in\mathbb{N}$ (which depends on K) we have

$$u_m \in C^{2+\gamma}(K),\ Lu_m(x)\geq 0,\ x\in K,\ u_m(x)\searrow u(x),\ x\in K. \qquad (4.162)$$

Here we consider only the continuous weak $L-$subsolutions u in the domain D. By Dini's theorem the above convergence is uniform

$$\sup_{x\in K}|u_m(x)-u(x)| \xrightarrow{m\to\infty} 0.$$

Now we will show that any continuous weak $L-$subsolution u in the domain D has all first order weak derivatives $\frac{\partial u(x)}{\partial x_i},\ i=1,2,\ldots,n$.

Theorem 4.21 *Let the assumption* (4.129) *and the uniform ellipticity condition* (4.87) *are satisfied. Suppose also that the coefficients of operator L are smooth as in* (4.88)*. Then any continuous weak L-subsolution u possesses the first order weak partial derivatives $\frac{\partial u(x)}{\partial x_i}, i=1,\ldots,n$, in the domain D.*

Proof. Consider the sequence u_m that satisfies (4.162) approximating the weak L-subsolution u, and the weight function h_{k+1} which is the solution of the Dirichlet problem (4.130). Let us write the inequality (4.140) for the functions $u_1 = u_m,\ u_2 = u_l$ and the domain D_{k+1}. Then we get

$$\int\limits_{D_{k+1}} \left|\operatorname{grad} u_m(x) - \operatorname{grad} u_l(x)\right|^2 h_{k+1}(x)\,dx$$
$$\leq \frac{c_{k+1}}{\alpha}\Big[2\|u_m - u_l\|_{L^\infty(D_{k+1})}\big(\|u_m\|_{L^\infty(D_{k+1})} + \|u_l\|_{L^\infty(D_{k+1})}\big) + \|u_m - u_l\|^2_{L^\infty(D_{k+1})}\Big], \tag{4.163}$$

where

$$c_{k+1} = \int\limits_{D_{k+1}} \Big(1 + |c(x)|h_{k+1}(x)\Big)\,dx. \tag{4.164}$$

Denote

$$d_{k+1} = \inf_{x\in\overline{D}_k} h_{k+1}(x). \tag{4.165}$$

It is clear that $d_{k+1} > 0$.
If we restrict the integral on the left hand side of (4.163) over the domain D_k, then we have

$$d_{k+1}\int\limits_{D_k} \left|\operatorname{grad} u_m(x) - \operatorname{grad} u_l(x)\right|^2 dx \tag{4.166}$$
$$\leq \frac{c_{k+1}}{\alpha}\Big[2\|u_m - u_l\|_{L^\infty(D_{k+1})}\big(\|u_m\|_{L^\infty(D_{k+1})} + \|u_l\|_{L^\infty(D_{k+1})}\big) + \|u_m - u_l\|^2_{L^\infty(D_{k+1})}\Big].$$

Since the sequence of the functions u_m converges to u in the norm $L^\infty(D_{k+1})$, we can write

$$\|u_m - u_l\|_{L^\infty(D_{k+1})} \longrightarrow 0 \ \text{ if } \ m,l \to \infty.$$

Passing to the limit in the inequality (4.166) as $m,l \to \infty$, we obtain

$$\lim_{m,l\to\infty} \sum_{i=1}^{n} \int\limits_{D_k} \left(\frac{\partial u_m(x)}{\partial x_i} - \frac{\partial u_l(x)}{\partial x_i}\right)^2 dx = 0. \tag{4.167}$$

By the completeness of the space $L^2(D_k)$, there exists a family of measurable functions $g_{k,i}(x)$, $i = 1,\ldots,n$, $k = 1,2,\ldots$, such that $g_{k,i}(x) \in L^2(D_k)$, $i = 1,\ldots,n$, and

$$\lim_{m\to\infty} \sum_{i=1}^{n} \int\limits_{D_k} \left(\frac{\partial u_m(x)}{\partial x_i} - g_{k,i}(x)\right)^2 dx = 0,\ \ k = 1,2,\ldots. \tag{4.168}$$

Let us extend the functions $g_{k,i}$ trivially outside of the domain D_k as follows

$$g_{k,i}(x) = \begin{cases} g_{k,i}(x) & \text{for } \ x \in D_k, \\ 0 & \text{for } \ x \in D \setminus D_k \end{cases}$$

and define the functions g_i, $i = 1,\dots,n$, on the domain D by

$$g_i(x) = \limsup_{k\to\infty} g_{k,i}(x), \quad i = 1,\dots,n. \tag{4.169}$$

It is obvious that the functions $g_{k+l,i}$, $l = 0,1,2,\dots$, agree on the domain D_k and therefore

$$g_i(x) = g_{k,i}(x) \ \text{(a.e.) on a domain } D_k. \tag{4.170}$$

Thus the functions g_i, $i = 1,\dots,n$, are locally square integrable in domain D.
Let us check that g_i, $i = 1,\dots,n$, represent the weak partial derivatives of the function u. Take any continuously differentiable function φ with compact support in D (i.e. $\varphi \in C_0^1(D)$). Then $\operatorname{supp}\varphi \subset D_k$ for some k, and we have

$$\int_{D_k} \frac{\partial u_m(x)}{\partial x_i}\varphi(x)\,dx = -\int_{D_k} u_m(x)\,\frac{\partial \varphi(x)}{\partial x_i}\,dx.$$

But u_m converges uniformly to u on D_k, and $\frac{\partial u_m}{\partial x_i}$ converges to g_i in $L^2(D_k)$. Hence, passing to the limit, as $m \to \infty$, we obtain the equality

$$\int_{D_k} g_i(x)\varphi(x)\,dx = -\int_{D_k} u(x)\,\frac{\partial \varphi(x)}{\partial x_i}\,dx, \tag{4.171}$$

which means that g_i, $i = 1,\dots,n$, are indeed the first order weak partial derivatives of the function u. □

Let us introduce the weighted Sobolev space $H^1(D;h)$ with the help of the weight function h, which is the unique solution of Dirichlet problem (4.130). The space $H^1(D;h)$ consists of the functions u for which the following integral is finite

$$\int_D u^2(x)\,dx + \sum_{i=1}^{n}\int_D \left(\frac{\partial u(x)}{\partial x_i}\right)^2 h(x)\,dx \equiv \|u\|^2_{H^1(D;h)}. \tag{4.172}$$

Theorem 4.22 *Suppose that the assumption* (4.129) *and the uniform ellipticity condition* (4.87) *are satisfied. Assume also that the coefficients of the operator L are smooth as in* (4.88). *Let h be the unique smooth solution of the Dirichlet problem* (4.130). *Then any continuous bounded weak L-subsolution u in the domain D belongs to the weighted Sobolev space* $H^1(D;h)$.

Proof. Consider the sequence of the functons u_m with (4.162) properties, which approximates the function u. If we write the inequality (4.140) for the functions

$$u_1(x) = 0, \quad u_2(x) = u_m(x),$$

and the domain D_{k+l}, we get

$$\int_{D_{k+l}} |\operatorname{grad} u_m(x)|^2 h_{k+l}(x)\,dx \le \frac{c_{k+l}}{\alpha} 3\|u_m\|^2_{L^\infty(D_{k+l})}. \tag{4.173}$$

Next passing to the limit as $m \to \infty$, we get

$$\int_{D_{k+l}} |\operatorname{grad} u(x)|^2 h_{k+l}(x)\, dx \leq \frac{c_{k+l}}{\alpha} 3\|u\|^2_{L^\infty(D_{k+l})}. \tag{4.174}$$

Restricting the integral on the left hand side of (4.174) over the domain D_k and letting the integer l go to infinity, we obtain

$$\int_{D_k} |\operatorname{grad} u(x)|^2 h(x)\, dx \leq \frac{c}{\alpha} 3\|u\|^2_{L^\infty(D)} < \infty, \tag{4.175}$$

where we have taken into account the limit relation (4.135).
Since the left hand side of (4.175) is increasing with respect to k and bounded, it has finite limit so that

$$\int_{D} |\operatorname{grad} u(x)|^2 h(x)\, dx \leq \frac{3c}{\alpha} \|u\|^2_{L^\infty(D)}. \tag{4.176}$$

where

$$c = \int_D \Big(|1 + |c(x)|h(x)\Big)\, dx. \tag{4.177}$$

□

Theorem 4.23 *Let the assumption* (4.129) *and the uniform ellipticity condition* (4.87) *are satisfied. Suppose also that the coefficients of the operator L are smooth in the domain D as in* (4.88). *Let h be the unique smooth solution of the Dirichlet problem* (4.130). *Consider two weak L−subsolutions* $u_i, i = 1,2$, *in the domain D, such that*

$$u_i \in C(D) \cap L^\infty(D),\ i = 1,2. \tag{4.178}$$

Then the following reverse Poincaré inequality holds for the difference $u_2 - u_1$ *of two weak L−subsolutions* $u_i,\ i = 1,2,$

$$\begin{aligned} \|u_2 - u_1\|^2_{H^1(D;h)} \leq & \left(\frac{c}{\alpha} + measD\right) \\ & \times \Big[2\|u_2 - u_1\|_{L^\infty(D)} \big(\|u_1\|_{L^\infty(D)} + \|u_2\|_{L^\infty(D)}\big) + \|u_2 - u_1\|^2_{L^\infty(D)}\Big], \end{aligned} \tag{4.179}$$

where

$$c = \int_D \Big(1 + |c(x)|h(x)\Big)\, dx \tag{4.180}$$

and $\alpha > 0$ *is the constant of the uniform ellipticity.*

Proof. Consider the sequences of smooth L-subsolutions $u_{m,i}$, $i = 1,2$, $m \in \mathbb{N}$, converging on the domains D_{k+l} uniformly to weak L-subsolutions u_i, $i = 1,2$. By the assumptions of the theorem, the functions u_i, $i = 1,2$, are continuous and bounded in the domain D. Let us apply the inequality (4.140) for the functions $u_{m,1}$ and $u_{m,2}$ over the domain D_{k+l}, $k,l \in \mathbb{N}$. We get

$$\int\limits_{D_{k+l}} \left|\operatorname{grad} u_{m,2}(x) - \operatorname{grad} u_{m,1}(x)\right|^2 h_{k+l}(x)\,dx \tag{4.181}$$
$$\leq \frac{c_{k+l}}{\alpha}\left[2\left\|u_{m,2}-u_{m,1}\right\|_{L^\infty(D_{k+l})}\left(\|u_{m,2}\|_{L^\infty(D_{k+l})}+\|u_{m,1}\|_{L^\infty(D_{k+l})}\right)+\left\|u_{m,2}-u_{m,1}\right\|^2_{L^\infty(D_{k+l})}\right].$$

Passing to the limit as $m \to \infty$ in (4.181), we obtain

$$\int\limits_{D_{k+l}} \left|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\right|^2 h_{k+l}(x)\,dx$$
$$\leq \frac{c_{k+l}}{\alpha}\left[2\|u_2-u_1\|_{L^\infty(D_{k+l})}\left(\|u_2\|_{L^\infty(D_{k+l})}+\|u_1\|_{L^\infty(D_{k+l})}\right)+\|u_2-u_1\|^2_{L^\infty(D_{k+l})}\right], \tag{4.182}$$

where we also used the assumption that $u_i, i=1,2$ are bounded on the domain D.
Now let us restrict the integral on the left hand side of (4.182) over the domain D_k and then let l tends to infinity,

$$\int\limits_{D_k} \left|\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)\right|^2 h(x)\,dx$$
$$\leq \frac{c}{\alpha}\left[2\|u_2-u_1\|_{L^\infty(D)}\left(\|u_2\|_{L^\infty(D)}+\|u_1\|_{L^\infty(D)}\right)+\|u_2-u_1\|^2_{L^\infty(D)}\right]. \tag{4.183}$$

From Theorem4.3 we know that the following energy integrals are finite

$$\int\limits_D |\operatorname{grad} u_i(x)|^2 h(x)\,dx < \infty,\ \ i=1,2. \tag{4.184}$$

Hence passing to limit as $k \to \infty$ in the inequality (4.183) we come to the desired result. □

4.9 The weighted reverse Poincaré type inequality for parabolic subsolutions

Our goal in this section is to establish the reverse Poincaré inequality for the second order uniformly parabolic partial differential operator in the cylindrical domain.

4.9.1 Mollification of the weak parabolic subsolutions

Let us consider a linear second order parabolic partial differential operator with constant coefficients in a cylinder Q, $Q = B(x_0,R)\times(0,T)$,

$$Lu(x,t) = \sum_{i,j=1}^n a_{ij}\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u(x,t)}{\partial x_i} + cu(x,t) - \frac{\partial u(x,t)}{\partial t}, \tag{4.185}$$

where $a_{ij} = a_{ji}$, $i, j = 1, \dots, n$, and its adjoint operator

$$L^*u(x,t) = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} b_i \frac{\partial u(x,t)}{\partial x_i} + cu(x,t) + \frac{\partial u(x,t)}{\partial t}, \tag{4.186}$$

It is assumed that the operator Lu is uniformly parabolic, that is

$$\sum_{i,j=1}^{n} a_{ij} y_i y_j \geq \alpha |y|^2, \;\; y \in R^n \tag{4.187}$$

with the constant of parabolicity $\alpha > 0$.

A bounded measurable function $u(x,t)$ defined in the cylinder $Q = B(x_0, R) \times (0, T)$ is said to be a weak parabolic subsolution of the equation

$$Lv(x,t) = 0 \;\text{ in the cylinder } Q \tag{4.188}$$

if for all nonnegative $v(x,t)$, belonging to the space $C_0^{2,1}(Q)$ the following inequality holds

$$\int_0^T \int_B u(x,t) L^* v(x,t) \, dx dt \geq 0. \tag{4.189}$$

It is a remarkable fact that this definition of the weak parabolic subsolution, which requires no a priori regularity, leads to the existence and the integrability of the weak (Sobolev) gradient of a continuous weak subsolution $u(x,t)$. This enables us to establish a new type of energy inequality for the difference of two arbitrary continuous weak parabolic subsolutions which is the main objective of this paper. We shall need to approximate weak subsolutions of the equation (4.188) by smooth ones and for this the classical approximation techniques (see, for example, Gilbarg, Trudinger [23, Chapter 7]) will be used.

Define

$$\rho_n(z) = \begin{cases} c \exp\left(\dfrac{1}{|z|^2 - 1}\right), & \text{if } \;|z| < 1, \\ 0, & \text{if } \;|z| \geq 1, \end{cases} \tag{4.190}$$

where $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $c > 0$ is a constant with $\int_{\mathbb{R}^n} \rho_n(z) \, dz = 1$.

Let us consider the mollification of the bounded measurable function $u(x,t)$ defined in the cylinder Q

$$u_h(x,t) = h^{-(n+1)} \int_0^T \int_B \rho_n\left(\frac{x-y}{h}\right) \rho_1\left(\frac{t-s}{h}\right) u(y,s) \, dy ds \tag{4.191}$$

for arbitrary $h > 0$. If we denote

$$\rho_h(x-y, t-s) = h^{-(n+1)} \rho_n\left(\frac{x-y}{h}\right) \rho_1\left(\frac{t-s}{h}\right),$$

then it is easy to see that the following equalities are valid

$$\frac{\partial}{\partial x_i}\rho_h(x-y,t-s) = -\frac{\partial}{\partial y_i}\rho_h(x-y,t-s),$$
$$\frac{\partial^2}{\partial x_i \partial x_j}\rho_h(x-y,t-s) = \frac{\partial^2}{\partial y_i \partial y_j}\rho_h(x-y,t-s),$$
$$\frac{\partial}{\partial t}\rho_h(x-y,t-s) = -\frac{\partial}{\partial s}\rho_h(x-y,t-s).$$

From the latter equalities we easily get

$$L_{x,t}\rho_h(x-y,t-s) = L^*_{y,s}\rho_h(x-y,t-s), \tag{4.192}$$

where $L_{x,t}$ and $L^*_{y,s}$ means taking differential operators with respect to arguments (x,t) and (y,s), respectively. Taking into account the relation (4.192) and the definition (4.191) we come to the interesting equality

$$L_{x,t}u_h(x,t) = \int_0^T \int_B u(y,s) L^*_{y,s}\rho_h(x-y,t-s)\, dy\, ds, \tag{4.193}$$

for arbitrary $h > 0$. Let us define the cylinders

$$Q_k = Q\Big(r_k, \frac{T}{k+2}\Big) = B(x_0, r_k) \times \left(\frac{T}{k+2}, \frac{k+1}{k+2}T\right), \tag{4.194}$$

where

$$r_k = \frac{k+1}{k+2}R, \;\; k \in \mathbb{N}.$$

The following theorem states that the functions $u_h(x,t)$ are smooth parabolic subsolutions in the cylinder Q_k for sufficiently small h.

Theorem 4.24 *Consider the weak parabolic subsolution $u(x,t)$ in the cylinder $Q = B(x_0,R) \times (0,T)$. Then for any $k \in \mathbb{N}$ there exists $\widehat{h} > 0$, such that if $0 < h < \widehat{h}$, each function $u_h(x,t)$ is the smooth parabolic subsolution in the cylinder Q_k, that is*

$$Lu_k(x,t) \geq 0, \;\; \textit{if} \;\; (x,t) \in Q_k. \tag{4.195}$$

Proof. Denote for fixed $k \in \mathbb{N}$

$$\widehat{h} = \min\left(\frac{R}{2(k+2)}, \frac{T}{2(k+2)}\right). \tag{4.196}$$

It is well-known that for arbitrary $h > 0$ the function $u_h(x,t)$ is infinitely differentiable with respect to its arguments in $\mathbb{R}^{n+1}$. Let us check that for arbitrary $(x,t) \in Q_k$ the function $\rho_h(x-y,t-s)$ has a compact support in the cylinder Q as a function of (y,s).
Consider the cylinder $\widehat{Q}_k$ defined as follows

$$\widehat{Q}_k = B\Big(x_0, \frac{2k+3}{2k+4}R\Big) \times \Big(\frac{T}{2k+4}, \frac{2k+3}{2k+4}T\Big). \tag{4.197}$$

If $(y,s) \notin \widehat{Q}_k$, then either $y \notin B\big(x_0, \frac{2k+3}{2k+4}R\big)$ or $s \notin \big(\frac{T}{2k+4}, \frac{2k+3}{2k+4}T\big)$. In the first case

$$|y-x| > \Big(\frac{2k+3}{2k+4} - \frac{2k+2}{2k+4}\Big)R = \frac{1}{2(k+2)}R > h,$$

while in the second case

$$|t-s| > \Big(\frac{2}{2k+4} - \frac{1}{2k+4}\Big)T > h.$$

Hence in both cases we have $\rho_h(x-y,t-s) = 0$. Therefore, the nonnegative smooth function $\rho_h(x-y,t-s)$ has a compact support in Q as a function of (y,s), if $h < \widehat{h}$ and by the definition of the weak parabolic subsolution $u(x,t)$ we have

$$\int_0^T \int_B u(y,s) L^*_{y,s} \rho_h(x-y,t-s)\, dy\, ds \geq 0. \tag{4.198}$$

From (4.193) we get $Lu_h(x,t) \geq 0$ if $(x,t) \in Q_k$ and $h < \widehat{h}$. □

4.9.2 The case of smooth parabolic subsolutions

We start with the classical Green's identity (see Friedman [18, Chapter 6, Section 4])

$$\begin{aligned} h(x,t)\,Lu(x,t) - u(x,t)\,L^*h(x,t) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\Bigg[\sum_{j=1}^n \Big(h(x,t)a_{ij}\frac{\partial u(x,t)}{\partial x_j} - u(x,t)a_{ij}\frac{\partial h(x,t)}{\partial x_j}\Big) \\ + b_i u(x,t)h(x,t)\Bigg] - \frac{\partial}{\partial t}\big(u(x,t)h(x,t)\big), \end{aligned} \tag{4.199}$$

where $h(x,t)$, $u(x,t)$ belong to $C^{2,1}(Q(r,s))$ for some cylinder $Q(r,s)$. In this chapter we shall consider a particular smooth weight function $h(x,t)$ of the following type

$$h(x,t) = \big(r^2 - |x-x_0|^2\big)(t-s)(T-s-t), \;\; x \in \overline{B}(x_0,r), \;\; s \leq t \leq T-s. \tag{4.200}$$

We have

$$\begin{cases} h(x,t) > 0, & \text{if } \;(x,t) \in Q(r,s), \\ h(x,t) = 0, & \text{if } \; x \in \partial B \;\text{ or }\; t \in \{s; T-s\}. \end{cases} \tag{4.201}$$

This section is devoted to the proof of the following

Proposition 4.3 *Consider two arbitrary smooth parabolic subsolutions $u_i(x,t)$, $i = 1,2$ in the cylinder $Q(r,s)$, i.e. $u_i(x,t) \in C^{2,1}(\overline{Q(r,s)})$, $i = 1,2$, and*

$$Lu_i(x,t) \geq 0 \;\text{ if }\; (x,t) \in Q(r,s), \;\; i = 1,2. \tag{4.202}$$

Let us assume that the uniform parabolicity condition (4.187) *is satisfied. Then the following energy inequality is valid*

$$\int_{Q(r,s)} \Big|\operatorname{grad} u_2(x,t) - \operatorname{grad} u_1(x,t)\Big|^2 h(x,t)\, dx\, dt \leq \frac{1}{\alpha} \int_{Q(r,s)} \Big(\big|L^*h(x,t)\big| + |c|h(x,t)\Big)\, dx\, dt$$

$$\times \left[2\|u_2-u_1\|_{L^\infty(Q(r,s))}\left(\|u_1\|_{L^\infty(Q(r,s))}+\|u_2\|_{L^\infty(Q(r,s))}\right)+\|u_2-u_1\|^2_{L^\infty(Q(r,s))}\right]. \tag{4.203}$$

Proof. Consider arbitrary smooth function $u(x,t) \in C^{2,1}(\overline{Q}(r,s))$ and let us integrate the identity (4.199) with respect to x over the ball $B(x_0,r)$ for fixed t, $s<t<T-s$. Then, by the Gauss–Ostrogradski divergence theorem we get

$$\begin{aligned}\int_B Lu(x,t)h(x,t)\,dx &= \int_B u(x,t)\,L^*h(x,t)\,dx \\ &+ \int_{\partial B}\sum_{i=1}^n\left[\sum_{j=1}^n\left(h(x,t)a_{ij}\frac{\partial u(x,t)}{\partial x_j}-u(x,t)a_{ij}\frac{\partial h(x,t)}{\partial x_j}\right)n_i(x)\right. \\ &\left.+b_iu(x,t)h(x,t)n_i(x)\right]d\sigma-\int_B\frac{\partial}{\partial t}\big(u(x,t)h(x,t)\big)\,dx, \end{aligned} \tag{4.204}$$

where $n(x)=(n_i(x))_{i=1,\dots,n}$ is the outward pointing unit normal vector at $x\in\partial B$, and $d\sigma$ is an $(n-1)$-dimensional surface measure of the ball $B(x_0,r)$. Denote $\nu_a(x)=(\nu_{ai}(x))_{i=1,\dots,n}$, where

$$\nu_{ai}(x)=\sum_{j=1}^n a_{ji}n_j(x),\ \ i=1,\dots,n.$$

We have by the uniform parabolicity condition (4.187)

$$\big(\nu_a(x),n(x)\big)=\sum_{i,j=1}^n a_{ij}n_i(x)n_j(x)\ge\alpha|n(x)|^2=\alpha>0.$$

Therefore for arbitrary $x\in\partial B$ we have

$$\big(\operatorname{grad}h(x,t),\nu_a(x)\big)=\lim_{s\downarrow 0}\frac{h(x,t)-h(x-s\nu_a(x),t)}{s}\le 0. \tag{4.205}$$

Let us write (4.204) in a convenient form taking into account that on the boundary ∂B the weight function $h(x,t)$ is vanishing, then we obtain

$$\begin{aligned}\int_B Lu(x,t)h(x,t)\,dx &= \int_B u(x,t)\,L^*h(x,t)\,dx-\int_{\partial B}u(x,t)\big(\operatorname{grad}h(x,t),\nu_a(x)\big)\,d\sigma \\ &-\int_B\frac{\partial}{\partial t}\big(u(x,t)h(x,t)\big)\,dx, \end{aligned} \tag{4.206}$$

where $s<t<T-s$. Integrating (4.206), with respect to t, over the time interval $(s,T-s)$ we get

$$\int_{Q(r,s)} Lu(x,t)h(x,t)\,dxdt=\int_{Q(r,s)}u(x,t)\,L^*h(x,t)\,dxdt$$

$$-\int_s^{T-s}\int_{\partial B} u(x,t)\big(\operatorname{grad} h(x,t), \nu_a(x)\big)\, d\sigma\, dt - \int_B \big[u(x,t)h(x,t)\big]\Big|_s^{T-s} dx$$

and since $h(x,s) = h(x,T-s) = 0$, we come to the Green's second formula

$$\int_{Q(r,s)} Lu(x,t)h(x,t)\, dx dt = \int_{Q(r,s)} u(x,t)\, L^* h(x,t)\, dx dt - \int_s^{T-s}\int_{\partial B} u(x,t)\big(\operatorname{grad} h(x,t), \nu_a(x)\big)\, d\sigma\, dt.$$

Take $u(x,t) = 1$ in the latter formula, then we get the equality

$$\int_s^{T-s}\int_{\partial B} \big(\operatorname{grad} h(x,t), \nu_a(x)\big)\, d\sigma\, dt = \int_{Q(r,s)} \big(L^* h(x,t) - ch(x,t)]\big)\, dx dt.$$

Now taking $u^2(x,t)$ instead of $u(x,t)$ in the Green's second formula, we have

$$\begin{aligned}\int_{Q(r,s)} Lu^2(x,t)h(x,t)\, dx dt &= \int_{Q(r,s)} u^2(x,t)\, L^* h(x,t)\, dx dt \\ &\quad - \int_s^{T-s}\int_{\partial B} u^2(x,t)\big(\operatorname{grad} h(x,t), \nu_a(x)\big)\, d\sigma\, dt. \qquad (4.207)\end{aligned}$$

It is easy to calculate

$$Lu^2(x,t) = 2\sum_{i,j=1}^{n} a_{ij}\frac{\partial u(x,t)}{\partial x_i}\frac{\partial u(x,t)}{\partial x_j} + 2u(x,t)\, Lu(x,t) - cu^2(x,t). \qquad (4.208)$$

Hence, from (4.207) we obtain the following inequality

$$\begin{aligned}&2\alpha \int_{Q(r,s)} |\operatorname{grad} u(x,t)|^2 h(x,t)\, dx dt \\ &\le 2\|u\|_{L^\infty(Q(r,s))}\int_{Q(r,s)} |Lu(x,t)|h(x,t)\, dx dt + \|u^2\|_{L^\infty(Q(r,s))}\int_{Q(r,s)} \Big(\big|L^* h(x,t)\big| + |c|h(x,t)\Big)\, dx dt \\ &+ \|u^2\|_{L^\infty(Q(r,s))}\int_s^{T-s}\int_{\partial B} \Big|\big(\operatorname{grad} h(x,t), \nu_a(x)\big)\Big|\, d\sigma\, dt. \qquad (4.209)\end{aligned}$$

From (4.205) we know that for $x \in \partial B$

$$\big(\operatorname{grad} h(x,t), -\nu_a(x)\big) \ge 0,$$

therefore, from (4.9.2) we obtain

$$\int_s^{T-s}\int_{\partial B} \Big|\big(\operatorname{grad} h(x,t), \nu_a(x)\big)\Big|\, d\sigma\, dt = \int_{Q(r,s)} \big(-L^* h(x,t) + ch(x,t)\big)\, dx dt. \qquad (4.210)$$

From the relations (4.209) and (4.210) we derive the estimate

$$\alpha \int\limits_{Q(r,s)} \left|\operatorname{grad} u(x,t)\right|^2 h(x,t)\, dx dt \le \|u\|_{L^\infty(Q(r,s))} \int\limits_{Q(r,s)} |Lu(x,t)| h(x,t)\, dx dt$$
$$+ \|u^2\|_{L^\infty(Q(r,s))} \int\limits_{Q(r,s)} \Big(\left|L^* h(x,t)\right| + |c| h(x,t)\Big)\, dx dt. \tag{4.211}$$

Up to now $u(x,t)$ was arbitrary function from the space $C^{2,1}(\overline{Q(r,s)})$, from now on we shall take

$$u(x,t) = u_2(x,t) - u_1(x,t), \tag{4.212}$$

where $u_i(x,t)$, $i = 1,2$ are smooth parabolic subsolutions.
If so, from

$$|Lu(x,t)| = \left|Lu_2(x,t) - Lu_1(x,t)\right| \le L\big(u_1(x,t) + u_2(x,t)\big), \tag{4.213}$$

we conclude

$$\int\limits_{Q(r,s)} |Lu(x,t)| h(x,t)\, dx dt \le \int\limits_{Q(r,s)} L\big(u_1(x,t) + u_2(x,t)\big) h(x,t)\, dx dt. \tag{4.214}$$

We can write from the Green's second formula

$$\int\limits_{Q(r,s)} L\big(u_1(x,t) + u_2(x,t)\big) h(x,t)\, dx dt \le \|u_1 + u_2\|_{L^\infty(Q(r,s))} \int\limits_{Q(r,s)} \left|L^* h(x,t)\right| dx dt$$
$$+ \|u_1 + u_2\|_{L^\infty(Q(r,s))} \int\limits_{Q(r,s)} \Big(\left|L^* h(x,t)\right| + |c| h(x,t)\Big)\, dx dt$$
$$\le 2\|u_1 + u_2\|_{L^\infty(Q(r,s))} \int\limits_{Q(r,s)} \Big(\left|L^* h(x,t)\right| + |c| h(x,t)\Big)\, dx dt. \tag{4.215}$$

From the estimates (4.211) and (4.215) we obtain the desired inequality (4.203). □

4.9.3 The existence and the integrability of the Sobolev gradient

Again, we consider cylinders $Q_k,\ k \in \mathbb{N}$

$$Q_k = Q\Big(r_k, \frac{T}{k+2}\Big) = B(x_0, r_k) \times \Big(\frac{T}{k+2}, \frac{k+1}{k+2} T\Big), \tag{4.216}$$

where $r_k = \frac{k+1}{k+2} R$. Let us introduce corresponding smooth weight functions

$$h_k(x,t) = \big(r_k^2 - |x - x_0|^2\big)\Big(t - \frac{T}{k+2}\Big)\Big(\frac{k+1}{k+2} T - t\Big), \tag{4.217}$$

$(x,t) \in \overline{Q}_k,\ k \in \mathbb{N}$. Also, we introduce the basic smooth weight function $h(x,t)$ for a cylinder $Q = B(x_0,R) \times (0,T)$

$$h(x,t) = \left(R^2 - |x-x_0|^2\right) t\,(T-t),\ \ (x,t) \in \overline{Q}. \tag{4.218}$$

Now we will show that any continuous weak parabolic subsolution $u(x,t)$ in the cylinder Q possesses all first order weak (Sobolev) derivatives

$$\frac{\partial u(x,t)}{\partial x_i},\ \ i = 1,\ldots,n.$$

Moreover, the gradient of the function $u(x,t)$ turns out to be square integrable with respect to the weight function $h(x,t)$.

Theorem 4.25 *Suppose that the condition* (4.187) *is . Then any continuous weak parabolic subsolution $u(x,t)$ has weak partial derivatives $\frac{\partial u(x,t)}{\partial x_i}$, $i = 1,\ldots,n$, in the cylinder $Q = B(x_0,R) \times (0,T)$, and they are square integrable with respect to the weight function $h(x,t)$, i.e.*

$$\int_Q \left|\operatorname{grad} u(x,t)\right|^2 h(x,t)\,dx\,dt < \infty. \tag{4.219}$$

Proof. Consider the mollification $u_h(x,t)$ defined by 4.191 of the weak parabolic subsolution $u(x,t)$. If the function $u(x,t)$ is continuous in the cylinder Q, then it is well-known fact (see Evans [14, Appendix C]) that on any compact subset K, $K \subset Q$ we have the uniform convergence

$$\sup_{(x,t)\in K} \left|u_h(x,t) - u(x,t)\right| \xrightarrow{h\to 0} 0.$$

Denote by $u_m(x,t)$ the mollification $u_h(x,t)$ for $h = \frac{1}{m}$, $m \in \mathbb{N}$. Then the latter uniform convergence takes the following form

$$\sup_{(x,t)\in K} \left|u_m(x,t) - u(x,t)\right| \xrightarrow{m\to\infty} 0. \tag{4.220}$$

Since the cylinders Q_k are compactly imbedded in the original cylinder Q, we get from Theorem 4.24 that for any $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$, such that each function $u_m(x,t)$ is the smooth parabolic subsolution in the cylinder Q_k for any $m \geq m(k)$. Consider the cylinder Q_{k+l} for some k and l. If we write the inequality (4.203) for

$$u_1(x,t) = u_m(x,t),\ \ u_2(x,t) = u_p(x,t),\ \ m,p \geq m(k+l)$$

and for the cylinder Q_{k+l}, then we get

$$\int_{Q_{k+l}} \left|\operatorname{grad} u_p(x,t) - \operatorname{grad} u_m(x,t)\right|^2 h_{k+l}(x,t)\,dx\,dt \leq \frac{1}{\alpha} \int_{Q_{k+l}} \left(\left|L^* h_{k+l}(x,t)\right| + |c| h_{k+l}(x,t)\right) dx\,dt$$
$$\times \left[2\|u_p - u_m\|_{L^\infty(Q_{k+l})}\left(\|u_m\|_{L^\infty(Q_{k+l})} + \|u_p\|_{L^\infty(Q_{k+l})}\right) + \|u_p - u_m\|^2_{L^\infty(Q_{k+l})}\right]. \tag{4.221}$$

Denote

$$c_{k+l} = \int\limits_{Q_{k+l}} \Big(\big| L^* h_{k+l}(x,t) \big| + |c| h_{k+l}(x,t) \Big)\, dx\,dt$$

and

$$\widehat{c}_{k+l} = \inf_{(x,t)\in Q_k} h_{k+l}(x,t) > 0, \;\; k,l = 1,2,\ldots.$$

Hence, if we restrict the integral on the left-hand side of (4.221) over the cylinder Q_k, then we get

$$\begin{aligned} &\widehat{c}_{k+l} \int\limits_{Q_k} \Big| \operatorname{grad} u_p(x,t) - \operatorname{grad} u_m(x,t) \Big|^2 \, dx\,dt \\ &\quad \le \frac{1}{\alpha} c_{k+l} \Big[2\|u_p - u_m\|_{L^\infty(Q_{k+l})} \Big(\|u_m\|_{L^\infty(Q_{k+l})} + \|u_p\|_{L^\infty(Q_{k+l})} \Big) + \|u_p - u_m\|^2_{L^\infty(Q_{k+l})} \Big]. \end{aligned} \tag{4.222}$$

We have from (4.220) that

$$\|u_p - u_m\|_{L^\infty(Q_{k+l})} \longrightarrow 0 \;\text{ if }\; m,p \to \infty.$$

Passing to the limit in the inequality (4.222), as $m,p \to \infty$, we obtain

$$\lim_{m,p\to\infty} \sum_{i=1}^n \int\limits_{Q_k} \left(\frac{\partial u_p(x,t)}{\partial x_i} - \frac{\partial u_m(x,t)}{\partial x_i} \right)^2 dx\,dt = 0. \tag{4.223}$$

By the completeness of the space $L^2(Q_k)$, there exists a family of measurable functions $v_{k,i}(x,t) \in L^2(Q_k)$, $i = 1,\ldots,n$, such that

$$\lim_{m\to\infty} \sum_{i=1}^n \int\limits_{Q_k} \left(\frac{\partial u_m(x,t)}{\partial x_i} - v_{k,i}(x,t) \right)^2 dx\,dt = 0, \;\; k \in \mathbb{N}. \tag{4.224}$$

Let us extend the functions $v_{k,i}(x,t)$ outside Q_k trivially by 0 and then define the functions $v_i(x,t)$, $i = 1,\ldots,n$, on the original cylinder Q by

$$v_i(x,t) = \limsup_{k\to\infty} v_{k,i}(x,t), \;\; i = 1,\ldots,n. \tag{4.225}$$

It is obvious that the functions $v_{k+l,i}(x,t)$, $l \in \mathbb{N}$ agree on the cylinder Q_k and therefore

$$v_i(x,t) = v_{k,i}(x,t) \;\text{ (a.e. } dx \times dt) \;\text{ on a cylinder } Q_k. \tag{4.226}$$

Thus the functions $v_i(x,t)$, $i = 1,\ldots,n$, are locally square integrable on the cylinder Q. Let us check that $v_i(x,t)$, $i = 1,\ldots,n$, represent the weak (Sobolev) partial derivatives of the function $u(x,t)$. Take arbitrary infinitely differentiable function $\varphi(x,t)$ with compact support in Q (i.e. $\varphi(x,t) \in C_0^\infty(Q)$). Then $\sup \varphi(x,t) \subset Q_k$ for some k. We have

$$\int\limits_{Q_k} \frac{\partial u_m(x,t)}{\partial x_i} \varphi(x,t)\, dx\,dt = -\int\limits_{Q_k} u_m(x,t) \frac{\partial \varphi_m(x,t)}{\partial x_i}\, dx\,dt$$

for any $m \geq m(k)$. But $u_m(x,t)$ converges uniformly to $u(x,t)$ on Q_k, and $\frac{\partial u_m(x,t)}{\partial x_i}$ converges to $v_i(x,t)$ in $L^2(Q_k)$. Hence, passing to the limit as $m \to \infty$ we obtain the following equality

$$\int_{Q_k} v_i(x,t)\varphi(x,t)\,dx\,dt = -\int_{Q_k} u(x,t)\frac{\partial\varphi(x,t)}{\partial x_i}\,dx\,dt, \tag{4.227}$$

which means that $v_i(x,t)$, $i = 1,\dots,n$, are indeed the weak partial derivatives of the function $u(x,t)$. We write again the inequality (4.203), this time for the functions $u_1(x,t) = 0$ and $u_2(x,t) = u_m(x,t)$ for $m \geq m(k+l)$ and the cylinder Q_{k+l}. We have

$$\int_{Q_{k+l}} \left|\operatorname{grad} u_m(x,t)\right|^2 h_{k+l}(x,t)\,dx\,dt \leq \frac{c_{k+l}}{\alpha}\cdot 3\|u_m\|^2_{L^\infty(Q_{k+l})}. \tag{4.228}$$

Passing to the limit as $m \to \infty$ in the latter inequality, we get

$$\int_{Q_{k+l}} \left|\operatorname{grad} u(x,t)\right|^2 h_{k+l}(x,t)\,dx\,dt \leq \frac{c_{k+l}}{\alpha}\cdot 3\|u\|^2_{L^\infty(Q_{k+l})}.$$

Let us restrict the integral on the left-hand side of this inequality over the cylinder Q_k and afterwards make the integer l tend to infinity, then we obtain

$$\int_{Q_k} \left|\operatorname{grad} u(x,t)\right|^2 h(x,t)\,dx\,dt \leq \frac{c_\infty}{\alpha}\cdot 3\|u\|^2_{L^\infty(Q)} < \infty, \tag{4.229}$$

where

$$c_\infty = \int_Q \Big(\left|L^* h(x,t)\right| + |c| h(x,t)\Big)\,dx\,dt. \tag{4.230}$$

Since the left-hand side of (4.229) is increasing with respect to k and bounded, it has the finite limit, so that

$$\int_Q \left|\operatorname{grad} u(x,t)\right|^2 h(x,t)\,dx\,dt \leq \frac{3c_\infty}{\alpha}\,\|u(x,t)\|^2_{L^\infty(Q)} < \infty. \tag{4.231}$$

□

Next we formulate the main result of this chapter.

Remark 4.7 *The inequality* (5.109) *gives us possibility of estimating the weighted L^2-distance between the gradients of two continuous weak parabolic subsolutions in terms of the uniform distance between subsolutions themself.*

Theorem 4.26 (THE WEIGHTED REVERSE POINCARÉ INEQUALITY) *Assume that the uniform parabolicity condition* (4.187) *is satisfied. Consider two arbitrary continuous weak parabolic subsolutions $u_i(x,t)$, $i = 1,2$, in the cylinder Q, $Q = B(x_0,R)\times(0,T)$.*

Then the following weighted reverse Poincaré type inequality holds for the difference $u_2(x,t) - u_1(x,t)$ *of two weak subsolutions*

$$\int_Q \left|\operatorname{grad} u_2(x,t) - \operatorname{grad} u_1(x,t)\right|^2 h(x,t)\,dx\,dt \le \frac{1}{\alpha}\int_Q \Big(\left|L^* h(x,t)\right| + |c| h(x,t)\Big)\,dx\,dt \times$$
$$\times \Big[2\|u_2 - u_1\|_{L^\infty(Q)}\big(\|u_1\|_{L^\infty(Q)} + \|u_2\|_{L^\infty(Q)}\big) + \|u_2 - u_1\|^2_{L^\infty(Q)}\Big]. \quad (4.232)$$

Proof. Consider mollifications $u_{m,i}(x,t)$, $i = 1,2$, of the continuous weak parabolic subsolutions $u_i(x,t)$, $i = 1,2$. We already know that for a cylinder Q_{k+l} there exists integer m_{k+l} such that each function $u_{m,i}(x,t)$, $i = 1,2$, is the smooth parabolic subsolution in the cylinder Q_{k+l} if $m \ge m_{k+l}$. We have also the following uniform convergence

$$\left\|u_{m,i} - u_i\right\|_{L^\infty(Q_{k+l})} \xrightarrow{m\to\infty} 0, \;\; i = 1,2.$$

Let us apply the inequality (4.203) to the functions $u_{m,1}(x,t)$ and $u_{m,2}(x,t)$ and the cylinder Q_{k+l}. We have

$$\int_{Q_{k+l}} \left|\operatorname{grad} u_{m,2}(x,t) - \operatorname{grad} u_{m,1}(x,t)\right|^2 h_{k+l}(x,t)\,dx\,dt$$
$$\le \frac{c_{k+l}}{\alpha}\Big[2\|u_{m,2} - u_{m,1}\|_{L^\infty(Q_{k+l})}\big(\|u_{m,1}\|_{L^\infty(Q_{k+l})} + \|u_{m,2}\|_{L^\infty(Q_{k+l})}\big) + \|u_{m,2} - u_{m,1}\|^2_{L^\infty(Q_{k+l})}\Big]. \quad (4.233)$$

Passing to the limit as $m \to \infty$ in the latter inequality we get

$$\int_{Q_{k+l}} \left|\operatorname{grad} u_2(x,t) - \operatorname{grad} u_1(x,t)\right|^2 h_{k+l}(x,t)\,dx\,dt$$
$$\le \frac{c_{k+l}}{\alpha}\Big[2\|u_2 - u_1\|_{L^\infty(Q_{k+l})}\big(\|u_1\|_{L^\infty(Q_{k+l})} + \|u_2\|_{L^\infty(Q_{k+l})}\big) + \|u_2 - u_1\|^2_{L^\infty(Q_{k+l})}\Big]. \quad (4.234)$$

Restricting the integral on the left-hand side of (4.234) over the cylinder Q_k and then passing to the limit as $l \to \infty$, we obtain

$$\int_{Q_k} \left|\operatorname{grad} u_2(x,t) - \operatorname{grad} u_1(x,t)\right|^2 h(x,t)\,dx\,dt$$
$$\le \frac{c_\infty}{\alpha}\Big[2\|u_2 - u_1\|_{L^\infty(Q)}\big(\|u_1\|_{L^\infty(Q)} + \|u_2\|_{L^\infty(Q)}\big) + \|u_2 - u_1\|^2_{L^\infty(Q)}\Big]. \quad (4.235)$$

By Theorem 4.25 we have

$$\int_Q \left|\operatorname{grad} u_i(x,t)\right|^2 h(x,t)\,dx\,dt < \infty, \;\; i = 1,2. \quad (4.236)$$

Passing now to the limit in the inequality (4.235) as $k \to \infty$ we obtain the desired estimate (4.232). □

Chapter 5

The weighted energy inequalities for subsolution of higher order partial differential equations

5.1 The weighted square integral inequalities for smooth and weak subsolution of fourth order Laplace equation

The fourth order Laplace equation with n variables is given as

$$\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_2^4} + \ldots + \frac{\partial^4 u}{\partial x_n^4} = 0 \tag{5.1}$$

Let us denote

$$\Delta^4 \equiv \frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} + \ldots + \frac{\partial^4}{\partial x_n^4}$$

It can also be easily prove that Δ^4 is self-adjoint operator i.e.

$$\Delta^4 = \Delta^{*4}.$$

Now (5.1) becomes as

$$\Delta^4 u = 0. \tag{5.2}$$

The function $u \in C^4(B)$ is called subsolution of fourth order Laplace equation if

$$\Delta^4 u \geq 0. \tag{5.3}$$

The function $u \in C^4(B)$ is called suppersolution of forth order Laplace equation if

$$\Delta^4 u \leq 0. \tag{5.4}$$

The second order Laplace equation represent a large number of practical problem, is a particular case of second order elliptic equation. Let L denote the second order elliptic differential operator having the form, either

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u \tag{5.5}$$

or

$$Lu = -\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i x_j}\right) + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u. \tag{5.6}$$

where (5.5) and (5.6) are the divergence and non divergence forms respectively. The differential operator L is uniformly elliptic if there exist constant $\theta > 0$ so that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta\,|\xi|^2, \quad for\ a.e.\ x \in U$$

and all $\xi \in \mathbb{R}^n$ i.e. uniform ellipticity means that the symmetric matrix $A(x)$ is positive definite and the smallest eigenvalue is more or equal to θ. The energy estimates for the weak subsolution of uniformly elliptic operator are derived in [63]. So it is also interesting to derived weighted energy estimates for the weak subsolution of fourth order Laplace equation.

The bounded measurable function u is called weak subsolution if u satisfy

$$\int_B u\Delta^{*4}\psi(x)dx \geq 0 \tag{5.7}$$

Throughout the chapter we will use the following notations

$$\mathrm{grad\ u} = \left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}_1}, \frac{\partial \mathrm{u}}{\partial \mathrm{x}_2}, \ldots, \frac{\partial \mathrm{u}}{\partial \mathrm{x}_\mathrm{n}}\right)$$

$$\mathrm{grad}^2\ \mathrm{u} = \left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}_1^2}, \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}_2^2}, \ldots, \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}_\mathrm{n}^2}\right)$$

5.1.1 The weighted energy estimates for the smooth subsolution for the fourth order Laplace equation

Theorem 5.1 *Let $u \in C^4(B)$ be the n-dimension smooth subsolution of the fourth order Laplace equation that satisfies $\frac{\partial^2 u}{\partial x_j^2} \geq 0 \;\; j = 1,2,\ldots,n$. Then, we have the following estimates*

$$\int_B \left|\mathrm{grad}^2\mathrm{u}\right|^2 h(x)dx \leq \int_B \left(\frac{u^2(x)}{2} - \sup_{x\in D}|u|\,u\right) \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx \tag{5.8}$$

where h is the non-negative weight function which satisfies

$$h(x) = \frac{\partial h(x)}{\partial x_i} = \frac{\partial^2 h(x)}{\partial x_i^2} = 0, \; and \; \frac{\partial^2 h(x)}{\partial x_i^2} \leq 0, \;\; for\, x \in \partial D. \tag{5.9}$$

Proof. Take

$$\int_D \left|\mathrm{grad}^2\mathrm{u}\right|^2 h(x)dx = \int_D \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 + \ldots + \left(\frac{\partial^2 u}{\partial x_n^2}\right)^2\right] h(x)dx$$

$$= \int_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 h(x)\,dx + \int_D \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 h(x)\,dx + \ldots + \int_D \left(\frac{\partial^2 u}{\partial x_n^2}\right)^2 h(x)\,dx \tag{5.10}$$

where $D \subseteq \mathbb{R}^n$. Let us denote

$$I = \int_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 h(x)\,dx + \int_D \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 h(x)\,dx + \ldots + \int_D \left(\frac{\partial^2 u}{\partial x_n^2}\right)^2 h(x)dx \tag{5.11}$$

and

$$I_i = \int_D \left(\frac{\partial^2 u}{\partial x_i^2}\right)^2 h(x)\,dx, \; i = 1, \ldots n.$$

On the integral

$$I_1 = \int_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 h(x)\,dx = \int_D \frac{\partial^2 u}{\partial x_1^2}\left(\frac{\partial^2 u}{\partial x_1^2} h(x)\right) dx$$

we apply the definition of weight function and integration by parts

$$I_1 = -\int_D \frac{\partial u}{\partial x_1}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial^2 u}{\partial x_1^2} h(x)\right)\right] dx = -\int_D \frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3} h(x)dx - \int_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1} dx.$$

Again, we us integration by parts on the first integral on the right side

$$I_1 = \int_D u\left[\frac{\partial}{\partial x_1}\left(\frac{\partial^3 u}{\partial x_1^3} h(x)\right)\right] dx - \int_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1} dx$$

$$= \int\limits_D u\frac{\partial^4 u}{\partial x_1^4}h(x)dx + \int\limits_D u\frac{\partial^3 u}{\partial x_1^3}\frac{\partial h(x)}{\partial x_1} - \int\limits_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1}dx. \tag{5.12}$$

We use integration by parts on the middle integral of (5.12)

$$\begin{aligned}\int\limits_D u\frac{\partial^3 u}{\partial x_1^3}\frac{\partial h(x)}{\partial x_1}dx &= -\int\limits_D \frac{\partial^2 u}{\partial x_1^2}\left[\frac{\partial}{\partial x_1}\left(u\frac{\partial h(x)}{\partial x_1}\right)\right]dx \\ &= -\int\limits_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1}dx - \int\limits_D u\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 h(x)}{\partial x_1^2}dx. \end{aligned}\tag{5.13}$$

If we use (5.13) in (5.12)

$$I_1 = \int\limits_D u\frac{\partial^4 u}{\partial x_1^4}h(x)dx - \int\limits_D u\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 h(x)}{\partial x_1^2}dx - 2\int\limits_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1}dx. \tag{5.14}$$

Now we calculate the integral

$$\begin{aligned}2\int\limits_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1}dx &= \int\limits_D \left[\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)^2\right]\frac{\partial h(x)}{\partial x_1}dx \\ &= -\int\limits_D \left(\frac{\partial u}{\partial x_1}\right)^2\frac{\partial^2 h(x)}{\partial x_1^2}dx = -\int\limits_D \frac{\partial u}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\frac{\partial^2 h(x)}{\partial x_1^2}\right)dx \\ &= \int\limits_D u\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\frac{\partial^2 h(x)}{\partial x_1^2}\right)\right]dx \\ &= \int\limits_D u\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 h(x)}{\partial x_1^2}dx + \int\limits_D u\frac{\partial u}{\partial x_1}\frac{\partial^3 h(x)}{\partial x_1^3}dx \\ &= \int\limits_D u\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 h(x)}{\partial x_1^2}dx + \frac{1}{2}\int\limits_D \frac{\partial u^2(x)}{\partial x_1}\frac{\partial^3 h(x)}{\partial x_1^3} \\ &= \int_D u\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 h(x)}{\partial x_1^2}dx - \frac{1}{2}\int\limits_D u^2(x)\frac{\partial^4 h(x)}{\partial x_1^4}. \end{aligned}\tag{5.15}$$

Now we have

$$I_1 = \int\limits_D u\frac{\partial^4 u}{\partial x_1^4}h(x)dx - 2\int\limits_D u\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 h(x)}{\partial x_1^2}dx + \frac{1}{2}\int\limits_D u^2(x)\frac{\partial^4 h(x)}{\partial x_1^4}. \tag{5.16}$$

Using similar calculation for $I_2, \ldots, I_n$, we have

$$I = \sum_{j=1}^{n} I_j$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\left(\int_D u\frac{\partial^4 u}{\partial x_j^4}h(x)dx - 2\int_D u\frac{\partial^2 u}{\partial x_j^2}\frac{\partial^2 h(x)}{\partial x_j^2}dx + \frac{1}{2}\int_D u^2(x)\frac{\partial^4 h(x)}{\partial x_j^4}dx\right)\\
&\leq \int_D |u|\left|\sum_{j=1}^{n}\frac{\partial^4 u}{\partial x_j^4}\right|h(x)dx + 2\int_D |u|\left|\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}\frac{\partial^2 h(x)}{\partial x_j^2}\right|dx + \frac{1}{2}\int_D u^2(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}dx\\
&\leq \sup_{x\in D}|u|\int_D\left|\sum_{j=1}^{n}\frac{\partial^4 u}{\partial x_j^4}\right|h(x)dx + 2\sup_{x\in D}|u|\int_D\left|\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}\frac{\partial^2 h(x)}{\partial x_j^2}\right|dx + \frac{1}{2}\int_D u^2(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}dx\\
&= \sup_{x\in D}|u|\int_D\sum_{j=1}^{n}\frac{\partial^4 u}{\partial x_j^4}h(x)dx - 2\sup_{x\in D}|u|\int_D\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}\frac{\partial^2 h(x)}{\partial x_j^2}dx + \frac{1}{2}\int_D u^2(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}dx.
\end{aligned}
\tag{5.17}
$$

Since

$$
\sum_{j=1}^{n}\frac{\partial^4 u}{\partial x_j^4}\geq 0,\qquad \frac{\partial^2 u}{\partial x_j^2}\geq 0\quad\text{and}\quad\frac{\partial^2 h(x)}{\partial x_j^2}\leq 0,\quad j=1,2,\ldots,n
$$

$$
I \leq \sup_{x\in D}|u|\int_D\sum_{j=1}^{n}\frac{\partial^4 u}{\partial x_j^4}h(x)dx - 2\sup_{x\in D}|u|\int_D\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}\frac{\partial^2 h(x)}{\partial x_j^2}dx + \frac{1}{2}\int_D u^2(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}dx,
\tag{5.18}
$$

using the definition of weight function and integration by parts, we have

$$
\int_D\sum_{j=1}^{n}\frac{\partial^4 u}{\partial x_j^4}h(x)\,dx = \int_D u\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx \text{ and}
$$

$$
\int_D\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}\frac{\partial^2 h(x)}{\partial x_j^2}\,dx = \int_D u\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx,
$$

and then

$$
I \leq \sup_{x\in D}|u|\int_D u\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}dx - 2\sup_{x\in D}|u|\int_D u\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}dx + \frac{1}{2}\int_D u^2(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}dx
$$

$$
\int_B\left|\mathrm{grad}^2\mathrm{u}\right|^2\,h(x)dx \leq \int_B\left(\frac{u^2(x)}{2}-\sup_{x\in D}|u|\,u\right)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx.
$$

□

Now we will prove similler inequality for the difference of two smooth subsolutions for the fourth order Laplace equation.

Theorem 5.2 *Let $u_i \in C^4(B)$, $i = 1,2$, be the smooth subsolutions of* (5.3) *over the ball $B = D \subseteq \mathbb{R}^n$. Then we have the following energy estimate for the difference*

$$\int_B \left|\mathrm{grad}^2\left(\mathrm{u}_2(\mathrm{x}) - \mathrm{u}_1(\mathrm{x})\right)\right|^2 h(x)dx$$
$$\leq \int_B \left(\frac{(u_2(x)-u_1(x))^2}{2} - \sup_{x\in D}|(u_2(x)-u_1(x))|\,(u_2(x)+u_1(x))\right) \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx \quad (5.19)$$

where h is the non-negative smooth weight function with compact support i.e. $\frac{\partial^2 h(x)}{\partial x_j^2} \leq 0$, $j = 1,2,\ldots,n$

Proof. Let $u = u_2 - u_1$. Denote

$$I = \int_D \left|\mathrm{grad}^2 \mathrm{u}\right|^2 h(x)dx$$
$$= \int_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 h(x)dx + \int_D \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 h(x)dx + \ldots + \int_D \left(\frac{\partial^2 u}{\partial x_n^2}\right)^2 h(x)dx, \quad (5.20)$$

and

$$I_i = \int_D \left(\frac{\partial^2 u}{\partial x_i^2}\right)^2 h(x)\,dx,\ i = 1,\ldots,n.$$

Now(4.196) becomes

$$I = I_1 + I_2 + \ldots + I_n. \quad (5.21)$$

Observe

$$I_1 = \int_D \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 h(x)\,dx = \int_D \frac{\partial^2 u}{\partial x_1^2}\left(\frac{\partial^2 u}{\partial x_1^2} h(x)\right) dx,$$

and then using integration by parts and the definition of weight function, we get

$$I_1 = -\int_D \frac{\partial u}{\partial x_1}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial^2 u}{\partial x_1^2} h(x)\right)\right] dx = -\int_D \frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3} h(x)dx - \int_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1} dx.$$

Using integration by parts on first integral

$$I_1 = \int_D u\left[\frac{\partial}{\partial x_1}\left(\frac{\partial^3 u}{\partial x_1^3} h(x)\right)\right] dx - \int_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1} dx$$
$$= \int_D u\frac{\partial^4 u}{\partial x_1^4} h(x)dx + \int_D u\frac{\partial^3 u}{\partial x_1^3}\frac{\partial h(x)}{\partial x_1} dx - \int_D \frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial h(x)}{\partial x_1} dx. \quad (5.22)$$

Now using integration by parts on the middle integral of (5.22)

$$\int_D u \frac{\partial^3 u}{\partial x_1^3} \frac{\partial h(x)}{\partial x_1} dx = -\int_D \frac{\partial^2 u}{\partial x_1^2} \left[\frac{\partial}{\partial x_1} \left(u \frac{\partial h(x)}{\partial x_1} \right) \right] dx$$
$$= -\int_D \frac{\partial u}{\partial x_1} \frac{\partial^2 u}{\partial x_1^2} \frac{\partial h(x)}{\partial x_1} dx - \int_D u \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} dx. \tag{5.23}$$

Now using (5.23) in (5.22) we get

$$I_1 = \int_D u \frac{\partial^4 u}{\partial x_1^4} h(x) dx - \int_D u \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} dx - 2\int_D \frac{\partial u}{\partial x_1} \frac{\partial^2 u}{\partial x_1^2} \frac{\partial h(x)}{\partial x_1} dx. \tag{5.24}$$

$$I_1 = \int_D u \frac{\partial^4 u}{\partial x_1^4} h(x) dx - \int_D u \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} dx - \int_D \left[\frac{\partial}{\partial x_1} \left(\frac{\partial u}{\partial x_1} \right)^2 \right] \frac{\partial h(x)}{\partial x_1} dx \tag{5.25}$$

Now take the integral

$$\int_D \left[\frac{\partial}{\partial x_1} \left(\frac{\partial u}{\partial x_1} \right)^2 \right] \frac{\partial h(x)}{\partial x_1} dx = -\int_D \left(\frac{\partial u}{\partial x_1} \right)^2 \frac{\partial^2 h(x)}{\partial x_1^2} dx = -\int_D \frac{\partial u}{\partial x_1} \left(\frac{\partial u}{\partial x_1} \frac{\partial^2 h(x)}{\partial x_1^2} \right) dx$$
$$= \int_D u \left[\frac{\partial}{\partial x_1} \left(\frac{\partial u}{\partial x_1} \frac{\partial^2 h(x)}{\partial x_1^2} \right) \right] dx = \int_D u \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} dx + \int_D u \frac{\partial u}{\partial x_1} \frac{\partial^3 h(x)}{\partial x_1^3} dx$$
$$= \int_D u \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} dx + \frac{1}{2} \int_D \frac{\partial u^2(x)}{\partial x_1} \frac{\partial^3 h(x)}{\partial x_1^3} = \int_D u \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} dx - \frac{1}{2} \int_D u^2(x) \frac{\partial^4 h(x)}{\partial x_1^4} dx. \tag{5.26}$$

Now, if we put (5.26) in (5.23) we get

$$I_1 = \int_D u \frac{\partial^4 u}{\partial x_1^4} h(x) dx - 2\int_D u \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 h(x)}{\partial x_1^2} dx + \frac{1}{2} \int_D u^2(x) \frac{\partial^4 h(x)}{\partial x_1^4} dx. \tag{5.27}$$

With a similar calculation for $I_2 \ldots, I_n$, we finally have

$$I = \sum_{j=1}^{n} \left(\int_D u \frac{\partial^4 u}{\partial x_j^4} h(x) dx - 2\int_D u \frac{\partial^2 u}{\partial x_j^2} \frac{\partial^2 h(x)}{\partial x_j^2} dx + \frac{1}{2} \int_D u^2(x) \frac{\partial^4 h(x)}{\partial x_j^4} dx \right)$$
$$\leq \int_D |u| \left| \sum_{j=1}^{n} \frac{\partial^4 u}{\partial x_j^4} \right| h(x) dx + 2\int_D |u| \left| \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \frac{\partial^2 h(x)}{\partial x_j^2} \right| dx + \frac{1}{2} \int_D u^2(x) \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx$$
$$\leq \sup_{x \in D} |u| \int_D \left| \sum_{j=1}^{n} \frac{\partial^4 u}{\partial x_j^4} \right| h(x) dx + 2 \sup_{x \in D} |u| \int_D \left| \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \frac{\partial^2 h(x)}{\partial x_j^2} \right| dx + \frac{1}{2} \int_D u^2(x) \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx. \tag{5.28}$$

Now using $u = u_2 - u_1$ we obtain

$$\int_D \left|\mathrm{grad}^2\,[\mathrm{u}_2(\mathrm{x}) - \mathrm{u}_1(\mathrm{x})]\right|^2 h(x)dx$$

$$\leq \sup_{x\in D} |(u_2(x) - u_1(x))| \int_D \left|\sum_{j=1}^{n} \frac{\partial^4 (u_2(x) - u_1(x))}{\partial x_j^4}\right| h(x)dx$$

$$+ 2\sup_{x\in D} |\,(u_2(x) - u_1(x))\,| \int_D \left|\sum_{j=1}^{n} \frac{\partial^2 (u_2(x) - u_1(x))}{\partial x_j^2} \frac{\partial^2 h(x)}{\partial x_j^2}\right| dx$$

$$+ \frac{1}{2} \int_D (u_2(x) - u_1(x))^2 \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx$$

$$\leq \sup_{x\in D} |u_2(x) - u_1(x)| \sum_{j=1}^{n} \int_D \left(\frac{\partial^4 u_2(x)}{\partial x_j^4} + \frac{\partial^4 u_1(x)}{\partial x_j^4}\right) h(x)dx$$

$$- 2\sup_{x\in D} |u_2(x) - u_1(x)| \sum_{j=1}^{n} \int_D \left(\frac{\partial^2 u_2(x)}{\partial x_j^2} + \frac{\partial^4 u_1(x)}{\partial x_j^4}\right) \frac{\partial^2 h(x)}{\partial x_j^2} dx$$

$$+ \frac{1}{2} \int\limits_D (u_2(x) - u_1(x))^2 \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx \tag{5.29}$$

Again using the definition of weight function and integration by parts , we have

$$\int_D \sum_{j=1}^{n} \frac{\partial^4 u_i(x)}{\partial x_j^4} h(x)\,dx = \int_D u_i(x) \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx \quad i = 1,2$$

and

$$\int_D \sum_{j=1}^{n} \frac{\partial^2 u_i(x)}{\partial x_j^2} \frac{\partial^2 h(x)}{\partial x_j^2} dx = \int_D u_i(x) \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx \;\; i = 1,2$$

the above (5.29) becomes

$$\int_B \left|\mathrm{grad}^2\,[\mathrm{u}_2(\mathrm{x}) - \mathrm{u}_1(\mathrm{x})]\right|^2 h(x)dx$$

$$\leq \int_B \left[\frac{[u_2(x) - u_1(x)]^2}{2} - \sup_{x\in D} |u_2(x) - u_1(x)|\,(u_2(x) + u_1(x))\right] \sum_{j=1}^{n} \frac{\partial^4 h(x)}{\partial x_j^4} dx.$$

□

Remark 5.1 *Taking the supremum norm on above inequality we obtained*

$$\int_B \left|\mathrm{grad}^2\,(\mathrm{u}_2(\mathrm{x}) - \mathrm{u}_1(\mathrm{x}))\right|^2 h(x)dx \leq \tag{5.30}$$

$$\left[\frac{1}{2} \|u_2(x) - u_1(x)\|_{L^\infty}^2 + \|u_2(x) - u_1(x)\|_{L^\infty} \left(\|u_1(x)\|_{L^\infty} + \|u_2(x)\|_{L^\infty}\right)\right] \int_B \Delta^4 h(x)dx.$$

5.1.2 The weighted energy estimates for the weak subsolution using smooth ones for the fourth order Laplace equation

The continuous function u is said to be weak subsolution of (5.3) if

$$\int_B u\,\Delta^4\psi(x)\,dx \geq 0, \text{ for } \psi \in C_c^\infty(B). \tag{5.31}$$

Now we will approximate the weak subsolution of (5.2) by the smooth ones. For this we again use mollification technique.
For

$$\eta(x) = \begin{cases} C\ exp\left(\frac{1}{x^2-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{5.32}$$

and $C > 0$ is such that

$$\int_{\mathbb{R}^n} \eta(x)\,dx = 1. \tag{5.33}$$

We define

$$u_\varepsilon(x) = \varepsilon^{-n} \int_B \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,dy \tag{5.34}$$

Let us denote,

$$\eta_\varepsilon(x-y) = \varepsilon^{-n}\ \eta\left(\frac{x-y}{\varepsilon}\right) \tag{5.35}$$

It is easy to see that

$$\frac{\partial^4 \eta_\varepsilon(x-y)}{\partial x_i^4} = \frac{\partial^4 \eta_\varepsilon(x-y)}{\partial y_i^4}, \qquad i = 1,2,\ \ldots\ ,n. \tag{5.36}$$

Hence,

$$\Delta_x^4\, u_\varepsilon(x) = p^{-n} \int_B u_y\, \Delta_y^4\, \eta_\varepsilon(x-y)\,dy. \tag{5.37}$$

where Δ_x^4 and Δ_y^4 are the fourth order Laplace operator with respect to x and y respectively.
Let the ball $B_k = B(x_0, r_k)$ with $r_k = \frac{k+1}{k+2}r\ \ , k \in \mathbb{N}$.

Theorem 5.3 *Let u be the continuous weak, convex, subsolution of (5.2) over ball $B(x_0,r)$. Then for any $k \in \mathbb{N}$ there exists $\hat{\varepsilon} > 0$ such that for every ε, $0 < \varepsilon < \hat{\varepsilon}$, each function $u_\varepsilon(x)$ is smooth convex over the ball B_k and also*

$$\Delta^4 u_\varepsilon(x) \geq 0 \ \ if \ \ x \in B_k.$$

Proof. For fixed $k \in \mathbb{N}$, let

$$\hat{h} = \frac{\gamma}{2(k+2)}. \tag{5.38}$$

It is trivially by the definition of u_ε is infinitely differentiable and also u_ε is smooth convex for each of its arguments.
Now we check that for arbitrary $x \in B_k,\ \eta_\varepsilon(x-y)$ has compact support in the ball D.
Let us take another ball $\hat{B}_k$ in the following way

$$\hat{B}_k = B(x_0, \frac{2k+3}{2k+4}r) \tag{5.39}$$

If $y \notin \hat{B}_k$, then

$$|y-x| > \left| \frac{2k+3}{2k+4} - \frac{2k+2}{2k+4} \right| = \frac{1}{2(k+2)} r > \varepsilon \tag{5.40}$$

$$\Rightarrow \quad \eta_\varepsilon(x-y) = 0, \tag{5.41}$$

so $\eta_\varepsilon(x-y)$ has compact support.
By the definition of weak subsolution and also using (5.41), we have

$$\int_B u(y)\, \Delta_y^4\, \eta_\varepsilon(x-y)\, dy \geq 0. \tag{5.42}$$

□

Theorem 5.4 *Let u be the continuous weak, convex, subsolution of* (5.2) *and also u. Then it posses the following weak partial derivatives $\frac{\partial^2 u}{\partial x_i^2},\ i = 1,\ldots,n$ over the ball B.*

Proof. For the existence of first derivative $\frac{\partial u}{\partial x_i}$ $i = 1,2,\ldots,n$ one can see [46]. Let us suppose the mollification $u_\varepsilon(x)$ defined in (5.34) for the weak subsolution of fourth order Laplace equation u.
For the continuous function u, the ball B, it is well-known fact that on compact set $K \subseteq B$ we have the following uniform-convergence

$$\sup_{x\in K} |u_\varepsilon(x) - u(x)| \xrightarrow{\varepsilon \to 0} 0.$$

Let us denote u_m for u_ε, $\varepsilon = \frac{1}{m}$, $m \in \mathbb{N}$ so above becomes

$$\sup_{x\in K} |u_m(x) - u(x)| \xrightarrow{m \to \infty} 0. \tag{5.43}$$

The balls $B_k,\ k \in \mathbb{N}$ are compactly contained in the original ball B.
From the Theorem 5.3, we know that for any $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that u_m is

smooth subsolution of (5.2)
Take the ball B_{k+l} and write the inequality (5.30) for $u_1 = u_m$ and $u_2 = u_p$

$$\int_{B_{k+l}} \left|\mathrm{grad}^2 \mathrm{u_p} - \mathrm{grad}^2 \mathrm{u_m}\right|^2 h_{k+l} dx$$
$$\leq \left[\frac{1}{2}\left\|u_p - u_m\right\|_{L^\infty}^2 + \left\|u_p - u_m\right\|_{L^\infty}\left(\left\|u_p\right\|_{L^\infty} + \|u_m\|_{L^\infty}\right)\right] \int_{B_{k+l}} \left|\Delta^4 h_{k+l}\right| dx. \quad (5.44)$$

Let us denote

$$\alpha_{k+l} = \int_{B_{k+l}} \left|\Delta^4 h_{k+l}\right| dx,\ \hat{\alpha} = \inf\, h_{k+l},\ x \in B_{k+l}.$$

Then

$$\hat{\alpha} \int_{B_{k+l}} \left|\mathrm{grad}^2 \mathrm{u_p} - \mathrm{grad}^2 \mathrm{u_m}\right|^2 dx$$
$$\leq \alpha_{k+l} \left[\frac{1}{2}\left\|u_p - u_m\right\|_{L^\infty}^2 + \left\|u_p - u_m\right\|_{L^\infty}\left(\left\|u_p\right\|_{L^\infty} + \|u_m\|_{L^\infty}\right)\right] dx. \quad (5.45)$$

Writing the left hand side integral for the smaller ball B_k, we have

$$\hat{\alpha} \int_{B_k} \left|\mathrm{grad}^2 \mathrm{u_p} - \mathrm{grad}^2 \mathrm{u_m}\right|^2 dx$$
$$\leq \alpha_{k+l} \left[\frac{1}{2}\left\|u_p - u_m\right\|_{L^\infty}^2 + \left\|u_p - u_m\right\|_{L^\infty}\left(\left\|u_p\right\|_{L^\infty} + \|u_m\|_{L^\infty}\right)\right] dx \quad (5.46)$$

From (5.43), we have

$$\left\|u_p - u_m\right\|_{L^\infty_{(B_{k+l})}} \longrightarrow 0 \text{ as } \quad m, p \longrightarrow \infty$$

so (5.46) becomes

$$\lim_{m,p\to\infty} \sum_{i=1}^{n} \int_{B_k} \left(\frac{\partial^2 u_p}{\partial x_i^2} - \frac{\partial^2 u_m}{\partial x_i^2}\right) dx = 0.$$

The completeness of $L^2(B_k)$ ensure the convergence of above sequence. So there exist a class of measurable functions $v_{k,i}(x) \in L^2(B_k)$ such that

$$\sum_{i=1}^{n} \int_{B_k} \left(\frac{\partial^2 u_m}{\partial x_i^2}(x) - v_{k,i}(x)\right)^2 dx \xrightarrow{m\to\infty} 0,\ k \in \mathbb{N}.$$

We extend $v_{k,i}$ trivially outside the ball B_k by 0.
Let us denote

$$v_i(x) = \lim_{k\to\infty} \sup_{x\in D}\ v_{k,i},\ i = 1, 2, \ldots, n$$

It can be checked easily that $v_i(x) = v_{k,i}(x)$ a.e. on the ball B_k.
Next we claim that v_i represent the weak second order partial derivative $\frac{\partial^2 u}{\partial x_i^2}$ of u.

Take ψ an arbitrary function having compact support in B. Then suppose supp $\psi \subset B_k$ from some $k \in \mathbb{N}$.
We have

$$\int_B \frac{\partial^2 u_m}{\partial x_i^2}\,\psi(x) = \int_B u_m\,\frac{\partial^2 \psi}{\partial x_i^2}\,dx$$

for the integers $m \geq m_k$.
But we have the following convergence

$$\|u_m - u\|_{L^\infty(B_k)} \xrightarrow{m\to\infty} 0,$$

and

$$\left\|\frac{\partial^2 u_m}{\partial x_i^2} - v_i\right\|_{L^2(B_k)} \xrightarrow{m\to\infty} 0.$$

Using this, we have

$$\int_{B_k} v_i(x)\;\psi(x)\;dx = \int_{B_k} u\frac{\partial^2 \psi}{\partial x_i^2}\;dx.$$

This shows that $v_i,\; i = 1,\ldots,n$ are the weak partial derivative of u. Rewriting the inequality (5.30) for the functions $u_1(x) = 0$ and $u_2(x) = u_m(x)$ for $m \geq m_{k+l}$ over the ball B_{k+l}, we get

$$\int_{B_{k+l}} \left|\mathrm{grad}^2 \mathrm{u_m(x)}\right|^2 p_{k+l}(x) \leq \frac{3}{2}\alpha_{k+l}\,\|u_m\|^2_{L^\infty(B_{k+l})}.$$

Taking limit $m \to \infty$, the above becomes

$$\int_{B_{k+l}} \left|\mathrm{grad}^2 \mathrm{u_m(x)}\right|^2 p_{k+l}(x) \leq \frac{3}{2}c_{k+l}\,\|u\|^2_{L^\infty(B_{k+l})}.$$

Now restricting the left hand side on the smaller ball B_k, we have

$$\int_{B_k} \left|\mathrm{grad}^2 \mathrm{u_m(x)}\right|^2 p_{k+l}(x) \leq \frac{3}{2}c_{k+l}\,\|u\|^2_{L^\infty(B_{k+l})}$$

Now taking limit as $m \to \infty$, we obtain

$$\int_{B_k} \left|\mathrm{grad}^2 \mathrm{u}\right|^2 h(x) \leq \frac{3}{2}c_\infty\,\|u\|^2_{L^\infty(B)} < \infty.$$

The left hand side of above increases as k increases and is also bounded. Using dominated convergence theorem

$$\int_{D} \left|\mathrm{grad}^2 \mathrm{u}\right|^2 h(x) \leq \frac{3}{2}c_\infty\,\|u\|^2_{L^\infty(B)} < \infty.$$

□

Now we prove the inequality for the weak subsolution of forth order Laplace equation.

Theorem 5.5 *Let u be the continuous weak subsolution of* (5.2)*, that satisfies*

$$\frac{\partial^2 u}{\partial x_j^2} \geq 0, \quad j = 1, 2, \dots, n$$

Then the following is valid

$$\int_B \left|\mathrm{grad}^2 \mathrm{u}_2(\mathrm{x}) - \mathrm{grad}^2 \mathrm{u}_1(\mathrm{x})\right|^2 h(x)dx \leq \tag{5.47}$$
$$\left[\frac{1}{2}\|u_2(x)-u_1(x)\|^2_{L^\infty(B)} + \|(u_2(x)-u_1(x))\|_{L^\infty(B)}\left(\|u_1(x)\|_{L^\infty(B)}+\|u_2(x)\|_{L^\infty(B)}\right)\right]\int_B \left|\Delta^4 h(x)\right| dx,$$

where h is the weight function satisfying (5.9)*.*

Proof. We take mollification $u_{m,i}$, $i = 1,2$, of the continuous weak subsolution u_i, for $i = 1,2$ respectively.
Since for the ball B_{k+l}, exists an integer m_{k+l}, such that each function $u_{m,i}$, for $i = 1,2$ is the smooth subsolution in the ball B_{k+l}, if $m > m_{k+l}$. Also we have the following convergence

$$\|u_{m,i} - u_i\|_{L^\infty(B_{k+l})} \xrightarrow{m\to\infty} 0, \;\; i = 1,\; 2.$$

Now writing the inequalities for the functions $u_{m,i}$, i=1,2, on the cylinder B_{k+l} , we get

$$\int_B \left|\mathrm{grad}^2 \mathrm{u}_{\mathrm{m},2} - \mathrm{grad}^2 \mathrm{u}_{\mathrm{m},1}\right|^2 h_{k+l}(x)dx \tag{5.48}$$
$$\leq \alpha_{k+l}\left[\frac{1}{2}\|u_{m,2}-u_{m,1}\|^2_{L^\infty(B_{k+l})} + \|u_{m,2}-u_{m,1}\|_{L^\infty(B_{k+l})}\left(\|u_{m,1}\|_{L^\infty(B_{k+l})}+\|u_{m,2}\|_{L^\infty(B_{k+l})}\right)\right].$$

Taking limit as $m \to \infty$, the inequality (5.48) becomes

$$\int_{B_{k+l}} \left|\mathrm{grad}^2 \mathrm{u}_2(\mathrm{x}) - \mathrm{grad}^2 \mathrm{u}_1(\mathrm{x})\right|^2 h_{k+l}(x)dx$$
$$\leq \alpha_{k+l}\left[\frac{1}{2}\|u_2-u_1\|^2_{L^\infty(B_{k+l})} + \|u_2-u_1\|_{L^\infty(B_{k+l})}\left(\|u_1\|_{L^\infty(B_{k+l})}+\|u_2\|_{L^\infty(B_{k+l})}\right)\right].$$

Again writing the above inequality, the left hand side for the smaller ball B_k and taking limit $l \to \infty$, we get, we finally obtain

$$\int_{B_k} \left|\mathrm{grad}^2 \mathrm{u}_2(\mathrm{x}) - \mathrm{grad}^2 \mathrm{u}_1(\mathrm{x})\right|^2 h_{k+l}(x)dx \tag{5.49}$$
$$\leq \alpha_\infty\left[\frac{1}{2}\|u_2-u_1\|^2_{L^\infty(B)} + \|u_2-u_1\|_{L^\infty(B)}\left(\|u_1\|_{L^\infty(B)}+\|u_2\|_{L^\infty(B)}\right)\right].$$

□

5.2 The weighted energy estimate for the smooth subsolution of *n*-dimensional *beam equation*

The fourth order beam equation with n variables is

$$\frac{\partial^4 u(x,t)}{\partial x_1^4}+\frac{\partial^4 u(x,t)}{\partial x_2^4}+...+\frac{\partial^4 u(x,t)}{\partial x_n^4}+\frac{\partial u(x,t)}{\partial t}=0, \tag{5.50}$$

where $x=(x_1,x_2,\ldots,x_n)$.
Let us denote

$$L=\frac{\partial^4}{\partial x_1^4}+\frac{\partial^4}{\partial x_2^4}+...+\frac{\partial^4}{\partial x_n^4}+\frac{\partial}{\partial t}. \tag{5.51}$$

Then (5.50) becomes

$$Lu(x,t)=0.$$

Let us denote cylinder $Q=B\times(0,T)$, where $B=B(x_0,r)$. The function $u(x,t)\in C^4(Q)$ is called subsolution of fourth order beam equation if

$$Lu(x,t)\geq 0.$$

The function $u(x,t)\in C^4(Q)$ is called supersolution of fourth order beam equation if

$$Lu(x,t)\leq 0.$$

The bounded measurable function $u(x,t)$ is called weak subsolution of beam equation if it satisfies

$$\int_0^T\int_B u(x,t)L^*\phi(x,t)dxdt\geq 0. \tag{5.52}$$

Theorem 5.6 *Let $u(x,t)\in C^4(Q)$ be $n+1$ dimension smooth subsolution of fourth order beam equation*

$$\frac{\partial^4 u(x,t)}{\partial x_1^4}+\frac{\partial^4 u(x,t)}{\partial x_2^4}+...+\frac{\partial^4 u(x,t)}{\partial x_n^4}+\frac{\partial u(x,t)}{\partial t}=0,$$

which satisfies $\frac{\partial^2}{\partial x_j^2}u(x,t)\geq 0$, $j=0,1,\ldots,n$.
Then the following estimate holds

$$\int_0^T\int_B \left|\mathrm{grad}^2\mathrm{u(x,t)}\right|^2 h(x,t)dxdt\leq \tag{5.53}$$

$$\int_0^T\int_B \left[\frac{(u(x,t))^2}{2} - \sup|u(x,t)|\,u(x,t)\right]\left(\sum_{i=1}^n \frac{\partial^4 h(x,t)}{\partial x_i^4} + \frac{\partial h(x,t)}{\partial t}\right)dxdt,$$

where $h(x,t)$ is non negative concave smooth weight function with compact support.

Proof. Denote

$$I = \int_0^T\int_B \left|\mathrm{grad}^2\mathrm{u(x,t)}\right|^2 h(x,t)dxdt =$$

$$\int_0^T\int_B \left[\left(\frac{\partial^2}{\partial x_1^2}u(x,t)\right)^2 + \ldots + \left(\frac{\partial^2}{\partial x_n^2}u(x,t)\right)^2\right] h(x,t)dxdt,$$

and

$$I_j = \int_0^T\int_B \left(\frac{\partial^2}{\partial x_1^2}u(x,t)\right)^2 h(x,t)dxdt,\ j = 1,\ldots,n.$$

We calculate

$$\begin{aligned} I_1 &= \int_0^T\int_B \left(\frac{\partial^2}{\partial x_1^2}u(x,t)\right)^2 h(x,t)dxdt \\ &= \int_0^T\int_B \frac{\partial^2}{\partial x_1^2}u(x,t)[\frac{\partial^2}{\partial x_1^2}u(x,t)]h(x,t)dxdt \end{aligned}$$

we use the definition of weight function and integration by parts

$$\begin{aligned} I_1 &= -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial}{\partial x_1}[\frac{\partial^2}{\partial x_1^2}u(x,t)h(x,t)]dxdt \\ &= -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^3}{\partial x_1^3}u(x,t)h(x,t)]dxdt \\ &-\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)]dxdt. \end{aligned}$$

Again, using integration by parts on first integral

$$I_1 = \int_0^T\int_B u(x,t)\frac{\partial}{\partial x_1}[\frac{\partial^3}{\partial x_1^3}u(x,t)h(x,t)]dxdt$$

$$-\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)]dxdt$$

$$=\int_0^T\int_B u(x,t)\frac{\partial^4}{\partial x_1^4}u(x,t)h(x,t)dxdt$$

$$+\int_0^T\int_B u(x,t)\frac{\partial^3}{\partial x_1^3}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt$$

$$-\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt.$$

We apply the same on the middle integral

$$\int_0^T\int_B u(x,t)\frac{\partial^3}{\partial x_1^3}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt$$

$$=-\int_0^T\int_B \frac{\partial^2}{\partial x_1^2}u(x,t)\left[\frac{\partial}{\partial x_1}\left(u(x,t)\frac{\partial}{\partial x_1}h(x,t)\right)\right]dxdt$$

$$=-\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt$$

$$-\int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt.$$

Now

$$\begin{aligned}
I_1 &= \int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_1^4}h(x,t)dxdt \\
&-\int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&-2\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)]dxdt \\
&=\int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_1^4}h(x,t)dxdt \\
&-\int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt
\end{aligned}$$

$$-\int_0^T\int_B \left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}u(x,t)\right)^2\right]\frac{\partial}{\partial x_1}h(x,t)dxdt. \tag{5.54}$$

Further, we evaluate

$$\begin{aligned}
&\int_0^T\int_B \left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}u(x,t)\right)^2\right]\frac{\partial}{\partial x_1}h(x,t)dxdt \\
&= -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\left(\frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)\right)dxdt \\
&= \int_0^T\int_B u(x,t)\left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)\right)\right]dxdt \\
&= \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&+\int_0^T\int_B u(x,t)\frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^3}{\partial x_1^3}h(x,t)dxdt \\
&= \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B \frac{\partial}{\partial x_1}(u(x,t))^2\frac{\partial^3}{\partial x_1^3}h(x,t)dxdt \\
&= \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&-\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial^4}{\partial x_1^4}h(x,t)dxdt.
\end{aligned}$$

Now we have

$$\begin{aligned}
I_1 &= \int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_1^4}h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial^4}{\partial x_1^4}h(x,t)dxdt
\end{aligned} \tag{5.55}$$

Similarly calculating the values of $I_2,\ldots,I_n$, we have

$$\begin{aligned}
I &= \sum_{i=1}^{n}\left(\int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_i^4}h(x,t)dxdt\right.\\
&\quad -2\int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_i^2}u(x,t)\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt\\
&\quad \left.+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt\right)\\
I &= \int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^4 u(x,t)}{\partial x_i^4}h(x,t)dxdt\\
&\quad -2\int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt\\
&\quad +\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^{n}\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt.
\end{aligned}\tag{5.56}$$

Adding and subtracting $\frac{\partial}{\partial t}u(x,t)$ in first integral

$$\begin{aligned}
I &= \int_0^T\int_B u(x,t)\left(\sum_{i=1}^{n}\frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)-\frac{\partial}{\partial t}u(x,t)\right)h(x,t)dxdt\\
&\quad -2\int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt\\
&\quad +\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^{n}\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt\\
&= \int_0^T\int_B u(x,t)\left(\sum_{i=1}^{n}\frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)\right)h(x,t)dxdt\\
&\quad -\int_0^T\int_B u(x,t)\frac{\partial u(x,t)}{\partial t}h(x,t)dxdt\\
&\quad -2\int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2 \sum_{i=1}^n \frac{\partial^4}{\partial x_i^4} h(x,t)dxdt \\
&= \int_0^T\int_B u(x,t)\left(\sum_{i=1}^n \frac{\partial^4 u(x,t)}{\partial x_i^4} + \frac{\partial}{\partial t}u(x,t)\right) h(x,t)dxdt \\
&-\frac{1}{2}\int_0^T\int_B \left(\frac{\partial}{\partial t}u(x,t)^2\right) h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2 \sum_{i=1}^n \frac{\partial^4}{\partial x_i^4} h(x,t)dxdt.
\end{aligned} \tag{5.57}
$$

Integrating second term with respect to variable t and applying the definition of weight function

$$
\begin{aligned}
I &= \int_0^T\int_B u(x,t)\left(\sum_{i=1}^n \frac{\partial^4 u(x,t)}{\partial x_i^4} + \frac{\partial}{\partial t}u(x,t)\right) h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B u(x,t)^2 \frac{\partial}{\partial t} h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2 \sum_{i=1}^n \frac{\partial^4}{\partial x_i^4} h(x,t)dxdt \\
&\leq \int_0^T\int_B |u(x,t)|\left|\sum_{i=1}^n \frac{\partial^4 u(x,t)}{\partial x_i^4} + \frac{\partial}{\partial t}u(x,t)\right| h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B u(x,t)^2 \frac{\partial}{\partial t} h(x,t)dxdt \\
&+2\int_0^T\int_B |u(x,t)|\left|\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}u(x,t)\right|\left|\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}h(x,t)\right| dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2 \sum_{i=1}^n \frac{\partial^4}{\partial x_i^4} h(x,t)dxdt
\end{aligned}
$$

$$\le \sup|u(x,t)|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)\right|h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B u(x,t)^2\frac{\partial}{\partial t}h(x,t)dxdt$$

$$+2\sup|u(x,t)|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\right|\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)\right|dxdt$$

$$+\frac{1}{2}\int_0^T\int_B(u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt. \tag{5.58}$$

Using the conditions

$$\sum_{i=1}^n\frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)\ge 0,$$

$$\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\ge 0,\ \ \sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)\le 0,$$

we have in fact

$$\begin{aligned}
I\ &\le \sup|u(x,t)|\int_0^T\int_B\left[\sum_{i=1}^n\frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)\right]h(x,t)dxdt\\
&+\frac{1}{2}\int_0^T\int_B u(x,t)^2\frac{\partial}{\partial t}h(x,t)dxdt\\
&-2\sup|u(x,t)|\int_0^T\int_B\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt\\
&+\frac{1}{2}\int_0^T\int_B(u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt\\
&\le \sup|u(x,t)|\int_0^T\int_B\sum_{i=1}^n\frac{\partial^4 u(x,t)}{\partial x_i^4}h(x,t)dxdt\\
&+\int_0^T\int_B\left(\frac{\partial}{\partial t}u(x,t)\right)h(x,t)dxdt+\frac{1}{2}\int_0^T\int_B u(x,t)^2\frac{\partial}{\partial t}h(x,t)dxdt\\
&-2\sup|u(x,t)|\int_0^T\int_B\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt
\end{aligned}$$

$$
+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt
$$

$$
\leq \sup|u(x,t)|\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^4 h(x,t)}{\partial x_i^4}dxdt
$$

$$
-\int_0^T\int_B u(x,t)\frac{\partial h(x,t)}{\partial t}dxdt+\frac{1}{2}\int_0^T\int_B u(x,t)^2\frac{\partial}{\partial t}h(x,t)dxdt
$$

$$
-2\sup|u(x,t)|\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt
$$

$$
+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt
$$

$$
\leq -\sup|u(x,t)|\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^4 h(x,t)}{\partial x_i^4}dxdt
$$

$$
-\int_0^T\int_B u(x,t)\frac{\partial h(x,t)}{\partial t}dxdt+\frac{1}{2}\int_0^T\int_B u(x,t)^2\frac{\partial}{\partial t}h(x,t)dxdt
$$

$$
+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt
$$

$$
\leq \int_0^T\int_B \frac{(u(x,t))^2}{2}\left(\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)+\frac{\partial h(x,t)}{\partial t}\right)dxdt
$$

$$
-\sup|u(x,t)|\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n \frac{\partial^4 h(x,t)}{\partial x_i^4}+\frac{\partial h(x,t)}{\partial t}\right)dxdt, \tag{5.59}
$$

and our the proof is done. □

Theorem 5.7 *Let $u_i(x,t)\in C^4(Q), i=1,2,$ be the smooth, convex subsolution of beam equation*

$$
\frac{\partial^4 u(x,t)}{\partial x_1^4}+\frac{\partial^4 u(x,t)}{\partial x_2^4}+\ldots+\frac{\partial^4 u(x,t)}{\partial x_n^4}+\frac{\partial u(x,t)}{\partial t}=0
$$

over the cylinder $Q\subseteq\mathbb{R}^n$ and $h(x,t)$ is non negative concave smooth weight function with compact support.. Then the following energy estimate for the difference of the functions is *valid*

$$\int_0^T\int_B \left|\mathrm{grad}^2\left(\mathrm{u}_2(\mathrm{x},\mathrm{t})-\mathrm{u}_1(\mathrm{x},\mathrm{t})\right)\right|^2 h(x,t)dxdt$$
$$\leq \int_0^T\int_B \left[\frac{(u(x,t))^2}{2} - \sup|u(x,t)|\left(u_2(x,t)+u_1(x,t)\right)\right]\times$$
$$\left(\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)+\frac{\partial}{\partial t}h(x,t)\right)dxdt. \tag{5.60}$$

Proof. Let $u(x,t) = u_2(x,t) - u_1(x,t)$, and denote

$$I = \int_0^T\int_B \left|\mathrm{grad}^2\mathrm{u}(\mathrm{x},\mathrm{t})\right|^2 h(x,t)dxdt$$

and

$$I_j = \int_0^T\int_B \left(\frac{\partial^2}{\partial x_1^2}u(x,t)\right)^2 h(x,t)dxdt,\ j=1,\ldots,n. \tag{5.61}$$

We calculate

$$\begin{aligned} I_1 &= \int_0^T\int_B \frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)h(x,t)dxdt \\ &= -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial}{\partial x_1}\left(\frac{\partial^2}{\partial x_1^2}u(x,t)h(x,t)\right)dxdt \\ &= -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^3}{\partial x_1^3}u(x,t)h(x,t)dxdt \\ &\quad -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt. \end{aligned} \tag{5.62}$$

Again, using integration by parts on the first integral

$$\begin{aligned} I_1 &= \int_0^T\int_B u(x,t)\frac{\partial}{\partial x_1}[\frac{\partial^3}{\partial x_1^3}u(x,t)h(x,t)]dxdt \\ &\quad -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)]dxdt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T\int_B u(x,t)\frac{\partial^4}{\partial x_1^4}u(x,t)h(x,t)dxdt \\
&+ \int_0^T\int_B u(x,t)\frac{\partial^3}{\partial x_1^3}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt \\
&- \int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)]dxdt.
\end{aligned}
\tag{5.63}
$$

Now, integrating the middle integral of of above equation

$$
\begin{aligned}
&\int_0^T\int_B u(x,t)\frac{\partial^3}{\partial x_1^3}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt \\
&= -\int_0^T\int_B \frac{\partial^2}{\partial x_1^2}u(x,t)\left[\frac{\partial}{\partial x_1}u(x,t)\left(u(x,t)\frac{\partial}{\partial x_1}h(x,t)\right)\right]dxdt \\
&= -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)dxdt \\
&- \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt.
\end{aligned}
\tag{5.64}
$$

Now,

$$
\begin{aligned}
I_1 &= \int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_1^4}h(x,t)dxdt \\
&- \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&-2\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial}{\partial x_1}h(x,t)]dxdt \\
&= \int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_1^4}h(x,t)dxdt \\
&- \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&- \int_0^T\int_B \left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}u(x,t)\right)^2\right]\frac{\partial}{\partial x_1}h(x,t)dxdt.
\end{aligned}
\tag{5.65}
$$

Now, we evaluate integral

$$\begin{aligned}
&\int_0^T\int_B \left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}u(x,t)\right)^2\right]\frac{\partial}{\partial x_1}h(x,t)dxdt \\
&= -\int_0^T\int_B \frac{\partial}{\partial x_1}u(x,t)\left(\frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)\right)dxdt
\end{aligned} \tag{5.66}$$

$$= \int_0^T\int_B u(x,t)\left[\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)\right)\right]dxdt \tag{5.67}$$

$$\begin{aligned}
&= \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&+\int_0^T\int_B u(x,t)\frac{\partial}{\partial x_1}u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^3}{\partial x_1^3}h(x,t)dxdt \\
&= \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B \frac{\partial}{\partial x_1}(u(x,t))^2\frac{\partial^3}{\partial x_1^3}h(x,t)dxdt \\
&= \int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&-\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial^4}{\partial x_1^4}h(x,t)dxdt.
\end{aligned} \tag{5.68}$$

Finally, we have

$$\begin{aligned}
I_1 &= \int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_1^4}h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_1^2}u(x,t)\frac{\partial^2}{\partial x_1^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial^4}{\partial x_1^4}h(x,t)dxdt.
\end{aligned} \tag{5.69}$$

Similarly, calculating the values of $I_2,\ldots,I_n$, we finally obtain

$$
\begin{aligned}
I &= \sum_{i=1}^{n}\Bigg(\int_0^T\int_B u(x,t)\frac{\partial^4 u(x,t)}{\partial x_i^4}h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\frac{\partial^2}{\partial x_i^2}u(x,t)\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial^4}{\partial x_i^4}h(x,t)\Bigg)dxdt \\
&= \int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^4 u(x,t)}{\partial x_i^4}h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^{n}\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt.
\end{aligned}
\tag{5.70}
$$

Adding and subtracting $u(x,t)\frac{\partial}{\partial t}u(x,t)$ in first integral

$$
\begin{aligned}
I &= \int_0^T\int_B u(x,t)\left(\sum_{i=1}^{n}\frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)-\frac{\partial}{\partial t}u(x,t)\right)h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt \\
&+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^{n}\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt \\
&= \int_0^T\int_B u(x,t)\left(\sum_{i=1}^{n}\frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)\right)h(x,t)dxdt \\
&-\int_0^T\int_B u(x,t)\frac{\partial u(x,t)}{\partial t}h(x,t)dxdt \\
&-2\int_0^T\int_B u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt
\end{aligned}
$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt$$

$$=\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n \frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)\right)h(x,t)dxdt$$

$$-\frac{1}{2}\int_0^T\int_B \frac{\partial\,(u(x,t))^2}{\partial t}h(x,t)dxdt$$

$$-2\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt.$$

Integrating the second integral with respect to variable t

$$I=\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n \frac{\partial^4 u(x,t)}{\partial x_i^4}+\frac{\partial}{\partial t}u(x,t)\right)h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial}{\partial t}h(x,t)dxdt$$

$$-2\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt.$$

If we replace $u(x,t)=u_2(x,t)-u_1(x,t)$ in the first integral

$$I=\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n \frac{\partial^4\,(u_2(x,t)-u_1(x,t))}{\partial x_i^4}+\frac{\partial\,(u_2(x,t)-u_1(x,t))}{\partial t}\right)h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial}{\partial t}h(x,t)dxdt-2\int_0^T\int_B u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4}h(x,t)dxdt$$

$$=\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n \frac{\partial^4 u_2(x,t)}{\partial x_i^4}-\sum_{i=1}^n \frac{\partial^4 u_1(x,t)}{\partial x_i^4}+\frac{\partial\,(u_2(x,t))}{\partial t}-\frac{\partial\,(u_2(x,t))}{\partial t}\right)h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\,\frac{\partial}{\partial t}h(x,t)dxdt-2\int_0^T\int_B u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt$$

$$=\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}-\sum_{i=1}^n\frac{\partial^4 u_1(x,t)}{\partial x_i^4}+\frac{\partial\,(u_2(x,t))}{\partial t}-\frac{\partial\,(u_1(x,t))}{\partial t}\right)h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\,\frac{\partial}{\partial t}h(x,t)dxdt-2\int_0^T\int_B u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt$$

$$=\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}+\frac{\partial\,(u_2(x,t))}{\partial t}\right)h(x,t)dxdt$$

$$-\int_0^T\int_B u(x,t)\left(\sum_{i=1}^n\frac{\partial^4 u_1(x,t)}{\partial x_i^4}+\frac{\partial\,(u_1(x,t))}{\partial t}\right)h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\,\frac{\partial}{\partial t}h(x,t)dxdt$$

$$-2\int_0^T\int_B u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt$$

$$\int_0^T\int_B \left|\mathrm{grad}^2\,(\mathrm{u}_2(\mathrm{x},\mathrm{t})-\mathrm{u}_1(\mathrm{x},\mathrm{t}))\right|^2 h(x,t)dxdt$$

$$\le |u(x,t)|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}+\frac{\partial\,(u_2(x,t))}{\partial t}\right|h(x,t)dxdt$$

$$+|u(x,t)|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^4 u_1(x,t)}{\partial x_i^4}+\frac{\partial\,(u_1(x,t))}{\partial t}\right|h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial}{\partial t}h(x,t)dxdt+2\left|u(x,t)\right|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\right|\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)\right|dxdt$$
$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt$$
$$\leq \sup\left|u(x,t)\right|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}+\frac{\partial\left(u_2(x,t)\right)}{\partial t}\right|h(x,t)dxdt$$
$$+\sup\left|u(x,t)\right|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^4 u_1(x,t)}{\partial x_i^4}+\frac{\partial\left(u_1(x,t)\right)}{\partial t}\right|h(x,t)dxdt$$
$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\frac{\partial}{\partial t}h(x,t)dxdt+2\sup\left|u(x,t)\right|\int_0^T\int_B\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\right|\left|\sum\frac{\partial^2}{\partial x_i^2}h(x,t)\right|dxdt$$
$$+\frac{1}{2}\int_0^T\int_B (u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt. \tag{5.71}$$

Using theorem assumptions

$$\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}+\frac{\partial\left(u_2(x,t)\right)}{\partial t}\geq 0,\ \sum_{i=1}^n\frac{\partial^4 u_1(x,t)}{\partial x_i^4}+\frac{\partial\left(u_1(x,t)\right)}{\partial t}\geq 0$$

and

$$\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\geq 0,\ \sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)\leq 0.$$

Now

$$\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u(x,t)\right|=\left|\sum_{i=1}^n\frac{\partial^2\left(u_2(x,t)-u_1(x,t)\right)}{\partial x_i^2}\right|$$
$$\leq\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_2(x,t)\right|+\left|\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_1(x,t)\right|$$
$$=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_2(x,t)+\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_1(x,t),$$

and then,

$$\int_0^T\int_B\left|\mathrm{grad}^2\left(\mathrm{u}_2(\mathrm{x,t})-\mathrm{u}_1(\mathrm{x,t})\right)\right|^2 h(x,t)dxdt$$
$$\leq \sup\left|u(x,t)\right|\int_0^T\int_B\left(\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}+\frac{\partial\left(u_2(x,t)\right)}{\partial t}\right)h(x,t)dxdt$$

$$+\sup|u(x,t)|\int_0^T\int_B\left(\sum_{i=1}^n\frac{\partial^4 u_1(x,t)}{\partial x_i^4}+\frac{\partial\,(u_1(x,t))}{\partial t}\right)h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B(u(x,t))^2\frac{\partial}{\partial t}h(x,t)dxdt$$

$$-2\sup|u(x,t)|\int_0^T\int_B\left(\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_2(x,t)+\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_1(x,t)\right)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B(u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt$$

$$\int_0^T\int_B\left|\mathrm{grad}^2\left(\mathrm{u}_2(\mathrm{x},\mathrm{t})-\mathrm{u}_1(\mathrm{x},\mathrm{t})\right)\right|^2h(x,t)dxdt$$

$$\le\sup|u(x,t)|\int_0^T\int_B\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}h(x,t)dxdt+\sup|u(x,t)|\int_0^T\int_B\frac{\partial\,(u_2(x,t))}{\partial t}h(x,t)dxdt$$

$$+\sup|u(x,t)|\int_0^T\int_B\sum_{i=1}^n\frac{\partial^4 u_1(x,t)}{\partial x_i^4}h(x,t)dxdt+\sup|u(x,t)|\int_0^T\int_B\frac{\partial\,(u_1(x,t))}{\partial t}h(x,t)dxdt$$

$$+\frac{1}{2}\int_0^T\int_B(u(x,t))^2\frac{\partial}{\partial t}h(x,t)dxdt-2\sup|u(x,t)|\int_0^T\int_B\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_2(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt$$

$$-2\sup|u(x,t)|\int_0^T\int_B\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}u_1(x,t)\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}h(x,t)dxdt+\frac{1}{2}\int_0^T\int_B(u(x,t))^2\sum_{i=1}^n\frac{\partial^4}{\partial x_i^4}h(x,t)dxdt.$$

In the integral

$$\int_0^T\int_B\sum_{i=1}^n\frac{\partial^4 u_2(x,t)}{\partial x_i^4}h(x,t)dxdt \tag{5.72}$$

we apply integration by parts four times and use definition of weight function

$$\int_0^T\int_B u_2(x,t)\sum_{i=1}^n\frac{\partial^4 h(x,t)}{\partial x_i^4}dxdt. \tag{5.73}$$

Similarly

$$\int_0^T\int_B\frac{\partial\,(u_2(x,t))}{\partial t}h(x,t)dxdt=-\int_0^T\int_B u_2(x,t)\frac{\partial h(x,t)}{\partial t}dxdt$$

and

$$\int_0^T \int_B \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u_2(x,t) \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} h(x,t) dxdt = \int_0^T \int_B u_2(x,t) \sum_{i=1}^n \frac{\partial^4}{\partial x_i^4} h(x,t) dxdt.$$

Finally,

$$\begin{aligned}
&\int_0^T \int_B \left|\mathrm{grad}^2 \left(\mathrm{u}_2(\mathrm{x},\mathrm{t}) - \mathrm{u}_1(\mathrm{x},\mathrm{t})\right)\right|^2 h(x,t) dxdt \\
&\leq -\sup|u(x,t)| \int_0^T \int_B u_2(x,t) \left(\sum_{i=1}^n \frac{\partial^4 h(x,t)}{\partial x_i^4} + \frac{\partial\,(h(x,t))}{\partial t} \right) dxdt \\
&-\sup|u(x,t)| \int_0^T \int_B u_1(x,t) \left(\sum_{i=1}^n \frac{\partial^4 h(x,t)}{\partial x_i^4} + \frac{\partial h(x,t)}{\partial t} \right) dxdt \\
&+\frac{1}{2} \int_0^T \int_B (u(x,t))^2 \left(\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4} h(x,t) + \frac{\partial}{\partial t} h(x,t) \right) dxd \\
&= \int_0^T \int_B \left[\frac{(u(x,t))^2}{2} - \sup|u(x,t)|\,(u_2(x,t)+u_1(x,t)) \right] \left(\sum_{i=1}^n \frac{\partial^4}{\partial x_i^4} h(x,t) + \frac{\partial}{\partial t} h(x,t) \right) dxdt.
\end{aligned}$$

□

Theorem 5.8 *Every continuous weak solution $u(x,t)$ of* beam equation *has weak partial derivative* $\frac{\partial^2 u(x,t)}{\partial x_i^2}$ $i = 1,2,\ldots,n$ *in the clynder* $Q \subseteq \mathbb{R}^n$ *and also they are weighted square integrable i.e*

$$\int_Q \left|\mathrm{grad}^2 \mathrm{u}(\mathrm{x},\mathrm{t})\right|^2 h(x,t) dxdt < \infty,$$

where $h(x,t)$ is non-negative weight function having compact support.

Proof. Take mollification $u_\varepsilon(x,t)$ of weak solution $u(x,t)$ of *beam equation*.
It is proved in Evan's [14], that on every compact sub-cylinder $Q_k,\ k \in \mathbb{N}$, for continuous function $u(x,t)$ we have the following convergence

$$\sup_{(x,t)\in Q_k} |u_m(x,t) - u(x,t)| \xrightarrow{m\to\infty} 0.$$

By the definition of clynder Q_k, it is clear that Q_k are completely embedded in the Q. By previous theorem for any $k \in \mathbb{N}$, there exist $m(k) \in \mathbb{N}$ such that each function $u_m(x,t)$ is smooth solution of the beam equation in Q_k for $m \geq m(k)$.

Now we write the inequality (5.74) for $u_m(x,t)$ and $u_p(x,t)$ instead of $u_1(x,t)$ and $u_2(x,t)$, respectively, on the clynder Q_{k+l}

$$\begin{aligned}
&\int\int_{Q_{k+l}} \left|\mathrm{grad}^2(\mathrm{u_p(x,t)}) - \mathrm{grad}^2(\mathrm{u_m(x,t)})\right|^2 h_{k+l}(x,t)dxdt \\
&\leq \left\|u_p - u_m\right\|_{L^\infty_{(Q_{k+l})}} \left(\left\|u_p\right\|_{L^\infty_{(Q_{k+l})}} + \|u_m\|_{L^\infty_{(Q_{k+l})}}\right) \int\int_{Q_{k+l}} |Lh_{k+l}(x,t)|\, dxdt \\
&+ \frac{1}{2}\left\|u_p - u_m\right\|^2_{L^\infty_{(Q_{k+l})}} \int\int_{Q_{k+l}} |Lh_{k+l}(x,t)|\, dxdt.
\end{aligned}$$

Let us denote

$$\begin{aligned}
c_{k+l} &= \int\int_{Q_{k+l}} |Lh_{k+l}(x,t)|\, dxdt, \\
\widetilde{c}_{k+l} &= \int\int_{Q_{k+l}} |Lh_{k+l}(x,t)|\, dxdt, \\
\hat{c}_{k+l} &\equiv \inf_{(x,t)\in Q_k} h_{k+l}(x,t) > 0.
\end{aligned}$$

Now we observe

$$\begin{aligned}
&\hat{c}_{k+l}\int\int_{Q_{k+l}} \left|\mathrm{grad}\,(u_p(x,t)) - \mathrm{grad}\,(\mathrm{u_m(x,t)})\right|^2 h_{k+l}(x,t)dx \qquad (5.74)\\
&\leq c_{k+l}\left(\left\|u_p - u_m\right\|_{L^\infty_{(Q_{k+l})}}\right)\left(\left\|u_p\right\|_{L^\infty_{(Q_{k+l})}} + \|u_m\|_{L^\infty_{(Q_{k+l})}}\right) + \frac{1}{2}\widetilde{c}_{k+l}\left\|u_p - u_m\right\|^2_{L_{(Q_{k+l})}}.
\end{aligned}$$

From (5.74), we have

$$\left\|u_p - u_m\right\|_{L^\infty_{(Q_{k+l})}} \to 0 \quad as\ m,p \to \infty$$

Letting $m,p \to \infty$, we obtain

$$\lim_{m,p\to\infty} \sum_{i=1}^{n} \int\int_{Q_k} \left(\frac{\partial u_p(x,t)}{\partial x_i} - \frac{\partial u_m(x,t)}{\partial x_i}\right)^2 dxdt = 0.$$

Since $L^2(Q_k)$ is complete, above sequence will converge. So there exist a class of functions $v_{k,i}(x,t) \in L^2(Q_k), \quad i = 1, ldots, n$ such that $v_{k,i}$ is measurable and satisfying

$$\sum_{i=1}^{n} \int_{Q_k} \left(\frac{\partial u_m(x,t)}{\partial x_i} - v_{k,i}(x,t)\right)^2 dxdt \xrightarrow{m\to\infty} 0, \quad k \in \mathbb{N}.$$

Now we trivially extended $v_{k,i}(x,t)$for the whole Q by zero outside of Q_k.
Denote

$$v_i(x,t) = \limsup_{k\to\infty}\, v_{k,i}(x,t),\ i = 1,2,\ldots,n.$$

It is clear that $v_i(x,t) = v_{k,l}(x,t)$, $a.e$, on the Q_k. Thus $v_i(x,T)$, $i = 1,2,\ldots,n$ are locally integrable on the ball Q_k, and they represents the weak partial derivatives of $u(x,t)$ with respect to x_i, $i = 1,2,\ldots,n$ respectively.
To see this take $\phi \in C_0^\infty(Q)$, then suppose supp $\phi \subset Q_k$, for some $k \in \mathbb{N}$.

$$\int\limits_{Q_k} \frac{\partial u_m(x,t)}{\partial x_i} \phi(x,t)dxdt = -\int\limits_{Q_k} u_m(x,t) \frac{\partial}{\partial x_i} \phi(x,t)dxdt,$$

for any $m \geq m(k)$.
Since

$$\|u_m - u\|_{Q_k} \xrightarrow{m\to\infty} 0,$$

and

$$\left\| \frac{\partial u_m}{\partial x_i} - v_i \right\|_{L^2_{Q_k}} \xrightarrow{m\to\infty} 0,$$

so above becomes

$$\int\limits_{Q_k} v_i(x,t)\phi(x,t)dxdt = -\int\limits_{Q_k} u(x,t)\frac{\partial \phi}{\partial x_i}(x,t)dxdt,$$

concluding that $v_i$are weak partial derivative of u.
Similarly, we have

$$\int\limits_{Q_{k+l}} |\mathrm{grad}\, u_m(x,t)|^2 h_{k+l}(x,t)dxdt \leq \|u_m\|_{L^\infty_{B_{k+l}}} c_{k+l} + \frac{1}{2} \|u_m\|_{L^\infty_{B_{k+l}}} \widetilde{c}_{k+l}.$$

Letting limit $m \to \infty$, we have

$$\int\limits_{Q_{k+l}} |\mathrm{grad}\, u(x,t)|^2 h_{k+l}(x,t)dxdt \leq \|u\|^2_{L^\infty_{Q_k}} (c_{k+l} + \frac{1}{2}\widetilde{c}_{k+l}).$$

Since $Q_k \subseteq Q_{k+l}$,

$$\int\limits_{Q_k} |\mathrm{grad}\, u(x,t)|^2 h_{k+l}(x,t)dxdt \leq \|u\|^2_{L^\infty_{Q_k}} (c_{k+l} + \frac{1}{2}\widetilde{c}_{k+l}).$$

If we let $l \longrightarrow \infty$, we obtain

$$\int\limits_{Q_{k+l}} |\mathrm{grad}\, u(x,t)|^2 h(x,t)dxdt \leq \|u\|^2_{L^\infty_{Q_k}} (c_\infty + \frac{1}{2}\widetilde{c}_\infty) < \infty.$$

Since above integer is bonded from every $k \in \mathbb{N}$, we have

$$\int_Q |\mathrm{grad}\, u(x,t)|^2\, h(x,t)dx < \infty.$$

□

Theorem 5.9 *Let u(x,t) be the convex, continious weak subsolution of beam equation. Then the following is valid*

$$\int_Q |\ \mathrm{grad}\, u_2(x,t) - \mathrm{grad}\, \mathrm{u_2(x)}\ |^2\ \mathrm{h(x,t)dx} \leq$$
$$\left[\frac{1}{2} \parallel u_2 - u_1 \parallel^2_{L_\infty(Q)} + \parallel u_2 - u_1 \parallel_{L_\infty(Q)} (\parallel u_1 \parallel_{L_\infty(Q)} + \parallel u_2 \parallel_{L_\infty(Q)})\right] \int_Q |\ \Delta h(x,t)\ |\ dxdt,$$

where $h(x,t)$ is the weight function defined in (5.8)*.*

Proof. We take $u_{m,i}(x,t), i = 1,2$, the mollification of weak subharmonic functions $u_i(x,t),\ i = 1,\ 2$.
By the definition of mollification, we know that for a cylinder Q_{k+l}, there exist integer m_{k+l} such that each function $u_{m,i}, i = 1,2$ is smooth subharmonic function on the ball Q_{k+l} if $m \geq m_{k+l}$.
Also we have the following convergence

$$\parallel u_{m,i} - u_i \parallel \xrightarrow{m\to\infty} 0,\ \ i = 1,2.$$

Now we write the inequality (2.4) for the functions $u_{m,1}(x,t)$ and $u_{m,2}(x,t)$ for the cylinder $Qk + l$. We have

$$\int_{Q_{k+l}} |\ \mathrm{grad}\, u_{m,2}(x,t)\ -\ \mathrm{grad}\, \mathrm{u_{m,1}(x,t)}\ |^2\ \mathrm{h_{k+1}(x,t)dxdt}$$
$$\leq\ \alpha_{k+l}[\frac{1}{2} \parallel u_{m,2} - u_{m,1} \parallel^2_{L_\infty(Q_{k+l})} + (\parallel u_{m,2} - u_{m,1} \parallel_{L_\infty(Q_{k+l})})$$
$$\times(\parallel u_{m,1} \parallel_{L_\infty(Q_{k+l})} + \parallel u_{m,2} \parallel_{L_\infty(Q_{k+l})})]. \tag{5.75}$$

Passing to the limit $m \to \infty$, we obtain

$$\int_{Q_{k+l}} |\ \mathrm{grad}\, u_2(x,t)\ -\ \mathrm{grad}\, \mathrm{u_1(x,t)}\ |^2\ \mathrm{h_{k+1}(x,t)dxdt}$$
$$\leq\ \alpha_{k+l}[\frac{1}{2} \parallel u_2 - u_1 \parallel^2_{L_\infty(Q_{k+l})} + (\parallel u_2 - u_1 \parallel_{L_\infty(Q_{k+l})})$$
$$\times(\parallel u_1 \parallel_{L_\infty(Q_{k+l})} + \parallel u_2 \parallel_{L_\infty(Q_{k+l})})]. \tag{5.76}$$

Since $Q_k \subseteq Q_{k+l}$, so writing the left hand side for the smaller ball and passing to the limit $l \longrightarrow \infty$, the above becomes

$$\int\limits_{Q_k} |\operatorname{grad} u_2(x,t) - \operatorname{grad} \mathrm{u}_1(\mathrm{x},\mathrm{t})|^2 \, \mathrm{h}_\infty(\mathrm{x},\mathrm{t})\mathrm{dx} \quad \leq c_\infty[\frac{1}{2} \| u_2 - u_1 \|^2_{L_\infty(Q)} \tag{5.77}$$

$$+(\| u_2 - u_1 \|_{L_\infty(Q)})(\| u_1 \|_{L_\infty(Q)} + \| u_2 \|_{L_\infty(Q)})].$$

By the Theorem 5.8, we have

$$\int_Q |\operatorname{grad} u_i(x,t)|^2 \, h(x,t)dxdt < \infty, \ i = 1, \ 2.$$

Passing to the limit as $k \longrightarrow \infty$, we obtain the required result. □

5.3 The weighted energy estimates for the smooth and weak sub-solutions of forth order partial differential equations

In this chapter we will develop the weighted energy estimates for the smooth and weak subsolution for the fourth order partial differential equation

$$\frac{\partial^4 u}{\partial x_1^2 \partial y_1^2} + \frac{\partial^4 u}{\partial x_2^2 \partial y_2^2} + \ldots + \frac{\partial^4 u}{\partial x_n^2 \partial y_n^2} = 0. \tag{5.78}$$

Also we calculate the estimates and some important differentiability properties of weak sub-solution of (5.78).
Let us define a linear operator

$$L = \frac{\partial^4}{\partial x_1^2 \partial y_1^2} + \frac{\partial^4}{\partial x_2^2 \partial y_2^2} + \ldots + \frac{\partial^4}{\partial x_n^2 \partial y_n^2}. \tag{5.79}$$

Now (5.78) becomes

$$Lu(x,y) = 0.$$

The smooth function $v(x,y), \ x,y \in \mathbb{R}^n$ is called smooth solution of (5.78) if

$$Lv(x,y) = 0.$$

And the function $v(x,y)$ is called smooth sub-solution (super-solution) of (5.78) if

$$Lv(x,y) \geq (\leq) 0. \tag{5.80}$$

The operator L is self-adjoint, i.e. $L = L^*$.
The continuous function $v(x,y)$ is called weak sub-solution of (5.78) if

$$\int v(x,y)L^*\phi(x,y)dxdy = 0, \tag{5.81}$$

for every $\phi(x,y) \in C_c^\infty(B)$.
In the next section we developed the result for smooth sub-solution. And also we mollify the weak sub-solution by smooth ones.In the last section we deal with weak sub-solution.

5.3.1 The weight energy inequality for smooth sub-solution and approximation of weak sub-solution

Theorem 5.10 *Let $u(x,y)$ is the smooth sub-solution of*

$$\frac{\partial^4 u}{\partial x_1^2 \partial y_1^2} + \frac{\partial^4 u}{\partial x_2^2 \partial y_2^2} + ... + \frac{\partial^4 u}{\partial x_n^2 \partial y_n^2} = 0, \tag{5.82}$$

such that

$$u_{x_i y_j} \geq 0, \ \ i,j = 1,\dots,n.$$

Then the following is valid:

$$\int_{B_1}\int_{B_2} |\mathrm{grad}_{xy} u(x,y)|^2 h(x,y)dxdy \leq \int_{B_1}\int_{B_2} \left(3u(x,y)\sup|u| + \frac{u^2(x,y)}{2}\right) Lh(x,y)dxdy.$$

Proof.

$$J = \int_{B_1}\int_{B_2} |\mathrm{grad}_{xy} u|^2 h(x,y)dxdy = \sum_{i=1}^{n} \int_{B_1}\int_{B_2} \left(\frac{\partial^2 u}{\partial x_i \partial y_i}\right)^2 h(x,y)dxdy. \tag{5.83}$$

Denote

$$J_i = \int_{B_1}\int_{B_2} \left(\frac{\partial^2 u}{\partial x_i \partial y_i}\right)^2 h(x,y)dxdy, \ i = 1,\dots,n.$$

Now, using integration by parts with respect to y_1,

$$\begin{aligned} J_1 \ &= \int_{B_1}\int_{B_2} \left(\frac{\partial^2 u}{\partial x_1 \partial y_1}\right)^2 h(x,y)dxdy = \int_{B_1}\int_{B_2} \frac{\partial^2 u}{\partial x_1 \partial y_1} \left(\frac{\partial^2 u}{\partial x_1 \partial y_1} h(x,y)\right) dxdy \\ &= -\int_{B_1}\int_{B_2} u_{x_1}(u_{x_1 y_1} h(x,y))_{y_1} dxdy \\ &= -\int_{B_1}\int_{B_2} u_{x_1} u_{x_1 y_1 y_1} h(x,y)dxdy - \int_{B_1}\int_{B_2} u_{x_1} u_{x_1 y_1}(x,y)h_{y_1} dxdy \end{aligned}$$

$$= -\int_{B_1}\int_{B_2} u_{x_1}u_{x_1y_1y_1}h(x,y)dxdy - \frac{1}{2}\int_{B_1}\int_{B_2}[(u_{x_1}^2)]_{y_1}h_{y_1}(x,y)dxdy. \tag{5.84}$$

Take the first integral of (5.84), and using formula of integration by parts, we get

$$\begin{aligned}
&-\int_{B_1}\int_{B_2} u_{x_1}u_{x_1y_1y_1}h(x,y)dxdy = \int_{B_1}\int_{B_2} u[u_{x_1y_1y_1}h(x,y)]_{x_1}dxdy\\
&= \int_{B_1}\int_{B_2} u[u_{x_1x_1y_1y_1}h(x,y) + u_{x_1y_1y_1}h_{x_1}(x,y)]dxdy\\
&= \int_{B_1}\int_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy + \int_{B_1}\int_{B_2} uu_{x_1y_1y_1}h_{x_1}(x,y)dxdy\\
&= \int_{B_1}\int_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy + \int_{B_1}\int_{B_2} u_{x_1y_1y_1}[uh_{x_1}(x,y)]dxdy.
\end{aligned} \tag{5.85}$$

Using integration by parts formula with respect to y_1 on second integral of (5.85), we have

$$\begin{aligned}
&= \int_{B_1}\int_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy - \int_{B_1}\int_{B_2} u_{x_1y_1}[uh_{x_1}(x,y)]_{y_1}dxdy\\
&= \int_{B_1}\int_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy - \int_{B_1}\int_{B_2} u_{x_1y_1}[u_{y_1}h_{x_1}(x,y) + uh_{x_1y_1}(x,y)]dxdy\\
&= \int_{B_1}\int_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy - \int_{B_1}\int_{B_2} u_{y_1}u_{x_1y_1}h_{x_1}(x,y)dxdy\\
&- \int_{B_1}\int_{B_2} uu_{x_1y_1}h_{x_1y_1}dxdy\\
&= \int_{B_1}\int_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy - \frac{1}{2}\int_{B_1}\int_{B_2}[u_{y_1}^2]_{x_1}h_{x_1}(x,y)dxdy\\
&- \int_{B_1}\int_{B_2} uu_{x_1y_1}h_{x_1y_1}(x,y)dxdy.
\end{aligned} \tag{5.86}$$

In the second integral of (5.86) we use integration by parts with respect to x_1 and then respect to y_1

$$\begin{aligned}
&-\frac{1}{2}\int_{B_1}\int_{B_2}[u_{y_1}^2]_{x_1}h_{x_1}(x,y)dxdy = \frac{1}{2}\int_{B_1}\int_{B_2} u_{y_1}^2h_{x_1x_1}(x,y)dxdy\\
&= \frac{1}{2}\int_{B_1}\int_{B_2} u_{y_1}(x,y)[u_{y_1}h_{x_1x_1}(x,y)]dxdy\\
&= -\frac{1}{2}\int_{B_1}\int_{B_2} u(x,y)[u_{y_1}h_{x_1x_1}]_{y_1}dxdy
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u(x,y)[u_{y_1y_1}h_{x_1x_1} + u_{y_1}h_{x_1x_1y_1}] \\
&= -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{y_1y_1}h_{x_1x_1}(x,y) - \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{y_1}h_{x_1x_1y_1}dxdy \\
&= -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{y_1y_1}h_{x_1x_1}(x,y)dxdy - \frac{1}{2}\cdot\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2}[u^2]_{y_1}h_{x_1x_1y_1}dxdy. \qquad (5.87)
\end{aligned}
$$

Take into account second integral of (5.87)

$$
-\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2}[u^2_{y_1}]_{x_1}h_{x_1}dxdy = -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{y_1y_1}h_{x_1x_1}(x,y)dxdy + \frac{1}{4}\int\limits_{B_1}\int\limits_{B_2} u^2h_{x_1x_1y_1y_1}dxdy. \qquad (5.88)
$$

Now using (5.88) in (5.84), we get

$$
\begin{aligned}
J_1 &= \int\limits_{B_1}\int\limits_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy - \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{y_1y_1}h_{x_1x_1}(x,y)dxdy \\
&+ \frac{1}{4}\int\limits_{B_1}\int\limits_{B_2} u^2h_{x_1x_1y_1y_1}(x,y)dxdy - \int\limits_{B_1}\int\limits_{B_2} uu_{x_1y_1}h_{x_1y_1}dxdy \\
&= \int\limits_{B_1}\int\limits_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy - \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{y_1y_1}h_{x_1x_1}dxdy \\
&+ \frac{1}{4}\int\limits_{B_1}\int\limits_{B_2} u^2h_{x_1x_1y_1y_1}dxdy - \int\limits_{B_1}\int\limits_{B_2} uu_{x_1y_1}h_{x_1y_1}dxdy - \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2}[u^2_{x_1}]_{y_1}h_{y_1}dxdy. \qquad (5.89)
\end{aligned}
$$

On the fifth integral of (5.89), we apply integration by parts over y_1

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2}[u^2_{x_1}]_{y_1}h_{y_1}(x,y)dxdy = \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u^2_{x_1}h_{y_1y_1}dxdy \\
&= \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u_{x_1}[u_{x_1}h_{y_1y_1}]dxdy \\
&= -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u[u_{x_1}h_{y_1y_1}]_{x_1}dxdy \\
&= -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u[u_{x_1x_1}h_{y_1y_1} + u_{x_1}h_{x_1y_1y_1}]dxdy \\
&= -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{x_1x_1}h_{y_1y_1}dxdy - \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{x_1}h_{x_1y_1y_1}dxdy \\
&= -\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} uu_{x_1x_1}h_{y_1y_1}dxdy - \frac{1}{2}\cdot\frac{1}{2}\int\limits_{B_1}\int\limits_{B_2}[u^2]_{x_1}h_{x_1y_1y_1}dxdy
\end{aligned}
$$

$$= -\frac{1}{2}\int_{B_1}\int_{B_2} uu_{x_1x_1}h_{y_1y_1}dxdy + \frac{1}{4}\int_{B_1}\int_{B_2} u^2 h_{x_1x_1y_1y_1}dxdy. \tag{5.90}$$

Using (5.90) in (5.84), we get

$$\begin{aligned} J_1 \;=\; & \int_{B_1}\int_{B_2} uu_{x_1x_1y_1y_1}h(x,y)dxdy - \frac{1}{2}\int_{B_1}\int_{B_2} uu_{y_1y_1}h_{x_1x_1}(x,y)dxdy \\ + \; & \frac{1}{2}\int_{B_1}\int_{B_2} u^2 h_{x_1x_1y_1y_1}(x,y)dxdy - \int_{B_1}\int_{B_2} uu_{x_1y_1}h_{x_1y_1}(x,y)dxdy \\ - \; & \frac{1}{2}\int_{B_1}\int_{B_2} uu_{x_1x_1}h_{y_1y_1}(x,y)dxdy. \end{aligned} \tag{5.91}$$

Similarly, solving other integrals of (5.84), we get

$$\begin{aligned} I \;=\; & \sum_{i=1}^{n}\Big[\int_{B_1}\int_{B_2} uu_{x_ix_iy_iy_i}h(x,y)dxdy - \frac{1}{2}\int_{B_1}\int_{B_2} uu_{y_iy_i}h_{x_ix_i}(x,y)dxdy \\ + \; & \frac{1}{2}\int_{B_1}\int_{B_2} u^2 h_{x_ix_iy_iy_i}h(x,y)dxdy - \int_{B_1}\int_{B_2} uu_{x_iy_i}h_{x_iy_i}(x,y)dxdy \\ - \; & \frac{1}{2}\int_{B_1}\int_{B_2} uu_{x_ix_i}h_{y_iy_i}(x,y)dxdy \\ = & \int_{B_1}\int_{B_2} u\sum_{i=1}^{n} u_{x_ix_iy_iy_i}h(x,y)dxdy - \frac{1}{2}\int_{B_1}\int_{B_2} u\sum_{i=1}^{n} u_{y_iy_i}h_{x_ix_i}(x,y)dxdy \\ + & \frac{1}{2}\int_{B_1}\int_{B_2} u^2\sum_{i=1}^{n} h_{x_ix_iy_iy_i}(x,y)dxdy - \int_{B_1}\int_{B_2} u\sum_{i=1}^{n} u_{x_iy_i}h_{x_iy_i}(x,y)dxdy \\ - & \frac{1}{2}\int_{B_1}\int_{B_2} u\sum_{i=1}^{n} u_{x_ix_i}h_{y_iy_i}(x,y)dxdy. \end{aligned} \tag{5.92}$$

Taking the modulus of (5.92), we have

$$\begin{aligned} I \leq & \int_{B_1}\int_{B_2} |u||\sum_{i=1}^{n} u_{x_ix_iy_iy_i}||h(x,y)|dxdy + \frac{1}{2}\int_{B_1}\int_{B_2} |u||\sum_{i=1}^{n} u_{y_iy_i}||h_{x_ix_i}(x,y)|dxdy \\ + \; & \frac{1}{2}\int_{B_1}\int_{B_2} u^2|\sum_{i=1}^{2} h_{x_ix_iy_iy_i}(x,y)|dxdy + \int_{B_1}\int_{B_2} |u||\sum_{i=1}^{n} u_{x_iy_i}||h_{x_iy_i}dxdy \\ + \; & \frac{1}{2}\int_{B_1}\int_{B_2} |u||\sum_{i=1}^{n} u_{x_ix_i}||h_{y_iy_i}(x,y)|dxdy. \end{aligned} \tag{5.93}$$

By using definition of $L-$operator, we have

$$
\begin{aligned}
I \leq\ & \sup|u| \int\limits_{B_1}\int\limits_{B_2} |Lu||h(x,y)|dxdy \\
+\ & \frac{1}{2}\sup|u| \int\limits_{B_1}\int\limits_{B_2} |\sum_{i=1}^{n} u_{y_iy_i}||h_{x_ix_i}(x,y)|dxdy + \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u^2(x,y)|Lh(x,y)|dxdy \qquad (5.94) \\
+\ & \sup|u| \int\limits_{B_1}\int\limits_{B_2} |\sum_{i=1}^{n} u_{x_iy_i}(x,y)||h_{x_iy_i}|dxdy + \frac{1}{2}\sup|u| \int\limits_{B_1}\int\limits_{B_2} |\sum_{i=1}^{n} u_{x_ix_i}(x,y)||h_{y_iy_i}|dxdy.
\end{aligned}
$$

Since, $u_{x_ix_i} \geq 0, u_{x_iy_i} \geq 0,\ u_{y_iy_i} \geq 0$ and $h_{x_ix_i} \geq 0,\ h_{y_iy_i} \geq 0,\ h_{x_iy_i} \geq 0$,

$$
\begin{aligned}
I \leq\ & \sup|u| \int\limits_{B_1}\int\limits_{B_2} L(u)h(x,y)dxdy + \frac{1}{2}\sup|u| \int\limits_{B_1}\int\limits_{B_2} \sum_{i=1}^{n} u_{y_iy_i}(x,y)h_{x_ix_i}(x,y)dxdy \\
+\ & \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u^2 L(h)dxdy + \sup|u| \int\limits_{B_1}\int\limits_{B_2} \sum_{i=1}^{n} u_{x_iy_i}(x,y)h_{x_iy_i}(x,y)dxdy \\
+\ & \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} \sum_{i=1}^{n} u_{x_ix_i}(x,y)h_{y_iy_i}(x,y)dxdy \\
\leq\ & \sup|u| \int\limits_{B_1}\int\limits_{B_2} L(u)h(x,y)dxdy + \frac{1}{2}\sup|u| \sum_{i=1}^{n}\int\limits_{B_1}\int\limits_{B_2} u(x,y)h_{x_ix_iy_iy_i}(x,y)dxdy \\
+\ & \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u^2(x,y)L(h)dxdy + \sup|u| \sum_{i=1}^{n}\int\limits_{B_1}\int\limits_{B_2} u(x,y)h_{x_ix_iy_iy_i}(x,y)dxdy \\
+\ & \frac{1}{2}\sup|u| \sum_{i=1}^{n}\int\limits_{B_1}\int\limits_{B_2} u(x,y)h_{x_ix_iy_iy_i}(x,y)dxdy \\
\leq\ & \sup|u| \int\limits_{B_1}\int\limits_{B_2} L(u)h(x,y)dxdy \\
+\ & 2\sup|u| \int\limits_{B_1}\int\limits_{B_2} u(x,y)\sum_{i=1}^{n} h_{x_ix_iy_iy_i}(x,y)dxdy + \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u^2(x,y)L(h)dxdy \\
\leq\ & \sup|u| \int\limits_{B_1}\int\limits_{B_2} L(u)h(x,y)dxdy + 2\sup|u| \int\limits_{B_1}\int\limits_{B_2} uL[h(x,y)]dxdy \\
+\ & \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u^2(x,y)L[h(x,y)]dxdy. \qquad (5.95)
\end{aligned}
$$

By using Gauss-Green Theorem

$$
I \leq \sup|u| \int\limits_{B_1}\int\limits_{B_2} u(x,y)L[h(x,y)]dxdy + 2\sup|u| \int\limits_{B_1}\int\limits_{B_2} u(x,y)L[h(x,y)]dxdy + \frac{1}{2}\int\limits_{B_1}\int\limits_{B_2} u^2(x,y)L[h(x,y)]dxd
$$

$$\leq 3\sup|u|\int_{B_1}\int_{B_2} u(x,y)L[h(x,y)]dxdy+\frac{1}{2}\int_{B_1}\int_{B_2} u^2(x,y)L[h(x,y)]dxdy. \quad (5.96)$$

Hence,

$$\int_{B_1}\int_{B_2} |\mathrm{grad}_{xy}u(x,y)|^2 h(x,y)dxdy \leq \int_{B_1}\int_{B_2}[3\sup|u|u(x,y)+\frac{u^2}{2}]Lh(x,y)dxdy. \quad (5.97)$$

□

The next result will give the similar inequality for the difference of smooth sub-solutions.

5.3.2 Existence of second order weak derivative and energy inequality for weak sub-solution

The next result tells us that if two sub-solution are closed in L_∞-norm, then the grad_{xy} is also closed in weighted L^2-norm.

Theorem 5.11 *Let $u_i(x,y), i=1,2$ be two smooth sub-solutions of*

$$\frac{\partial^4 u}{\partial x_1^2\partial y_1^2}+\frac{\partial^4 u}{\partial x_2^2\partial y_2^2}+...+\frac{\partial^4 u}{\partial x_n^2\partial y_n^2}=0.$$

such that

$$(u_1)_{x_iy_j}\geq 0,\ (u_2)_{x_iy_j}\geq 0,\ i,j=1,\dots,n.$$

Then the following hold:

$$\int_{B_1}\int_{B_2} |\mathrm{grad}_{xy}u_2(x,y)-\mathrm{grad}_{xy}u_1(x,y)|^2 h(x,y)dxdy$$
$$\leq \int_{B_1}\int_{B_2}[3\sup|u_2(x,y)-u_1(x,y)|(u_2(x,y)+u_1(x,y))+\frac{(u_2(x,y)-u_1(x,y))^2}{2}]Lh(x,y)dxdy.$$

Proof. Let

$$u(x,y)=u_2(x,y)-u_1(x,y).$$

Then using (5.94) we have

$$\begin{aligned} I \leq\ & \sup|u|\int_{B_1}\int_{B_2}|Lu||h(x,y)|dxdy \\ &+\frac{1}{2}\sup|u|\int_{B_1}\int_{B_2}|\sum_{i=1}^{n}u_{y_iy_i}||h_{x_ix_i}(x,y)|dxdy+\frac{1}{2}\int_{B_1}\int_{B_2}u^2(x,y)|Lh(x,y)|dxdy \end{aligned}$$

$$
\begin{aligned}
&+ \sup|u|\int_{B_1}\int_{B_2}|\sum_{i=1}^{n} u_{x_iy_i}(x,y)||h_{x_iy_i}|dxdy + \frac{1}{2}\sup|u|\int_{B_1}\int_{B_2}|\sum_{i=1}^{n} u_{x_ix_i}(x,y)||h_{y_iy_i}|dxdy \\
&\leq \sup|u_2-u_1|\int_{B_1}\int_{B_2}|L(u_2-u_1)|h(x,y)dxdy \\
&+ \frac{1}{2}\sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}|u_2-u_1|_{x_iy_i}h_{x_iy_i}(x,y)dxdy \\
&+ \frac{1}{2}\int_{B_1}\int_{B_2}(u_2-u_1)^2.|L[h(x,y)]|dxdy \\
&+ \sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}|u_2-u_1|_{x_iy_i}h_{x_iy_i}dxdy \\
&+ \frac{1}{2}\sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}|u_2-u_1|_{x_iy_i}h_{x_iy_i}(x,y)dxdy
\end{aligned} \tag{5.98}
$$

Since,

$$
|L(u_2-u_1)| \leq |L(u_2)|+|L(u_1)|,
$$

$$
\begin{aligned}
I &\leq \sup|u_2-u_1|\int_{B_1}\int_{B_2}[L(u_2)-L(u_1)]h(x,y)dxdy \\
&+ \frac{1}{2}\sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}[(u_2)_{x_iy_i}+(u_1)_{x_iy_i}]h_{x_iy_i}(x,y)dxdy \\
&+ \frac{1}{2}\int_{B_1}\int_{B_2}(u_2-u_1)^2L[h(x,y)]dxdy \\
&+ \sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}[(u_2)_{x_iy_i}+(u_1)_{x_iy_i}]h_{x_iy_i}(x,y)dxdy \\
&+ \frac{1}{2}\sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}[(u_2)_{x_iy_i}+(u_1)_{x_iy_i}]h_{x_iy_i}(x,y)dxdy.
\end{aligned} \tag{5.99}
$$

By Gauss-Green theorem, we have

$$
\begin{aligned}
I &\leq \sup|u_2-u_1|\int_{B_1}\int_{B_2}(u_2+u_1)L(h)dxdy \\
&+ \frac{1}{2}\sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}(u_2+u_1)h_{x_ix_iy_iy_i}dxdy \\
&+ \frac{1}{2}\int_{B_1}\int_{B_2}(u_2-u_1)^2L[h(x,y)]dxdy + \sup|u_2-u_1|\int_{B_1}\int_{B_2}\sum_{i=1}^{n}(u_2+u_1)h_{x_ix_iy_iy_i}dxdy
\end{aligned}
$$

$$+\ \frac{1}{2}\sup|u_2-u_1|\int\limits_{B_1}\int\limits_{B_2}\sum_{i=1}^{n}(u_2+u_1)h_{x_ix_iy_iy_i}dxdy.$$

Hence,

$$I\le\int\limits_{B_1}\int\limits_{B_2}\left(3\sup|u_2-u_1|(u_2+u_1)+\frac{(u_2-u_1)^2}{2}\right)|Lh(x,y)|dxdy. \tag{5.100}$$

Taking the norm of(5.100), we get

$$\int\limits_{B_1}\int\limits_{B_2}|\mathrm{grad}_{\mathrm{xy}}\mathrm{u}_2-\mathrm{grad}_{\mathrm{xy}}\mathrm{u}_1|^2\mathrm{h}(\mathrm{x},\mathrm{y})\mathrm{dxdy}\le \tag{5.101}$$

$$\le\left[3\|u_2-u_1\|_{L^\infty}\left(\|u_1\|_{L^\infty}+\|u_2\|_{L^\infty}\right)+\frac{1}{2}\|u_2-u_1\|^2_{L^\infty}\right]\int\limits_{B_1}\int\limits_{B_2}|Lh(x,y)|dxdy.$$

□

5.3.3 Existence and integrability of weak partial derivatives and weighted square inequalities for the difference of weak subsolutions

Now we will approximate the weak subsolution by smooth subsolution.

$$\eta_n(z)=\begin{cases}c\exp\frac{1}{|z|^2-1}, & |z|\le 1\\ 0, & |z|>1.\end{cases}$$

Now we use the mollification $u_\varepsilon(x,y)$ of bounded, measurable subsolution $u(x,y)$ in the following way:

$$u_\varepsilon(x,y)=\varepsilon\int_{B_1}\int_{B_2}\eta_n(x-t)\eta_n(y-s)u(t,s)dtds.$$

Let us denote

$$\eta_\varepsilon(x-t,y-s)=\varepsilon^{-(n+n)}\eta_n(x-t)\eta_n(y-s).$$

From above, the following is trivial,

$$\frac{\partial^4}{\partial x_i^2\partial y_i^2}\eta_\varepsilon(x-t,y-s)=\frac{\partial^4}{\partial t_i^2\partial s_i^2}\eta_\varepsilon(x-t,y-s).$$

This implies that

$$L_{x,y}\eta_\varepsilon(x-t,y-s)=L_{t,s}\eta_\varepsilon(x-y,t-s)=L^*_{t,s}\eta_\varepsilon(x-y,t-s).$$

$$
\begin{aligned}
L_{x,y}u_\varepsilon(x,y) &= L^*_{x,y}\varepsilon^n \int_{B_1}\int_{B_2} \eta_n(x-t)\eta_n(y-s)u(t,s)dtds \\
&= \varepsilon^n \int_{B_1}\int_{B_2} u(t,s)L^*_{x,y}[\eta_n(x-t)\eta_n(y-s)]dtds \\
&= \varepsilon^n \int_{B_1}\int_{B_2} u(t,s)[L^*_{t,s}\eta_n(x-t)\eta_n(y-s)]dtds.
\end{aligned}
$$

This implies that

$$L_{x,y}u_\varepsilon(x,y) \geq 0. \tag{5.102}$$

The following theorem tells about the existence of sequence of smooth sub-solutions.

Theorem 5.12 *Let $u(x,y)$ be the weak sub-solution of*

$$\frac{\partial^4 u}{\partial x_1^2 \partial y_1^2} + \frac{\partial^4 u}{\partial x_2^2 \partial y_2^2} + \ldots + \frac{\partial^4 u}{\partial x_n^2 \partial y_n^2} = 0 \tag{5.103}$$

on $B(x_0,r_1)\times B(y_0,r_2)$.
Then for any $k\in\mathbb{N}$ there exists $\hat{h}>0$, such that for any h, $0<h<\hat{h}$, each $u_h(x,y)$ is smooth sub-solution of (5.103) *over the ball $B(x_0,r_1)\times B(y_0,r_2)$.*

Proof. For fixed $k\in\mathbb{N}$, let

$$\hat{h} = \frac{r}{2(k+2)}. \tag{5.104}$$

It is clear for arbitrary $h>0$ the function $u_h(x,y)$ is infinitely differentiable. Now we check that for arbitrary $x,y\in B_k$, $\rho_h(x-t),(y-s)$ has a compact support in the ball $B(x_0,r_1)\times B(y_0,r_2)$.
Take the ball in the following way:

$$\hat{Q}_k = B_1\left(x_0, \frac{2k+3}{2k+4}R_1\right) \times B_2\left(y_0, \frac{2k+3}{2k+4}R_2\right). \tag{5.105}$$

Let's take an element $s,t \notin \hat{Q}_k$.
Then either,

$$s \notin B_1\left(x_0, \frac{2k+3}{2k+4}R_1\right),$$

or

$$t \notin B_2\left(y_0, \frac{2k+3}{2k+4}R_2\right).$$

In the first case:

$$|s-x| > \left(\frac{2k+3}{2k+4} - \frac{2k+2}{2k+4}\right)R_1 = \frac{1}{2(k+2)}R_1 > h.$$

Also,

$$|t-y| > \left(\frac{2k+3}{2k+4} - \frac{2k+2}{2k+4}\right)R_2 = \frac{1}{2(k+2)}R_2 > h.$$

Hence in the both cases, we have

$$\rho_h(x-y,t-s)=0.$$

Therefore, non-negative weight function $\rho_h(x-y,t-s)$ has compact support in Q as a function of t and s. If $h<\hat{h}$, by definition

$$\int_{B_1}\int_{B_2} u(y,s)L^*_{y,s}\rho_h(x-t,y-s)dydx \geq 0. \tag{5.106}$$

Otherwise we get form:

$$Lu_h(x,y)\geq 0,\ (x,y)\in Q_k.$$

□

Theorem 5.13 *Any continuous weak sub-solution possesses the second order weak partial derivatives*

$$\frac{\partial^2 u(x,y)}{\partial x_i \partial y_j},\quad i,j=1,2,\dots,n \tag{5.107}$$

for $(x,y)\in B^2_{k+l}=B_k\times B_l$ *where* $B_k=B(x_0,r_k),\ B_l=B(y_0,r_l)$.

Proof. First, we write the inequality (5.101) for $u_m(x,y)$ and $u_p(x,y)$ instead of $u_1(x,y)$ and $u_2(x,y)$, respectively on B^2_{k+l}

$$\begin{aligned}
&\int\limits_{B_k}\int\limits_{B_l} \left|\mathrm{grad}_{\mathrm{xy}}(\mathrm{u_p(x,y)})-\mathrm{grad}_{\mathrm{xy}}(\mathrm{u_m(x,y)})\right|^2 h_{k+l}(x,y)dxdy \\
&\leq \left\|u_p-u_m\right\|_{L^\infty(B^2_{k+l})}\left(\left\|u_p\right\|_{L^\infty(B^2_{k+l})}+\left\|u_m\right\|_{L^\infty(B^2_{k+l})}\right)\int\limits_{B_k}\int\limits_{B_l}|Lh_{k+l}(x,y)|\,dxdy \\
&\quad +\frac{1}{2}\left\|u_p-u_m\right\|^2_{L^\infty(B^2_{k+l})}\int\limits_{B_k}\int\limits_{B_l}|Lh_{k+l}(x,y)|\,dxdy.
\end{aligned} \tag{5.108}$$

Denote

$$c_{k+l}=\int\limits_{B_k}\int\limits_{B_l}|Lh_{k+l}(x,y)||\,dxdy,\ \widehat{c}_{k+l}=\inf_{(x,y)\in B^2_{k+l}} h_{k+l}(x,y)>0\quad k,l\in\mathbb{N}. \tag{5.109}$$

Now, from inequality (5.108)

$$\begin{aligned}
&\int\limits_{B_k}\int\limits_{B_l} \left|\mathrm{grad}_{\mathrm{xy}}(\mathrm{u_p(x,y)})-\mathrm{grad}_{\mathrm{xy}}(\mathrm{u_m(x,y)})\right|^2 h_{k+l}(x,y)dxdy \\
&\leq c_{k+l}\left[\left\|u_p-u_m\right\|_{L^\infty_{(B^2_{k+l})}}\left(\left\|u_p\right\|_{L^\infty_{(B^2_{k+l})}}+\left\|u_m\right\|_{L^\infty_{(B^2_{k+l})}}\right)\right]
\end{aligned}$$

$$+\frac{1}{2}\widetilde{c}_{k+l}\left\|u_p-u_m\right\|^2_{L_{(B^2_{k+l})}}. \tag{5.110}$$

Since

$$\left\|u_p-u_m\right\|_{L^{\infty}_{(B^2_{k+l})}} \to 0 \quad as\ m,p\to\infty, \tag{5.111}$$

we obtain

$$\lim_{m,p\to\infty}\sum_{i,j=1}^{n}\int_{B_k}\int_{B_l}\left(\frac{\partial^2 u_p(x,y)}{\partial x_i y_j}-\frac{\partial^2 u_m(x,y)}{\partial x_i y_j}\right)^2 dxdy=0. \tag{5.112}$$

By the completeness of the space $L^2(B^2_{k+l})$, there exists a sequence of measurable functions $v_{m;i,j}(x,y)\in L^2(B^2_{k+l}),\quad i,j=1,\ldots,n,\ m\in\mathbb{N},\quad$ such that $v_{m;i,j}$ is measurable and satisfying

$$\sum_{i,j=1}^{n}\int_{B_k}\int_{B_l}\left(\frac{\partial^2 u_m(x,y)}{\partial x_i y_j}-v_{m;i,j}(x,y)\right)^2 dxdy \xrightarrow{m\to\infty} 0,\quad u\in\mathbb{N}. \tag{5.113}$$

Let us define $v_{i,j}(x,y)$ in the following way:

$$v_{i,j}(x,y)=\limsup_{u\to\infty} v_{m;i,j}(x,y),\ i,j=1,\ldots,n. \tag{5.114}$$

Now we claim that $v_{i,j}(x,y),\ i,j=1,\ldots,n$ are Sobolev derivatives of functions $u(x,y)$. To prove this take $\phi(x,y)\in C_0^{\infty}(B)$, supp $\phi(x,y)\subset B^2_{k+l}$.
Now

$$\int_{B_k}\int_{B_l}\frac{\partial^2 u_m(x,y)}{\partial x_i y_j}\phi(x,y)dxdy=\int_{B_k}\int_{B_l}u_m(x,y)\frac{\partial^2}{\partial x_i y_j}\phi(x,y)dxdy \tag{5.115}$$

But $u_m(x,y)$ converges uniformly to $u(x,y)$ and $\frac{\partial^2 u_m(x,y)}{\partial x_i y_j}$ converges to $v_{i,j}(x,y)$ in $L^2(B^2_{k+l})$. Hence, if we let $m\to\infty$, we get desired result. □

Corollary 5.1 *If $u(x,y)$ is sub-solution on the ball then it is weighted square integrable i.e.*

$$\int_{B^2_{k+l}}\left|\mathrm{grad}_{xy}\mathrm{u}(\mathrm{x},\mathrm{y})\right|^2 h(x,y)dxdy<\infty, \tag{5.116}$$

where $h(x,y)$ is non-negative weight function having compact support.

5.4 The weighted square integral inequalities for smooth and weak subsolution of system of partial differential inequalities

Let $u(x),\ x \in \mathbb{R}^n$ be a solution of the following system of partial differential inequalities

$$\left.\begin{array}{r} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_j^2} \geq 0 \\ \frac{\partial^2 u}{\partial x_{j+1}^2} + \frac{\partial^2 u}{\partial x_{j+2}^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \leq 0 \end{array}\right\} \tag{5.117}$$

where $1 \leq j < n,\ n \geq 2$.

The bounded measurable function u is weak solution of the system (5.117) if for every $\phi(x) \in C_c^2(B)$, the following holds

$$\left.\begin{array}{r} \int\limits_B u(x)\Delta_{1,j}\,\phi(x)dx \geq 0 \\ \int\limits_B u(x)\Delta_{j+1,n}\,\phi(x)dx \leq 0 \end{array}\right\} \tag{5.118}$$

where

$$\Delta_{1,j} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_j^2} \tag{5.119}$$

and

$$\Delta_{j+1,n} = \frac{\partial^2}{\partial x_{j+1}^2} + \frac{\partial^2}{\partial x_{j+2}^2} + \ldots + \frac{\partial^2}{\partial x_n^2}. \tag{5.120}$$

It is trivial that $\Delta = \Delta_{1,j} + \Delta_{j+1,n}$ where $\Delta_{1,j}$ and $\Delta_{j+1,n}$ both operators are self adjoint operators.
The $\operatorname{grad} u(x)$ is n-dimensional vector given by

$$\operatorname{grad} u(x) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$$

We also introduce

$$\left.\begin{array}{r} \mathrm{grad}_{1,j}\mathrm{u}(\mathrm{x}) = \left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}_1}, \frac{\partial \mathrm{u}}{\partial \mathrm{x}_2}, \ldots, \frac{\partial \mathrm{u}}{\partial \mathrm{x}_j}\right) \\ \mathrm{grad}_{j+1,n}\mathrm{u}(\mathrm{x}) = \left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}_{j+1}}, \frac{\partial \mathrm{u}}{\partial \mathrm{x}_{j+2}}, \ldots, \frac{\partial \mathrm{u}}{\partial \mathrm{x}_n}\right) \end{array}\right\} \tag{5.121}$$

where $1 < j < n$.

5.4.1 The reverse Poincaré inequalities for smooth subsolution and approximation of weak subsolution by smooth ones

The following two lemmas for superharmonic functions and subharmonic functions are proved in [45].

Lemma 5.1 *Consider two arbitrary smooth superharmonic functions $u_i, i = 1,2$ over domain D, $D \subset \mathbb{R}^n$ i.e. $u_i \in C^2(\overline{D})$ and $\Delta u_i(x) \leq 0,\ x \in D,\ i = 1,2$.*
Then we have

$$\int_D |\operatorname{grad} u(x)|^2 w(x)dx \leq \int_D \left[\frac{u^2(x)}{2} - \| u(x) \|_{L^\infty} (u_2(x) + u_1(x))\right] \Delta w(x)dx.$$

where w is the non-negative weight function that satisfies

$$w(x) = \frac{\partial w(x)}{\partial x_i} = 0,\ i = 1,\ldots,n,\ x \in \partial D. \tag{5.122}$$

Lemma 5.2 *Consider two arbitrary smooth subharmonic functions $u_i,\ i = 1,2$ over domain D, $D \subset R^n$, i.e. $u_i \in \overline{C^2}(D)$ and $\Delta u_i(x) \leq 0,\ x \in D,\ i = 1,2$. Then the following holds*

$$\int_D |\operatorname{grad} u(x)|^2 w(x)dx \leq \int_D \left(\frac{u^2(x)}{2} + \| u(x) \|_{L^\infty} (u_2(x) + u_1(x))\right) \Delta w(x)dx.$$

where w is the non-negative weight function that satisfies (5.122).

Theorem 5.14 *Let $u_i,\ i = 1,\ 2$, be the two smooth solutions of system* (5.117) *over the domain $D \subseteq \mathbb{R}^n$, having smooth boundary and let w be the arbitrary non-negative smooth function on the domain D satisfying* (5.122) *then the following estimate holds*

$$\begin{aligned}\int_D |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 w(x)dx &\leq \|u_2 - u_1\|_{L^\infty}(\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}) \\ &\times \int_D \left|\widetilde{\Delta} w(x)\right| dx + \frac{1}{2}\|u_2 - u_1\|^2_{L^\infty} \int_D |\Delta w(x)|\, dx. \end{aligned} \tag{5.123}$$

where Δ is Laplace operator and $\widetilde{\Delta} = \Delta_{1,j} - \Delta_{j+1,n}$.

Proof. Let $u = u_2 - u_1$. Take

$$\begin{aligned}\int_D |\operatorname{grad} u(x)|^2 w(x)dx &= \int_D \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_j}\right)^2 + \left(\frac{\partial u}{\partial x_{j+1}}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)^2\right] w(x)dx. \\ &= \left(\int_D \left(\frac{\partial u}{\partial x_1}\right)^2 w(x)dx + \ldots + \int_D \left(\frac{\partial u}{\partial x_j}\right)^2 w(x)dx\right)\end{aligned}$$

$$+\left(\int_D \left(\frac{\partial u}{\partial x_{j+1}}\right)^2 w(x)dx + \ldots + \int_D \left(\frac{\partial u}{\partial x_n}\right)^2 w(x)dx\right). \tag{5.124}$$

Using (5.121) in equation (5.124), we obtain the the following

$$\int_D |\mathrm{grad}\, u(x)|^2 w(x)dx = \int_D |\mathrm{grad}_{1,j}\mathrm{u(x)}|^2 \mathrm{w(x)dx} + \int_D |\mathrm{grad}_{j+1,n}\mathrm{u(x)}|^2 \mathrm{w(x)dx}$$

Now using Lemma 5.2 on first integral and Lemma 5.1 on the second integral we obtain,

$$\int_D |\mathrm{grad}\, u(x)|^2 w(x)dx \le \int_D \left[\frac{(u_2-u_1)^2}{2} + \|u_2-u_1\|_{L^\infty}(u_2+u_1)\right]\Delta_{1,j}w(x)dx$$

$$+\int_D \left[\frac{(u_2-u_1)^2}{2} - \|u_2-u_1\|_{L^\infty}(u_2+u_1)\right]\Delta_{j+1,n}w(x)dx$$

$$\le \int_D \frac{(u_2-u_1)^2}{2}\big(\Delta_{1,j}w(x)+\Delta_{j+1,n}w(x)\big)dx+$$

$$\int_D \|u_2-u_1\|_{L^\infty}(u_2+u_1)\big(\Delta_{1,j}w(x)-\Delta_{j+1,n}w(x)\big)dx$$

$$\int_D |\mathrm{grad}\, u(x)|^2 w(x)dx \le \int_D \frac{(u_2-u_1)^2}{2}\Delta w(x)dx + \int_D \|u_2-u_1\|_{L^\infty}(u_2+u_1)\widetilde{\Delta} w(x)dx$$

where $\widetilde{\Delta} = \Delta_{1,j} - \Delta_{j+1,n}$.
Taking infinite norm on (5.125) we get the result (5.123). □

Remark 5.2 *The above theorem is also true for arbitrary ball B, $B = B(x_0,r)$ with center x_0 and radius r*

$$\int_{B(x_0,r)} |\mathrm{grad}\, u_2(x) - \mathrm{grad}\, u_1(x)|^2 w(x)dx$$

$$\le \|u_2-u_1\|_{L^\infty(B)}\left(\|u_1\|_{L^\infty(B)} + \|u_2\|_{L^\infty(B)}\right)\int_{B(x_0,r)} \left|\widetilde{\Delta} w(x)\right| dx$$

$$+\frac{1}{2}\|u_2-u_1\|^2_{L^\infty(B)}\int_{B(x_0,r)} |\Delta w(x)|\, dx. \tag{5.125}$$

From onward we will use $B(x_0,r)$ as a domain and the following particular weight function

$$w(x) = \left[r^2 - (x-x_0)^2\right]^2.$$

It is trivial that

$$\frac{\partial w}{\partial x_i}(x) = w(x) = 0, \quad x \in \partial B, \quad i = 1, 2, \ldots, n.$$

Now we prove that for weak solution of system of inequality (5.117), we may approximate it by system of smooth solutions, so we will use mollification technique, again.
Define

$$\varphi(x) = \begin{cases} Cexp\frac{1}{|x|^2-1}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $x \in \mathbb{R}^n$, and $C > 0$ such that

$$\int\limits_{\mathbb{R}^n} \varphi(x)dx = 1.$$

Now we define mollifier of bounded measurable solution $u(x)$ in the following way

$$u_h(x) = h^{-n} \int\limits_{B(x_0,r)} \varphi\left(\frac{x-y}{h}\right) u(y)dy.$$

Denote

$$\varphi_h(x-y) = h^{-n}\varphi\left(\frac{x-y}{h}\right).$$

It is trivial that

$$\frac{\partial^2}{\partial x_i^2}\varphi_h(x-y) = \frac{\partial^2}{\partial y_i^2}\varphi_h(x-y), \; i = 1, \ldots, n.$$

So

$$\Delta_x u_h(x) = h^{-n} \int\limits_{B(x_0,r)} u(y)\Delta_y \varphi_h(x-y)dy,$$

where Δ_x and Δ_y are the Laplace operator with respect to x and y.
We will define the smaller balls B_k, $k \in \mathbb{N}$, in the form

$$B_k = B(x_0, r_k) \;\; where \;\; r_k = \frac{k+1}{k+2}r, \;\; k \in \mathbb{N}$$

and the corresponding weight functions are

$$w_k(x) = \left[r_k^2 - (x-x_0)^2\right]^2.$$

The next theorem tells us that the function u_h defined above are smooth solutions of the system of inequality (5.123) over the ball B_k, for sufficiently small h.

Theorem 5.15 *Let u be the weak solution of system* ((5.117)) *on the ball B, $B = B(x_0, r)$. Then for any $k \in \mathbb{N}$, there exist $\widehat{h} > 0$, such that for any h, $0 < h < \widehat{h}$, each u_h is smooth solution of the system* (5.117) *over the ball B_k.*

Proof. For fixed, $k \in \mathbb{N}$, let

$$\widehat{h} = \frac{r}{2(k+2)}.$$

It is clear that for arbitrary $h > 0$ the function $u_h(x)$ is infinitely differentiable.
Now we check that for arbitrary $x \in B_k$, $\varphi_h(x-y)$ has compact support in the ball $B(x_0, r)$.
Take the ball $\widehat{B}_k$ in the following way

$$\widehat{B}_k = B\left(x_0, \frac{2k+3}{2k+4} r\right).$$

If $y \notin \widehat{B}_k$, then

$$|y - x| > \left|\frac{2k+3}{2k+4} - \frac{2k+2}{2k+4}\right| = \frac{r}{2(k+2)} > h \;\Rightarrow \varphi_h(x-y) = 0.$$

Hence $\varphi_h(x-y)$ has compact support in ball B as a function of y if $h < \widehat{h}$ and by the definition of weak solution u we have

$$\int\limits_B u(y)(\Delta_y)_{1,j}\, \varphi_h(x-y)dy \geq 0,$$

$$\int\limits_B u(y)(\Delta_y)_{j+1,n}\, \varphi_h(x-y)dy \leq 0,$$

which completes the proof. □

5.4.2 The existence and integrability of weak partial derivative and weighted square inequalities for the difference of weak subsolutions

The following theorem tells that continuous weak subsolution of system (5.117) possess all first order weak partial derivatives and also they are square integrable.

Theorem 5.16 *Every continuous weak solution u of system* (5.117) *has weak partial derivative $\frac{\partial u}{\partial x_i}$, $i = 1, \ldots, n$, in the ball $B(x_0, r) \subseteq \mathbb{R}^n$ and also they are weighted square integrable i.e.*

$$\int_B |\operatorname{grad} u(x)|^2\, w(x)dx < \infty,$$

where w is non-negative weight function having compact support.

Proof. The proof of the theorem can be made on similar lines, as proof of the Theorem 3.1 of [45], using inequality (5.125) instead of (3.5) of [45]. □

Next theorem will give us reverse Poincaré type inequalities for weak subsolution of system (5.117).

Theorem 5.17 *For any two arbitrary continuous weak solutions u_i, $i = 1,2$, for the system* (5.117) *in the ball $B(x_0,r)$, the following is valid*

$$\int\limits_{B(x_0,r)} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 w(x)dx \leq$$
$$\|u_2 - u_1\|_{L^\infty}(\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}) \int\limits_{B(x_0,r)} \left|\widetilde{\Delta} w(x)\right| dx + \frac{1}{2}\|u_2 - u_1\|_{L^\infty}^2 \int\limits_{B(x_0,r)} |\Delta w(x)|\, dx, \tag{5.126}$$

where Δ is Laplace operator and $\widetilde{\Delta} = \Delta_{1,j} - \Delta_{j+1,n}$.

Proof. For the continuous weak sub solutions u_i, $i = 1,2$ for system (5.117) we take smooth approximation $u_{m,i}$ $i = 1,2$. For the ball B_{k+l}, there exist integer m_{k+l} such that $u_{m,i}$ is smooth in the ball B_{k+l}, $m \geq m_{k+l}$, and $u_{m,i}$ converges uniformly to u_i, $i = 1,2.u_{m,1}$ and $u_{m,2}$ on the ball B_{k+l} .

$$\int\limits_{B_{k+l}} |\operatorname{grad} u_{m,2}(x) - \operatorname{grad} u_{m,1}(x)|^2 w_{k+l}(x)dx$$
$$\leq \widetilde{c}_{k+l} \|u_{m,2}(x) - u_{m,1}(x)\|_{L^\infty(B)} \left(\|u_{m,2}(x)\|_{L^\infty(B)} + \|u_{m,1}(x)\|_{L^\infty(B)} \right)$$
$$+\frac{1}{2} c_{k+l} \|u_{m,2}(x) - u_{m,1}(x)\|_{L^\infty(B)}^2,$$

where

$$\widetilde{c}_{k+l} = \int_B \left|\widetilde{\Delta} w(x)\right| dx, \quad c_{k+l} = \int_B |\Delta w(x)|\, dx.$$

Applying limt $m \to \infty$, we get

$$\int\limits_{B_{k+l}} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 w_{k+l}(x)dx$$
$$\leq \widetilde{c}_{k+l} \|u_2(x) - u_1(x)\|_{L^\infty_{(B_{k+l})}} \left(\|u_2(x)\|_{L^\infty_{(B_{k+l})}} + \|u_1(x)\|_{L^\infty_{(B_{k+l})}} \right)$$
$$+\frac{1}{2} c_{k+l} \|u_2(x) - u_1(x)\|_{L^\infty_{(B_{k+l})}}^2 .$$

Writing the left integral for the smaller ball $B_k \subseteq B_{k+l}$, and taking limit as $l \to \infty$, we obtain

$$\int\limits_{B_k} |\operatorname{grad} u_2(x) - \operatorname{grad} u_1(x)|^2 w(x)dx$$
$$\leq \widetilde{c}_\infty \|u_2(x) - u_1(x)\|_{L^\infty(B)} \left(\|u_2(x)\|_{L^\infty(B)} + \|u_1(x)\|_{L^\infty(B)} \right)$$

$$+\frac{1}{2}c_\infty \|u_2(x) - u_1(x)\|^2_{L^\infty(B)} .$$

By the last theorem (3.1), we have

$$\int_B |\text{grad}\, u_i(x)|^2 w(x) dx < \infty, \quad i = 1, 2, \tag{5.127}$$

and if we take limit, as $k \to \infty$, we obtain (5.126). □

Definitions

(i) **Convex function**
A function $f : I \to \mathbb{R}$ is said to be convex on I, if the following inequality hold:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \forall x, y \in I \; t \in [0,1]$$

(ii) $\phi-$**convex function**[15]
Let I be an interval in real line $\mathbb{R}$ and $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunction then a function $f : I \to \mathbb{R}$ is called ϕ-convex, if

$$f(\lambda x + (1-\lambda)y) \leq f(y) + \lambda \phi(f(x); f(y))$$

for all $x; y \in I$and $\lambda \in [0,1]$.

(iii) $\phi-$ **Quasiconvex function**[15]
A function f is called ϕ-quasiconvex, if $(\lambda x + (1-\lambda)y) \leq max f(y), f(y) + \phi(f(x), f(y))$ for all $x, y \in I$ and $\lambda \in [0,1]$.

(iv) $\phi-$**Affine function**[15]
A function f is called ϕ-affine if

$$f(\lambda x + (1-\lambda)y) = f(y) + \lambda \phi(f(x), f(y))$$

for all $x, y, \lambda \in \mathbb{R}$
Let $I \subset \mathbb{R}$ be the non-empty interior I^0. Two sub-intervals of I specied by the point $c \in I^0$ will be denoted by: $I_{x \geq c} = \{x \in I : x \geq c\}$ and$I_{x \leq c} = \{x \in I : x \leq c\}$

(v) **Right convex function**[73]
i.e A function $f : I \to R$ is right convex if it is convex on $I_X \geq c$ for some point $c \in I$

(vi) **Left convex function**[73]
A function $f : I \to \mathbb{R}$ is left convex if it is convex on $I_x \leq c$ for some $c \in I$
where

$$I_x \leq c = x \in I : x \leq c$$

(vii) **Half convex function**[73]
A function is half convex function if it is either right convex function or left convex function

(viii) **s-convex function**[48]
A function $f : [0,\infty) \to R$ is said to b s-convex in the second sense if the following inequality holds

$$f(tx+(1-t)y \leq t^s f(x)+(1-t)^s f(y)$$

for all $x,y \in [0,\infty)$ $t \in [0,1]$ and for some fixed $s \in (0,1]$
the class of s-convex function in the second sense is usually denoted by H_S^2

(ix) **h-convex function**[69]
Let $f,h : J \to \mathbb{R}$ be a positive or non negative function. Then f is said to be h-convex function or $f \in SX(h,I)$, if

$$f(tx+(1-t)) \leq h(t)f(x)+h(1-t)f(y)$$

$\forall x;y \in I \textit{ and } t \in (0,1)$

(x) **Modified h-convex function**[66]
Let $f,h : J \subset \mathbb{R} \to \mathbb{R}$ be a positive or non negative function. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be modified h-convex function if

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y)$$

and $\forall x,y \in J$ and $t \in [0,1]$

(xi) $(\alpha,1)$ **convex function**[67]
A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called an $(\alpha,1)$-convex function if $\forall x,y \in I$, we have

$$f(tx+(1-t)y) \leq t^\alpha f(x)+(1-t^\alpha)f(y)$$

(xii) **Wright Convex Function**[71]
A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is said to be Wright-convex if

$$f((1-t)x+ty)+f(tx+(1-t)y) \leq f(x)+f(y)$$

$\forall x,y \in D, t \in [0,1]$

(xiii) **m convex function**[65]
For $f : [0,b] \to \mathbb{R}$ and $m \in (0,1]$, if

$$f(tx+m(1-t)y \leq tf(x)+m(1-t)f(y)$$

is valid for all $x,y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an m-convex function on $[0,b]$.

(xiv) $(\alpha,m)-$**convex function**[38]
Let the $f:[0,b]\to\mathbb{R}$ is said to be (α,m) convex function, where $(\alpha,m)\in[0,1]$, if we have

$$f(tx+m(1-t)y\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y)$$

is valid for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that $f(x)$ is an (α,m)-convex function on [0, b].
The class of all (α,m)-convex functions on [0, b] for which $f(0)\le 0$ is denoted by K_m^{α}

(xv) **(h,m) convex function**[52]
Let $J\subseteq\mathbb{R}$ be an interval, $(0,1)\subseteq J,h:J\to\mathbb{R}$ be a nonnegative function. We say that $f:[0,b]\to\mathbb{R}$ is an (h,m)-convex function, or say, f belongs to the class $SMX((h,m),[0,b])$, if f is nonnegative and, for all $x,y\in[0,b]$ and $t\in[0,1]$ and for some $m\in(0;1]$, we have

$$f(tx+m(1-t)y)\le h(t)f(x)+mh(1-t)f(y)$$

(xvi) n**-convex vector**
The m-dimensional vector

$$F(x)=\big(f_1(x),f_2(x),\dots,f_m(x)\big)$$

is called smooth n-convex vector if

$$\frac{d^n}{dx^n}f_i(x)\ge 0\ \forall i=1,2,\dots,m$$

and smooth n-convex vector if

$$\frac{d^n}{dx^n}f_i(x)\ge 0\ \forall i=1,2,\dots,m$$

The vector $F(x)$ is arbitrary n-convex provided

$$f_i{}^{(n)}(\lambda x+(1-\lambda)y)\le\lambda f_i{}^{(n)}(x)+(1-\lambda)f_i^{(n)}(y)\ \forall i=1,2,\dots,m$$

for each $\lambda\in[0,1]$ and all x,y belongs to $\mathbb{R}$.

(xvii) **Geometrically convex function**
Let $[a_1,b_1]$ is a subset of $\mathbb{R}$. A mapping ψ from $[a_1,b_1]$ to $\mathbb{R}$ is geometrically convex if:

$$[\psi(p^rq^{1-r})\le[\psi(p)]^r[\psi(q)]^{1-r}$$

where, p,q belongs to $[a_1,b_1]$ and $r\in[0,1]$.

(xviii) **Starshaped funtion**
If we set $m=0$ in m-convex function then we obtain starshaped function on $[0,b_1]$, We recall that if ψ be a mapping from $[0,b_1]$ to $\mathbb{R}$ is starshaped if

$$\psi(rp)\le r\psi(p)$$

$\forall\ r$ belongs to $[0,1]$ and p belongs to $[0,b_1]$

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