MONOGRAPHS IN INEQUALITIES 16

Majorization Inequality and Information Theory

Selected topics of majorization and applications

Muhammad Adil Khan, Slavica Ivelić Bradanović,
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ELEMENT

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Preface

Mathematical inequalities make important part of mathematics. As a mathematical concept they were well known to ancient mathematicians. For example, the triangle inequality as a geometric fact, as well as the arithmetic-geometric mean inequality, were proved for the first time in the era of ancient Greece. When we look through the history, inequalities played an important role in supporting and developing other mathematical branches. It need a long time that inequalities become a discipline of study. Today, it becomes one of the central areas of mathematical analysis and is a fast growing discipline with increasing applications in different scientific fields. When we talk about the importance of inequalities, we could emphasize the role of inequalities for producing equalities, solving linear programming, solving optimization problems, provide a way of expressing the domain of a function, of solving limits etc. Further, inequalities have important applications in many other areas of science and engineering.

Now there is also a powerful and useful mathematical concept called majorization which are used for finding some nice and applicable inequalities. Majorization together with the strongly related concept of Schur-convexity gives an important characterization of convex functions. Moreover, the most important inequalities for convex functions as Jensen's inequality, the Hermite-Hadamard inequality, the arithmetic-geometric mean inequality can be easily derived by using an argument based on concept of majorization and the Schur-convex functions theory. Further, majorization theory is a key tool that allows us to transform complicated matrix-valued non-convex problem into a simple scalar problem. Majorization relation plays a key role in the design of linear MIMO transceivers, whereas the multiplicative majorization relation is the basis for nonlinear decision-feedback MIMO transceivers [139].

In the paper *Majorization: Here, There and Everywhere,* by Barry C. Arnold in 2007 [33], it is written that prior to the appearance of the celebrated volume *Inequalities: Theory of Majorization and its Applications* (Marshall and Olkin 1979) many researchers were unaware of a rich body of literature related to majorization that was scattered in journals in a wide variety of fields. Indeed, many majorization concepts had been reinvented and often rechristened in different research areas e.g., as Lorenz or dominance ordering in economics. In 2011, authors gave the second edition of this book which is also a great deal for researcher in the concept of majorization. They heroically had shifted the literature and endeavored to arrange new ideas about majorization and convexity, often providing a deeper understanding and also given multiple proofs and multiple view points on key results. Many of the key ideas relating to majorization was already discussed in the (also justly celebrated) volume entitled Inequalities by Hardy, Littlewood and Polyá ([79], 1934). We hope that our book is one of monographs in a series that will contribute to the further development and application of the concept of majorization.

In this book, we introduce new methods that allow us to establish a link between the concept of majorization with class of convex functions and theirs natural generalization, the class of convex functions of higher order, as well as classes of exponential and logarithmically convex functions. In obtaining such results we use interpolation by different classes of interpolating polynomials like the Abel-Gontscharoff polynomials, Lidstone's polynomials, Hermite's polynomials, Taylor's polynomials, as well as application of the well known identities as generalized Montgomery's identity and Fink's identity. We develop newly class of Green functions with nice properties which in combination with interpolation give a series of refinements and generalizations of Majorization theorem which plays important role in majorization theory. One of a version of Majorization theorem is given in the form of the well known Majorization inequality

$$\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i)$$

which holds for every convex function ϕ and real vectors $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m)$ such that "$\mathbf{y}$ is majorized by $\mathbf{x}$", in symbol $\mathbf{y} \prec \mathbf{x}$, which is given in Definition 1.7. This inequality in literature it is also known as Karamata's or the Hardy-Littlewood- Pólya inequality. It's weighted version is proved by Fuchs (see [74], [144, p.323]). A slight extension of Fuchs' theorem is given by J. Pečarić and S. Abramovich ([161], 1997). However, in this book, we present an aspect of development of Majorization theorem and its weighted versions, simultaneously in its discrete and integral version. We also obtain refinements and generalizations of one of the most important inequality for convex function known as Jensen's inequality and some closely related inequality in various spaces for different classes of functions. Further, our results involve Chebyshev functionals, the Grüss and Ostrowski type inequalities and inequalities involving measures of information entropy from Information theory. This book gives results that are infact great contributions and directions to the researchers in developing the notion of majorization.

The book is divided into three chapters. In the first chapter, we give a brief review of some fundamental results on the topics. We give the notion of majorization, additive as well as multiplicative majorization, and give results in discrete as well as continuous, i.e. integral case. We present a several basic motivating ideas. Given results we interpret in the form of the Lagrange and Cauchy type mean value theorems. As outcome we obtaine new classes of Chauchy type of means. Applying so called Exponential convexity method, established in [84], we interpret our results in the form of exponentially convex functions or in the special case logarithmically convex functions. We present vary on choice of a family in order to construct different examples of exponentially convex functions and construct some means. Further, we give majorization results for double integrals. We introduce majorization for matrices and give corresponding means.

In Chapter 2, we give the generalized results about majorization by using interpolation by different classes of interpolating polynomials like the Abel-Gontscharoff polynomials, Lidstone's polynomials, Hermite's polynomials, Taylor's polynomials, as well as application of the well known identities as generalized Montgomery's identity and Fink's identity in combination with newly developed class of Green functions with nice properties. We obtain related the Grüss and Ostrowski type inequalities. We also present n-exponential convex functions, exponential convex functions and log-convex functions to the corresponding functionals obtained by generalized results.

In Chapter 3 we show how the Shannon entropy is connected to the theory of majorization. They are both linked to the measure of disorder in a system. However, the theory of majorization usually gives stronger criteria than the entropic inequalities. We give some generalized results for majorization inequality using Csiszár f-divergence. This divergence we apply to some special convex functions reduces to the results for majorization inequality in the form of Shannon entropy and the Kullback-Leibler divergence. We give several applications by using the Zipf-Mandelbrot law (shorter Z-M law). We present the majorization inequalities for various distances obtaining by some special convex functions in the Csiszár f-divergence for Z-M law like the Rényi α-order entropy for Z-M law, variational distance for Z-M law, the Hellinger distance for Z-M law, χ^2-distance for Z-M law and triangular discrimination for Z-M law. We also give important applications of the Zipf's law in linguistics and obtain the bounds for the Kullback-Leibler divergence of the distributions associated to the English and the Russian languages. We consider the definition of "useful" Csiszár divergence and "useful" Zipf-Mandelbrot law associated with the real utility distribution to give the results for majorization inequalities by using monotonic sequences. We obtain the equivalent statements between continuous convex functions and Green functions via majorization inequalities, "useful" Csiszár functional and "useful" Zipf-Mandelbrot law. By considering "useful" Csisáar divergence in integral case, we give the results for integral majorization inequality.

The book can serve as a reference and a source of inspiration for researchers working in these and related areas. Applications of methods presented in book could be extended to the others class of functions as strongly convex, uniformly convex and superquadratic functions which contributed to the developments of methods and enabled the prospect of further applications. In the end of this preface we want to emphasize that this book integrates the whole variety of results from different papers which were previous published in journals by different authors. It was practically impossible to quite unite the notation in the book.

Authors

Contents

Chapter 1

Introduction

In this chapter, a brief review of some fundamental results on the topics in the sequel is given and a several basic motivating ideas are presented.

1.1 Convex Functions

Definition 1.1 *Let I be an real interval. Then $\phi : I \to \mathbb{R}$ is said to be* ***convex function*** *on I if for all $x, y \in I$ and every $\lambda \in [0,1]$, we have*

$$\phi\left((1-\lambda)x+\lambda y\right) \leq (1-\lambda)\phi(x)+\lambda\phi(y). \tag{1.1}$$

If (1.1) is strict for all $x, y \in I$, $x \neq y$ and every $\lambda \in (0,1)$, then ϕ is said to be ***strictly convex****.*
If in (1.1) the reverse inequality holds, then ϕ is said to be ***concave function****. If it is strict for all $x, y \in I$, $x \neq y$ and every $\lambda \in (0,1)$, then ϕ is said to be* ***strictly concave****.*

For convex functions the following propositions are valid which exactly define convex functions on equivalent ways.

Remark 1.1 *a) The inequality (1.1), for $x_1, x_2, x_3 \in I$, such that $x_1 \leq x_2 \leq x_3$, $x_1 \neq x_3$, we can write in the form*

$$\phi\left(x_2\right) \leq \frac{x_3-x_2}{x_3-x_1}\phi\left(x_1\right)+\frac{x_2-x_1}{x_3-x_1}\phi\left(x_3\right), \tag{1.2}$$

i.e.

$$(x_3 - x_2)\,\phi\,(x_1) + (x_1 - x_3)\,\phi\,(x_2) + (x_2 - x_1)\,\phi\,(x_3) \geq 0, \tag{1.3}$$

setting $x = x_1,\ y = x_3,\ \lambda = (x_2 - x_1)/(x_3 - x_1)$. *This inequality is often used as alternative definition of convexity.*
b) Another way of writing (1.3) is

$$\frac{\phi\,(x_2) - \phi\,(x_1)}{x_2 - x_1} \leq \frac{\phi\,(y_2) - \phi\,(y_1)}{y_2 - y_1}, \tag{1.4}$$

where $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2$ *and* $y_1 \neq y_2$.

The following two theorems concern derivatives of convex functions.

Theorem 1.1 *(see [144, p. 4]) Let I be an real interval. Let* $\phi : I \to \mathbb{R}$ *be convex. Then*

(i) ϕ *is Lipschitz on any closed interval in I;*

(ii) ϕ'_- *and* ϕ'_+ *exist and are increasing on I, and* $\phi'_- \leq \phi'_+$ *(if* ϕ *is strictly convex, then these derivatives are strictly increasing);*

(iii) ϕ' *exists, except possibly on a countable set, and on the complement of which it is continuous.*

Remark 1.2 *a) If* $\phi : I \to \mathbb{R}$ *is derivable function, then* ϕ *is convex iff a function* ϕ' *is increasing.*
b) If $\phi : I \to \mathbb{R}$ *is twice derivable function, then* ϕ *is convex iff* $\phi''(x) \geq 0$ *for all* $x \in I$. *If* $\phi''(x) > 0$, *then* ϕ *is strictly convex.*

Theorem 1.2 *(see [144, p. 5]) Let I be an open interval in* $\mathbb{R}$.

(i) $\phi : I \to \mathbb{R}$ *is convex iff there is at least one line of support for* ϕ *at each* $x_0 \in I$, *i.e. for all* $x \in I$ *we have*

$$\phi\,(x) \geq \phi\,(x_0) + \lambda\,(x - x_0),$$

where $\lambda \in \mathbb{R}$ *depends on* x_0 *and is given by* $\lambda = \phi'\,(x_0)$ *when* $\phi'\,(x_0)$ *exists, and* $\lambda \in \left[\phi'_-\,(x_0), \phi'_+\,(x_0)\right]$ *when* $\phi'_-\,(x_0) \neq \phi'_+\,(x_0)$.

(ii) $\phi : I \to \mathbb{R}$ *is convex if the function* $x \longmapsto \phi\,(x) - \phi\,(x_0) - \lambda\,(x - x_0)$, *(the difference between the function and its support) is decreasing for* $x < x_0$ *and increasing for* $x > x_0$.

Definition 1.2 *Let* $\phi : I \to \mathbb{R}$ *be a convex function. Then the subdifferential of* ϕ *at* x, *denoted by* $\partial\phi(x)$ *is defined by*

$$\partial\phi(x) = \{\alpha \in \mathbb{R} : \phi\,(y) - \phi(x) - \alpha\,(y - x) \geq 0,\ y \in I\}.$$

There is a connection between a convex function and its subdifferential. It is well-known that $\partial\phi(x) \neq 0$ for all $x \in IntI$. More precisely, at each point $x \in IntI$ we have $-\infty < \phi'_{-}(x) \leq \phi'_{+}(x) < \infty$ and

$$\partial\phi(x) \in \left[\phi'_{-}(x_0), \phi'_{+}(x_0)\right],$$

while the set on which ϕ is not differentiable is at most countable. Moreover, each function $\varphi : I \to \mathbb{R}$ such that $\varphi(x) \in \partial\phi(x)$, whenever $x \in IntI$, is increasing on $IntI$. For any such function φ and arbitrary $x \in IntI$, $y \in I$, we have

$$\phi(y) - \phi(x) - \varphi(x)(y-x) \geq 0$$

and

$$\begin{aligned}\phi(y) - \phi(x) - \varphi(x)(y-x) &= |\phi(y) - \phi(x) - \varphi(x)(y-x)| \\ &\geq \|\phi(y) - \phi(x)| - |\varphi(x)| \cdot |(y-x)\|.\end{aligned}$$

J. L. Jensen is considered generally as being the first mathematician whostudied convex functions in a systematic way. He defined the concept of convex functions using the inequality (1.5) that are listed in the following definition.

Definition 1.3 *A function $\phi : I \to \mathbb{R}$ is called **Jensen-convex** or **J-convex** if for all $x, y \in I$ we have*

$$\phi\left(\frac{x+y}{2}\right) \leq \frac{\phi(x) + \phi(y)}{2}. \tag{1.5}$$

Remark 1.3 *It can be easily proved that a convex function is J-convex. If $\phi : I \to \mathbb{R}$ is continuos function, then ϕ is convex iff it is J-convex.*

Inequality (1.1) can be extended to the convex combinations of finitely many points in I by mathematical induction. These extensions are known as **discrete Jensen's inequality**.

Theorem 1.3 (JENSEN'S INEQUALITY) *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ be a positive n-tuple. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i), \tag{1.6}$$

where

$$W_k = \sum_{i=1}^{k} w_i, \quad k = 1, \ldots, n. \tag{1.7}$$

If f is strictly convex, then inequality (1.6) is strict unless $x_1 = \cdots = x_n$.

The condition "$\boldsymbol{w}$ is a positive n-tuple" can be replaced by "$\boldsymbol{w}$ is a non-negative n-tuple and $W_n > 0$". Note that the Jensen inequality (1.6) can be used as an alternative definition of convexity.

It is reasonable to ask whether the condition "$\boldsymbol{w}$ is a non-negative n-tuple" can be relaxed at the expense of restricting $\boldsymbol{x}$ more severely. An answer to this question was given by Steffensen [161] (see also [144, p.57]).

Theorem 1.4 (THE JENSEN-STEFFENSEN INEQUALITY) *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{w} = (w_1, \ldots, w_n)$ a real n-tuple such that*

$$0 \le W_k \le W_n \,, \quad k = 1, \ldots, n-1, \quad W_n > 0, \tag{1.8}$$

is satisfied, where W_k are as in (1.7), then (1.6) holds. If f is strictly convex, then inequality (1.6) is strict unless $x_1 = \cdots = x_n$.

Inequality (1.6) under conditions from Theorem 1.4 is called **the Jensen-Steffensen inequality**.

1.2 Space of Integrable, Continuous and Absolutely Continuous Functions

Let $[a,b]$ be a finite interval in $\mathbb{R}$, where $-\infty \le a < b \le \infty$. We denote by $L_p[a,b]$, $1 \le p < \infty$, the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|^p dt < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}},$$

and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$ with

$$\|f\|_\infty = ess\sup\{|f(x)| : x \in [a,b]\}.$$

Theorem 1.5 (HOLDER'S INEQUALITY) *Let $p,q \in \mathbb{R}$ be such that $1 \le p,q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f,g : [a,b] \to \mathbb{R}$ be integrable functions such that $f \in L_p[a,b]$ and $g \in L_q[a,b]$. Then*

$$\int_a^b |f(t)g(t)|dt \le \|f\|_p \|g\|_q \,. \tag{1.9}$$

The equality in (1.9) holds iff $A|f(t)|^p = B|g(t)|^q$ almost everywhere (shortened to a.e.), where A and B are constants.

We denote by $C^n([a,b]), n \in \mathbb{N}_0$, the space of functions which are n times continuously differentiable on $[a,b]$, that is

$$C^n([a,b]) = \left\{ f : [a,b] \to \mathbb{R} : f^{(k)} \in C([a,b]), k = 0, 1, \ldots, n \right\}.$$

In particular, $C^0([a,b]) = C([a,b])$ is the space of continuous functions on $[a,b]$ with the norm

$$\|f\|_{C^n} = \sum_{k=0}^{n} \left\| f^{(k)} \right\|_C = \sum_{k=0}^{n} \max_{x \in [a,b]} \left| f^{(k)}(x) \right|,$$

and for $C([a,b])$

$$\|f\|_C = \max_{x\in[a,b]} |f(x)|.$$

Lemma 1.1 *The space $C^n([a,b])$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} c_k(x-a)^k, \tag{1.10}$$

where $\varphi \in C([a,b])$ and c_k are arbitrary constants ($k = 0,1,\ldots,n-1$).
Moreover,

$$\varphi(t) = f^{(n)}(t), \quad c_k = \frac{f^{(k)}(a)}{k!} \quad (k = 0,1,\ldots,n-1). \tag{1.11}$$

The space of **absolutely continuous functions** on an interval $[a,b]$ is denote by $AC([a,b])$. It is known that $AC([a,b])$ coincides with the space of primitives of Lebesgue integrable functions $L_1[a,b]$ (see [100]):

$$f \in AC([a,b]) \quad \Leftrightarrow \quad f(x) = f(a) + \int_a^x \varphi(t)dt, \quad \varphi \in L_1[a,b].$$

Therefore, an absolutely continuous function f has an integrable derivatives $f'(x) = \varphi(x)$ almost everywhere on $[a,b]$. We denote by $AC^n([a,b]), n \in \mathbb{N}$, the space

$$AC^n([a,b]) = \{f \in C^{n-1}([a,b]) : f^{(n-1)} \in AC([a,b])\}.$$

In particular, $AC^1([a,b]) = AC([a,b])$.

Lemma 1.2 *The space $AC^n([a,b])$ consists of those and only those functions which can be represented in the form (1.10), where $\varphi \in L_1[a,b]$ and c_k are arbitrary ($k = 0,1,..,n-1$). Moreover, (1.11) holds.*

The next theorem has numerous applications involving multiple integrals.

Theorem 1.6 (FUBINI'S THEOREM) *Let $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ be σ-finite measure space and f be $\mu\times\nu$-measurable function on $X\times Y$. If $f \geq 0$, then the next integrals are equal*

$$\int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y), \quad \int_X \left(\int_Y f(x,y)d\nu(y)\right)d\mu(x), \quad \int_Y \left(\int_X f(x,y)d\mu(x)\right)d\nu(y).$$

Remark 1.4 *The next equalities*

$$\begin{aligned}
\int_a^b \left(\int_c^d f(x,y)dy\right)dx &= \int_c^d \left(\int_a^b f(x,y)dx\right)dy,\\
\int_a^b \left(\int_c^x f(x,y)dy\right)dx &= \int_a^b \left(\int_y^b f(x,y)dx\right)dy,
\end{aligned}$$

are consequences of the previous theorem.

Theorem 1.7 (INTEGRAL JENSEN'S INEQUALITY) *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $\phi : \Omega \to \mathbb{R}$ be μ-integrable function. Let $f : I \to \mathbb{R}$ be a convex function such that $Im\,\phi \subseteq I$ anf $f \circ \phi$ is a μ – integrable function. Then*

$$f\left(\frac{1}{\mu(\Omega)}\int_\Omega \phi(x)d\mu(x)\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega f(\phi(x))d\mu(x). \tag{1.12}$$

If f is strictly convex, then (1.12) becomes equality iff ϕ is a constant μ-almost everywhere on Ω. If f is concave, then (1.12) is reversed.

Remark 1.5 *The discrete Jensen inequality (1.6) is obtained by means of the discrete measure μ on $\Omega = \{1,\ldots,n\}$, with $\mu(\{i\}) = p_i$ and $\phi(i) = x_i$.*

Another integral version of jensen's inequality is based on the notation of the Riemann-Stieltjes integral for which a brief outline is given here. One can find more information on the Riemann-Stieltjes integral in [153].

Let $[a,b] \subset \mathbb{R}$ and let $f, \phi : [a,b] \to \mathbb{R}$ be bounded functions. The each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$, such that $t_0 < t_1 < \cdots < t_{n-1} < t_n$, Stieltjes' integral sum

$$\sigma(f,\phi;D,\gamma_1,\ldots,\gamma_n) = \sum_{i=1}^{n} f(\gamma_i)(\phi(t_i) - \phi(t_{i-1}))$$

is assigned, where $\gamma_i \in [t_{i-1}, t_i]$, $i = 1,\ldots,n$. These sums will be denoted with $\sigma(f,\phi;D)$ in the sequel.

Definition 1.4 *Let $f, \phi : [a,b] \to \mathbb{R}$ be bounded functions. A function f is said to be Riemann-Stieltjes integrable regardung a function ϕ if there exists $I_f \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a decomposition D_0 of $[a,b]$ such that for every decomposition $D \supseteq D_0$ of $[a,b]$ anf for every sum $\sigma(f,\phi;D)$*

$$|\sigma(f,\phi;D) - I_f| < \varepsilon$$

holds. The unuque I_f is the Riemann-Stieltjes integral of the function F regarding the function ϕ and is denoted with

$$\int_a^b f(t)d\phi(t).$$

The Riemann-Stieltjes integral is a generalization of the Riemann integral and coincides with it when ϕ is an identity.

The notation of the Riemann-Stieltjes integral is narrowly related to the class of the function of bounded variation.

Definition 1.5 *Let $\phi : [a,b] \to \mathbb{R}$ be a real function. To each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$, such that $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, belongs the sum*

$$V(\phi;D) = \sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})|,$$

which is said to be a variation of the function ϕ regarding decomposition D.
A function ϕ is said to be a function of bounded variation if the set $\{V(\phi;D) : D \in \mathscr{D}\}$ is bounded, where $\mathscr{D}$ is a family of all decompositions of the interval $[a,b]$. Number

$$V(\phi) = \sup\{V(\phi;D) : D \in \mathscr{D}\}$$

is called a total variation of a function ϕ.

Theorem 1.8 *The following assertions hold:*

(i) *Every monotonic function $\phi : [a,b] \to \mathbb{R}$ is a function of bounded variation on $[a,b]$ and $V(\phi) = |\phi(b) - \phi(a)|$;*

(ii) *Every function of bounded variation is a bounded function;*

(iii) *If f and g are functions of bounded variation on $[a,b]$, then $f+g$ is a function of bounded variation on $[a,b]$.*

Theorem 1.9 *Let ϕ be a function of bounded variation on $[a,b]$. then:*

(i) *ϕ has at most countably many of step discontinuities on $[a,b]$;*

(ii) *ϕ can be presented as $\phi = s_\phi + g$, where step function s_ϕ and continuous function g are both functions of bounded variation on $[a,b]$.*

At the end of this section, we introduce two recently obtained results involving Čebyšev's functional that involve the Grüss and Ostrowski type inequalities.

Definition 1.6 *For two Lebesgue integrable functions $f,g : [\alpha,\beta] \to \mathbb{R}$, we define* ***Čebyšev's functional*** *as*

$$T(f,g) := \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)g(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt.$$

Theorem 1.10 *[57, Theorem 1] Let $f : [\alpha,\beta] \to \mathbb{R}$ be Lebesgue integrable and $g : [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(g')^2 \in L_1[\alpha,\beta]$. Then*

$$|T(f,g)| \le \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (x-\alpha)(\beta-x)[g'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{1.13}$$

The constant $\frac{1}{\sqrt{2}}$ in (1.13) is the best possible.

Theorem 1.11 *[57, Theorem 2] Let $g : [\alpha,\beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha,\beta]$. Then*

$$|T(f,g)| \le \frac{1}{2(\beta-\alpha)} \|f'\|_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dg(x). \tag{1.14}$$

The constant $\frac{1}{2}$ in (1.14) is the best possible.

1.3 About Majorization

In this section, we introduce the concepts of majorization and Schur-convexity in order to give some basic results from the theory of majorization that give an important characterization of convex functions. Majorization theorem for convex functions and the classical concept of majorization, due to Hardy et al. [79], have numerous applications in different fields of applied sciences (see the monograph [117]). In recent times, majorization type results has attracted the interest of several mathematicians which resulting with interesting generalizations and applications (see for example [4], [6], [5], [52], [137]-[136]). A complete and superb reference on the subject is the book by Marshall and Olkin [123]. The book by Bhatia (1997) [45] contains significant material on majorization theory as well. Other textbooks on matrix and multivariate analysis also include a section on majoriztion theory, e.g., [82, Sec.4.3], [24, Sec.8.10] and [144].

Majorization makes precise the vague notion that the components of a vector $\boldsymbol{y}$ are "less spread out" or "more nearly equal" than the components of a vector $\boldsymbol{x}$.
For fixed $n \geq 2$, let

$$\boldsymbol{x} = (x_1, \ldots, x_n), \quad \boldsymbol{y} = (y_1, \ldots, y_n)$$

denote two n-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq \ldots \geq y_{[n]},$$

$$x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}, \quad y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(n)}$$

be their ordered components.

Definition 1.7 ***Majorization****: (see [144, p.319])* $\mathbf{x}$ *is said to majorize* $\mathbf{y}$ *(or* $\mathbf{y}$ *is said to be majorized by* $\mathbf{x}$*), in symbol,* $\mathbf{x} \succ \mathbf{y}$*, if*

$$\sum_{i=1}^{m} y_{[i]} \leq \sum_{i=1}^{m} x_{[i]} \tag{1.15}$$

holds for $m = 1, 2, \ldots, n-1$ *and*

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i.$$

Note that (1.15) is equivalent to

$$\sum_{i=n-m+1}^{n} y_{(i)} \leq \sum_{i=n-m+1}^{n} x_{(i)}$$

holds for $m = 1, 2, \ldots, n-1$.

The following notion of Schur-convexity generalizes the definition of convex function via the notion of majorization.

Definition 1.8 ***Schur-convexity***: *A function $F : S \subseteq \mathbb{R}^m \to \mathbb{R}$ is called Schur-convex on S if*

$$F(\mathbf{y}) \leq F(\mathbf{x}) \tag{1.16}$$

for every $\mathbf{x}$, $\mathbf{y} \in S$ such that

$$\mathbf{y} \prec \mathbf{x}.$$

Definition 1.9 ***(Weakly Majorization)***: *For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say $\boldsymbol{y}$ is weakly majorized by $\boldsymbol{x}$ or $\mathbf{x}$ weakly majorizes $\mathbf{y}$ (denoted by $\mathbf{x}\ {}^{w}\!\succ \mathbf{y}$ or $\mathbf{x} \succ^{w} \mathbf{y}$) if*

$$\sum_{i=1}^{m} y_{(i)} \geq \sum_{i=1}^{m} x_{(i)}$$

holds for $m = 1, 2, \ldots, n-1, n$, or, equivalently,

$$\sum_{i=m}^{n} y_{[i]} \geq \sum_{i=m}^{n} x_{[i]}$$

holds for $m = 1, 2, \ldots, n-1, n$.

Note that $\boldsymbol{x} \succ \boldsymbol{y}$ implies $\boldsymbol{x}\ {}^{w}\!\succ \boldsymbol{y}$; in other words, majorization is a more restrictive definition than weakly majorization.
Observe that the original order of the elements of $\mathbf{x}$ and $\mathbf{y}$ plays no role in the definition of majorization. In other words,

$$\boldsymbol{x} \prec \Pi\boldsymbol{x}$$

for all permutation matrices Π.

Parallel to the concept of additive majorization is the notion of multiplicative majorization (also termed log-majorization).

Definition 1.10 ***(Multiplicative Majorization)***: *[139] Let $\mathbf{x}$, $\mathbf{y}$ be two positive n-tuples, $\mathbf{y}$ is said to be multiplicatively majorized by $\mathbf{x}$, denoted by $\mathbf{y} \prec_{\times} \mathbf{x}$ if*

$$\prod_{i=1}^{m} y_{[i]} \leq \prod_{i=1}^{m} x_{[i]} \tag{1.17}$$

holds for $m = 1, 2, \ldots, n-1$ and

$$\prod_{i=1}^{n} y_i = \prod_{i=1}^{n} x_i.$$

Note that (1.17) is equivalent to

$$\prod_{i=n-m+1}^{n} y_{(i)} \leq \prod_{i=n-m+1}^{n} x_{(i)}$$

holds for $m = 1, 2, \ldots, n-1$.
To differentiate the two types of majorization, we sometimes use the symbol $\prec_{+}$ rather than $\prec$ to denote (additive) majorization.

There are several equivalent characterizations of the majorization relation $\boldsymbol{x} \succ \boldsymbol{y}$ in addition to the conditions given in definition of majorization. One is actually the answer of a question posed and answered in 1929 by Hardy, Littlewood and Polya [80, 79] in the form of the following theorem well-known as **Majorization theorem** (see [123, p.11], [144, p.320]).

Theorem 1.12 (MAJORIZATION THEOREM) *Let I be an interval in $\mathbb{R}$, and let $\mathbf{x}$, $\mathbf{y}$ be two n-tuples such that x_i, $y_i \in I$ $(i = 1, \ldots, n)$. Then*

$$\sum_{i=1}^{n} \phi(y_i) \leq \sum_{i=1}^{n} \phi(x_i) \tag{1.18}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ if and only if $\mathbf{x} \succ \mathbf{y}$.
If ϕ is a strictly convex function, then equality in (1.18) is valid iff $x_{[i]} = y_{[i]}$, $i = 1, \ldots, n$.

Another interesting characterization of $\boldsymbol{x} \succ \boldsymbol{y}$, also by Hardy, Littlewood, and Polya [80, 79], is that $\boldsymbol{y} = \mathbf{P}\boldsymbol{x}$ for some double stochastic matrix $\mathbf{P}$. In fact, the previous characterization implies that the set of vectors $\boldsymbol{y}$ that satisfy $\boldsymbol{x} \succ \boldsymbol{y}$ is the convex hull spanned by the $n!$ points formed from the permutations of the elements of $\boldsymbol{y}$.

The previous Majorization theorem can be be slightly preformulate in the following form which gives a relation between one-dimensional convex function and m-dimensional Schur-convex function (see [144, p. 333]).

Theorem 1.13 *Let $I \subset \mathbb{R}$ be an interval and $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m) \in I^m$. Let $\phi : I \to \mathbb{R}$ be continuous function. Then a function $F : I^m \to \mathbb{R}$, defined by*

$$F(\mathbf{x}) = \sum_{i=1}^{m} \phi(x_i),$$

is Schur-convex on I^m iff ϕ is convex on I.

The following theorem can be regarded as a weighted version of Theorem 1.13 and is proved by Fuchs in ([74], [144, p.323]).

Theorem 1.14 (FUCHS'S THEOREM) *Let $\mathbf{x}$, $\mathbf{y}$ be two decreasing real n-tuples, $\mathbf{x}, \mathbf{y} \in I^n$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a real n-tuple such that*

$$\sum_{i=1}^{k} w_i y_i \leq \sum_{i=1}^{k} w_i x_i \text{ for } k = 1, \ldots, n-1, \tag{1.19}$$

and

$$\sum_{i=1}^{n} w_i y_i = \sum_{i=1}^{n} w_i x_i. \tag{1.20}$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$, we have

$$\sum_{i=1}^{n} w_i \phi(y_i) \leq \sum_{i=1}^{n} w_i \phi(x_i). \tag{1.21}$$

Remark 1.6 *Throughout this book, if in some results we have $\boldsymbol{x} = (x_1, x_2, \ldots, x_t)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_t)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_t)$ are t-tuples and g is associated function and we say that that these tuples are satisfying conditions (1.19), (1.20) and (1.21) holds, then we take $n = t$ and $\phi = g$ in (1.19), (1.20) and (1.21).*

The following theorem is valid ([133, p.32]):

Theorem 1.15 ([108]) *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on an interval I, $\mathbf{w}$ be a positive n-tuple and $\mathbf{x}$, $\mathbf{y} \in I^n$ such that satisfying (1.19) and (1.20)*

(i) If $\mathbf{y}$ is decreasing n-tuple, then (1.21) *holds.*

(ii) If $\mathbf{x}$ is increasing n-tuple, then reverse inequality in (1.21) *holds.*

If ϕ is strictly convex and $\boldsymbol{x} \neq \boldsymbol{y}$, then (1.21) *and reverse inequality in* (1.21) *are strict.*

Proof. As in [161] (see [133, p.32]), because of the convexity of ϕ

$$\phi(u) - \phi(v) \geq \phi'_+(v)\,(u - v)\,.$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{n} w_i \left[\phi\left(x_i\right) - \phi\left(y_i\right)\right] \\
&\quad \geq \sum_{i=1}^{n} w_i\, \phi'_+\left(y_i\right)\left(x_i - y_i\right) \\
&\quad = \phi'_+\left(y_n\right)\left(X_n - Y_n\right) \\
&\quad + \sum_{k=1}^{n-1}\left(X_k - Y_k\right)\left[\phi'_+\left(y_k\right) - \phi'_+\left(y_{k+1}\right)\right] \geq 0.
\end{aligned} \tag{1.22}$$

where $X_k = \sum_{i=1}^{k} w_i x_i$ and $Y_k = \sum_{i=1}^{k} w_i y_i$.
The last inequality follows from (1.19) and (1.20), $\boldsymbol{y}$ is decreasing and the convexity of ϕ. Similarly, we can prove the case when $\boldsymbol{x}$ is increasing.
If ϕ is strictly convex and $\boldsymbol{x} \neq \boldsymbol{y}$, then

$$\phi\left(x_i\right) - \phi\left(y_i\right) > \phi'_+\left(y_i\right)\left(x_i - y_i\right),$$

for at least one $i = 1, \ldots, n$. Which gives strict inequality in (1.21) and reverse inequality in (1.21). □

The following theorem is a slight extension of Theorem 1.14 proved by J. Pečarić and S. Abramovich [161].

Theorem 1.16 ([108]) *Let $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{y}$ be an positive n-tuples. Suppose $\psi, \phi : [0, \infty) \to \mathbb{R}$ are such that ψ is a strictly increasing function and ϕ is a convex function with respect to ψ i.e., $\phi \circ \psi^{-1}$ is convex. Also suppose that*

$$\sum_{i=1}^{k} w_i\, \psi\left(y_i\right) \leq \sum_{i=1}^{k} w_i\, \psi\left(x_i\right), \quad k = 1, \ldots, n-1, \tag{1.23}$$

and

$$\sum_{i=1}^{n} w_i \psi(y_i) = \sum_{i=1}^{n} w_i \psi(x_i). \tag{1.24}$$

(i) *If* $\mathbf{y}$ *is a decreasing n-tuple, then* (1.21) *holds.*

(ii) *If* $\mathbf{x}$ *is an increasing n-tuple, then the reverse inequality in* (1.21) *holds.*

If $\phi \circ \psi^{-1}$ *is strictly convex and* $\mathbf{x} \neq \mathbf{y}$, *then the strictly inequality holds in* (1.21).

Definition 1.11 ***(Integral majorization)*** *Let* x, y *be real valued functions defined on an interval* $[a,b]$ *such that* $\int_a^s x(\tau)d\tau$, $\int_a^s y(\tau)d\tau$ *both exist for all* $s \in [a,b]$. *[144, p.324]* $x(\tau)$ *is said to majorize* $y(\tau)$, *in symbol,* $x(\tau) \succ y(\tau)$, *for* $\tau \in [a,b]$ *if they are decreasing in* $\tau \in [a,b]$ *and*

$$\int_a^s y(\tau)\,d\tau \leq \int_a^s x(\tau)\,d\tau \quad for\, s \in [a,b], \tag{1.25}$$

and equality in (1.25) holds for $s = b$.

The following theorem can be regarded as **integral majorization theorem** [144, p.325].

Theorem 1.17 (INTEGRAL MAJORIZATION THEOREM) $x(\tau) \succ y(\tau)$ *for* $\tau \in [a,b]$ *iff they are decreasing in* $[a,b]$ *and*

$$\int_a^b \phi(y(\tau))\,d\tau \leq \int_a^b \phi(x(\tau))\,d\tau \tag{1.26}$$

holds for every ϕ *that is continuous and convex in* $[a,b]$ *such that the integrals exist.*

The following theorem is a simple consequence of Theorem 1 in [140] (see also [144, p.328]):

Theorem 1.18 *Let* $x(\tau), y(\tau) : [a,b] \to \mathbb{R}$, $x(\tau)$ *and* $y(\tau)$ *are continuous and increasing and let* $\mu : [a,b] \to \mathbb{R}$ *be a function of bounded variation.*

(a) *If*

$$\int_\nu^b y(\tau)\,d\mu(\tau) \leq \int_\nu^b x(\tau)\,d\mu(\tau) \;\; for\ every\ \ \nu \in [a,b], \tag{1.27}$$

and

$$\int_a^b y(\tau)\,d\mu(\tau) = \int_a^b x(\tau)\,d\mu(\tau) \tag{1.28}$$

hold, then for every continuous convex function ϕ, *we have*

$$\int_a^b \phi(y(\tau))\,d\mu(\tau) \leq \int_a^b \phi(x(\tau))\,d\mu(\tau). \tag{1.29}$$

(b) If (1.27) *holds, then* (1.29) *holds for every continuous increasing convex function* ϕ.

Definition 1.12 *Let* $F(\tau)$, $G(\tau)$ *be two continuous and increasing functions for* $\tau \geq 0$ *such that* $F(0) = G(0) = 0$ *and define*

$$\overline{F}(\tau) = 1 - F(\tau), \quad \overline{G}(\tau) = 1 - G(\tau) \quad for \quad \tau \geq 0. \tag{1.30}$$

(cf.[144], p.330) $\overline{F}(\tau)$ *is said to majorize* $\overline{G}(\tau)$, *in symbol,* $\overline{F}(\tau) \succ \overline{G}(\tau)$, *for* $\tau \in [0, +\infty)$ *if*

$$\int_0^s \overline{G}(\tau)\,d\tau \leq \int_0^s \overline{F}(\tau)\,d\tau \quad \text{for all } s > 0,$$

and

$$\int_0^\infty \overline{G}(\tau)\,d\tau = \int_0^\infty \overline{F}(\tau)\,d\tau < \infty.$$

The following result was obtained by Boland and Proschan (1986) [47] (see [144], p.331):

Theorem 1.19 $\overline{F}(\tau) \succ \overline{G}(\tau)$ *for* $\tau \in [0, +\infty)$ *holds iff*

$$\int_0^\infty \phi(\tau)\,dF(\tau) \leq \int_0^\infty \phi(\tau)\,dG(\tau) \tag{1.31}$$

holds for all convex functions ϕ, *provided the integrals are finite.*

The following theorem is a slight extension of Lemma 2 in [120] which is proved by L. Maligranda, J. Pečarić, L. E. Persson (1995):

Theorem 1.20 ([109]) *Let* w, x *and* y *be positive functions on* $[a,b]$. *Suppose that* $\phi : [0,\infty) \to \mathbb{R}$ *is a convex function and that*

$$\int_a^\nu y(t)\,w(t)\,dt \leq \int_a^\nu x(t)\,w(t)\,dt, \quad \nu \in [a,b] \quad and$$

$$\int_a^b y(t)\,w(t)\,dt = \int_a^b x(t)\,w(t)\,dt.$$

(i) If y *is a decreasing function on* $[a,b]$, *then*

$$\int_a^b \phi\,(y(t))\,w(t)\,dt \leq \int_a^b \phi\,(x(t))\,w(t)\,dt. \tag{1.32}$$

(i) If x *is an increasing function on* $[a,b]$, *then*

$$\int_a^b \phi\,(x(t))\,w(t)\,dt \leq \int_a^b \phi\,(y(t))\,w(t)\,dt. \tag{1.33}$$

If ϕ *is strictly convex function and* $x \neq y$ $(a.e.)$, *then* (1.32) *and* (1.33) *are strict.*

Proof. As in [120], if we prove the inequalities for $\phi \in C^1[0,\infty)$, then the general case follows from the pointwise approximation of ϕ by smooth convex functions.
Since ϕ is a convex function on $[0,\infty)$, it follows that

$$\phi(u_1) - \phi(u_2) \geq \phi'(u_2)(u_1 - u_2).$$

If we set

$$F(\nu) = \int_a^\nu [x(t) - y(t)]\, w(t)\, dt,$$

then $F(\nu) \geq 0$, $\nu \in [a,b]$, and $F(a) = F(b) = 0$.
Then

$$\begin{aligned}
&\int_a^b [\phi[x(t)] - \phi[y(t)]]\, w(t)\, dt \\
&\geq \int_a^b \phi'[y(t)]\, [x(t) - y(t)]\, w(t)\, dt \\
&= \int_a^b \phi'[y(t)]\, dF(t) \\
&= \left[\phi'[y(t)]\, F(t)\right]_a^b - \int_a^b F(t)\, d\left[\phi'[y(t)]\right] \\
&= -\int_a^b F(t)\, \phi''[y(t)]\, f'(t)\, dt \geq 0.
\end{aligned}$$

The last inequality follows from the convexity of ϕ and y being decreasing.
Similarly, we can prove the case when x is increasing.
If ϕ is strictly convex function and $x \neq y$ $(a.e.)$, then

$$\phi[x(t)] - \phi[y(t)] > \phi'[y(t)]\, [x(t) - y(t)] \quad \text{(a. e.)}.$$

Which gives strict inequality in (1.32) and (1.33). □

The following theorem (see [109]) is a slight extension of Theorem 2 in [161] which is proved by J. Pečarić and S. Abramovich (1997):

Theorem 1.21 ([109]) *Let w be a weight function on $[a,b]$ and let x and y be positive functions on $[a,b]$. Suppose ϕ, $\psi : [0,\infty) \to \mathbb{R}$ are such that ψ is a strictly increasing function and ϕ is a convex function with respect to ψ i.e., $\phi \circ \psi^{-1}$ is convex. Suppose also that*

$$\int_a^\nu \psi(y(t))\, w(t)\, dt \leq \int_a^\nu \psi(x(t))\, w(t)\, dt, \quad \nu \in [a,b] \tag{1.34}$$

and

$$\int_a^b \psi(y(t))\, w(t)\, dt = \int_a^b \psi(x(t))\, w(t)\, dt. \tag{1.35}$$

(i) If y is a decreasing function on $[a,b]$, then (1.32) holds.

(ii) If x is an increasing function on $[a,b]$, then (1.33) holds.

If $\phi \circ \psi^{-1}$ is strictly convex function and $x \neq y$ $(a.e.)$, then the strict inequality holds in (1.32) *and* (1.33).

1.4 Mean Value Theorems

A mean on I^n, where $I \subseteq \mathbb{R}$ is an interval, is every function $M : I^n \to \mathbb{R}$, with property

$$\min\{x_1, x_2, \ldots, x_n\} \le M(x_1, x_2, \ldots, x_n) \le \max\{x_1, x_2, \ldots, x_n\}$$

that holds for every choice of all $x_1, \ldots, x_n \in I$. For mean M we said that is symmetric if for every permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ we have $M(x_1, x_2, \ldots, x_n) = M(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$.

As examples, we present classes of means that follows from the well-known mean value theorems.

Theorem 1.22 (LAGRANGE'S MEAN VALUE THEOREM) *If a function $\varphi : [x, y] \to \mathbb{R}$ is continuous on a closed interval $[x, y]$ and differentiable on the open interval (x, y), then there is at least one point $\xi \in (x, y)$ such that*

$$\varphi'(\xi) = \frac{\varphi(y) - \varphi(x)}{y - x}.$$

Under assumption that a function φ' is invertible, from Lagrange's theorem it follows that there is a unique number

$$\xi = \left(\varphi'\right)^{-1} \left(\frac{\varphi(y) - \varphi(x)}{y - x} \right)$$

which we called **Lagrange's mean** of $[x, y]$.

Lagrange's mean we can generalize using Cauchy's mean value theorem.

Theorem 1.23 (CAUCHY'S MEAN VALUE THEOREM) *Let functions $\varphi, \psi : [x, y] \to \mathbb{R}$ be continuous on an interval $[x, y]$ and differentiable on (x, y) and let $\psi'(t) \neq 0$ for all $t \in (x, y)$. Then there is a point $\xi \in (x, y)$ such that*

$$\frac{\varphi'(\xi)}{\psi'(\xi)} = \frac{\varphi(y) - \varphi(x)}{\psi(y) - \psi(x)}.$$

Under assumption that a function $\frac{\varphi'}{\psi'}$ is invertible, from Cauchy's theorem it follows that there is a unique number

$$\xi = \left(\frac{\varphi'}{\psi'} \right)^{-1} \left(\frac{\varphi(y) - \varphi(x)}{\psi(y) - \psi(x)} \right).$$

which we called **Cauchy's mean** of interval $[x, y]$. Continuous expansion gives $\xi = x$ if $y = x$.

Remark 1.7 *If we take $\psi(x) = x$, as a special case of Cauchy's mean we get Lagrange's mean. Moreover, many well known means in mathematics we can get as special cases of Cauchy's mean. Under assumption that $x, y \in (0, \infty)$ and choosing $\varphi(x) = x^v$ and $\psi(x) = x^u$, $u, v \in \mathbb{R}$, $u \neq v$, $u, v \neq 0$, we obtain two-parameter mean*

$$E_{u,v}(x,y) = \left(\frac{u(y^v - x^v)}{v(y^u - x^u)}\right)^{\frac{1}{v-u}}$$

firstly introduced by Stolarsky ([162], [163]). It's usually known as ***Stolarsky's mean****. Stolarsky has also proved that $E(u,v;x,y)$ can be extended by continuity as follows:*

$$\begin{aligned}
&E_{u,v}(x,y) = \left(\frac{u(y^v - x^v)}{v(y^u - x^u)}\right)^{\frac{1}{v-u}}, \quad u \neq v,\, uv \neq 0,\, x \neq y,\\
&E_{u,0}(x,y) = E_{0,v}(x,y) = \left(\frac{y^u - x^u}{u(\ln y - \ln x)}\right)^{\frac{1}{u}}, \quad u \neq 0,\, x \neq y,\\
&E_{u,u}(x,y) = e^{-\frac{1}{u}}\left(\frac{y^{y^u}}{x^{x^u}}\right)^{\frac{1}{y^u - x^u}}, \quad x \neq y,\, u \neq 0,\\
&E_{0,0}(x,y) = \sqrt{xy}, \quad x \neq y,\\
&E_{u,v}(x,x) = x.
\end{aligned}$$

This mean is symmetric, i.e. $E_{u,v}(x,y) = E_{v,u}(x,y)$ holds for all choice of numbers $u,v \in \mathbb{R}$, $x,y \in (0,\infty)$. It is also monotonic in both parameters, i.e. for $r,s,u,v \in \mathbb{R}$, such that $r \leq s$, $u \leq v$, we have

$$E_{r,u}(x,y) \leq E_{s,v}(x,y).$$

As special cases of the Stolarsky's mean, we get basic means:

- *arithmetic mean:* $E_{1,2}(x,y) = \dfrac{x+y}{2}$,
- *geometric mean:* $E_{0,0}(x,y) = \sqrt{xy}$,
- *harmonic mean:* $E_{-2,-1}(x,y)$
- *power mean of order* r : $E_{r,2r}(x,y) = \left(\dfrac{x^r + y^r}{2}\right)^{\frac{1}{r}}$,
- *logarithmic mean:* $E_{1,0}(x,y) = \dfrac{y-x}{\ln y - \ln x}$,
- *identric mean:* $\lim_{u\to 1} E_{u,u}(x,y) = \dfrac{1}{e}\left(\dfrac{y^y}{x^x}\right)^{\frac{1}{y-x}}$, *etc.*

Definition of mean can be extend to the weighted version.

The weighted power mean of order $s \in \mathbb{R}$ of $\mathbf{x}$, where $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{w} = (w_1,\ldots,w_n)$ are positive real n-tuples such that $\sum_{i=1}^n w_i = 1$, is defined by

$$M_s(\mathbf{x}) = M_s(\mathbf{x},\mathbf{w}) := \begin{cases}
\left(\sum\limits_{i=1}^n w_i x_i^s\right)^{\frac{1}{s}}, & s \neq 0;\\
\prod\limits_{i=1}^n x_i^{w_i}, & s = 0.\\
\min\{x_1,\ldots,x_n\}, & s \to -\infty,\\
\max\{x_1,\ldots,x_n\} & s \to \infty.
\end{cases}$$

This mean are defined in the same manner, with the conditions

$$\min\{x_1,\ldots,x_n\} \leq M_s(\mathbf{x}) \leq \max\{x_1,\ldots,x_n\}$$

for all $x_1, \ldots, x_n$.

As a special case of the weight power mean we get:

- weighted arithmetic mean $M_1(\mathbf{x}) = A_n = \sum_{i=1}^n w_i x_i$
- weighted geometric mean $M_0(\mathbf{x}) = \prod_{i=1}^{n} x_i^{w_i}$
- weighted harmonic mean $M_{-1}(\mathbf{x}) = \frac{1}{\sum_{i=1}^n \frac{w_i}{x_i}}$

Theorem 1.24 (EXTREME VALUE THEOREM) *If $\varphi : [x,y] \to \mathbb{R}$ is continuous function on a closed interval $[x,y]$, then φ is bounded and attains a maximum and minimum value over that interval, i.e. $m \le \varphi(t) \le M$ for all $t \in [x,y]$, where $m = \min_{t\in[x,y]} \varphi(t)$ and $M = \max \varphi(t)$.*

Following the idea described before and using Extreme value theorem, we prove firstly the Lagrange type mean-value theorems and then deduce from them the Cauchy type mean-value theorems. As consequences we generate new Cauchy's type means.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ be real n-tuples. Throughut this section I_1 represents the interval defined by $I_1 = [m_1, M_1]$, where $m_1 = \min\{\min_i x_i, \min_i y_i\}$ and $M_1 = \max\{\max_i x_i, \max_i y_i\}$.

Theorem 1.25 ([26]) *Let $\mathbf{x}$ and $\mathbf{y}$ be two real n-tuples, $\mathbf{x} \succ \mathbf{y}$, and $\phi \in C^2(I_1)$. Then there exists $\xi \in I_1$ such that*

$$\sum_{i=1}^{n} \phi(x_i) - \sum_{i=1}^{n} \phi(y_i) = \frac{\phi''(\xi)}{2} \left\{ \sum_{i=1}^{n} x_i{}^2 - \sum_{i=1}^{n} y_i^2 \right\}. \tag{1.36}$$

Proof. Since ϕ'' is continuous on I_1, so $m \le \phi''(x) \le M$ for $x \in I_1$, where $m = \min_{x\in I_1} \phi''(x)$ and $M = \max_{x\in I_1} \phi''(x)$. Consider the function ϕ_1 and ϕ_2 defined on I_1 as

$$\phi_1(x) = \frac{Mx^2}{2} - \phi(x) \quad and \quad \phi_2(x) = \phi(x) - \frac{mx^2}{2} \quad for\ x \in I_1.$$

It is easily seen that

$$\phi_1''(x) = M - \phi''(x) \quad and \quad \phi_2''(x) = \phi''(x) - m \quad for\ x \in I_1.$$

So, ϕ_1 and ϕ_2 are convex.
Now by applying ϕ_1 for ϕ in Theorem 1.12, we have

$$\sum_{i=1}^{n} \phi_1(y_i) \le \sum_{i=1}^{n} \phi_1(x_i).$$

Hence, we get

$$\sum_{i=1}^{n} \phi(x_i) - \sum_{i=1}^{n} \phi(y_i) \le \frac{M_1}{2} \left\{ \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 \right\}. \tag{1.37}$$

Similarly, by applying ϕ_2 for ϕ in Theorem 1.12, we get

$$\sum_{i=1}^{n} \phi(x_i) - \sum_{i=1}^{n} \phi(y_i) \ge \frac{m_1}{2} \left\{ \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 \right\}. \tag{1.38}$$

If $\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 = 0$, then from (1.37) and (1.38) follows that for any $\xi \in I_1$, (1.36) holds.
If $\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 > 0$, it follows by combining (1.37) and (1.38) that

$$m_1 \le 2 \left(\frac{\sum_{i=1}^{n} \phi(x_i) - \sum_{i=1}^{n} \phi(y_i)}{\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2} \right) \le M_1.$$

Now using the fact that for $m \le \rho \le M$ there exists $\xi \in I_1$ such that $\phi''(\xi) = \rho$, we get (1.36). □

Theorem 1.26 ([26]) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two real n-tuples,* $\mathbf{x} \succ \mathbf{y}$ *and* $\phi, \varphi \in C^2(I_1)$. *Then there exists* $\xi \in I_1$ *such that*

$$\frac{\sum_{i=1}^{n} \phi(x_i) - \sum_{i=1}^{n} \phi(y_i)}{\sum_{i=1}^{n} \varphi(x_i) - \sum_{i=1}^{n} \varphi(y_i)} = \frac{\phi''(\xi)}{\varphi''(\xi)} \tag{1.39}$$

provided that the denominators are non-zero.

Proof. Let a function $k \in C^2(I_1)$ be defined as

$$k = c_1 \phi - c_2 \varphi,$$

where c_1 and c_2 are defined as

$$c_1 = \sum_{i=1}^{n} \varphi(x_i) - \sum_{i=1}^{n} \varphi(y_i),$$
$$c_2 = \sum_{i=1}^{n} \phi(x_i) - \sum_{i=1}^{n} \phi(y_i).$$

Then, using Theorem 1.25 with $f = k$, we have

$$0 = \left(c_1 \frac{\phi''(\xi)}{2} - c_2 \frac{\varphi''(\xi)}{2} \right) \left\{ \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 \right\}. \tag{1.40}$$

By using (1.36) for φ, left hand side of (1.36) is non-zero by our assumption, it follows that $\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 \neq 0$.
Therefore, (1.40) gives

$$\frac{c_2}{c_1} = \frac{\phi''(\xi)}{\varphi''(\xi)}.$$

After putting values of c_1 and c_2, we get (1.39). □

Corollary 1.1 ([26]) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two real n-tuples such that* $\mathbf{x} \succ \mathbf{y}$*, then for distinct* $s,t \in \mathbb{R}\backslash\{0,1\}$*, there exists* $\xi \in I_1$ *such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} x_i^t - \sum_{i=1}^{n} y_i^t}{\sum_{i=1}^{n} x_i^s - \sum_{i=1}^{n} y_i^s}. \tag{1.41}$$

Proof. Set $\phi(x) = x^t$ and $\varphi(x) = x^s$, $s,t \in \mathbb{R}\backslash\{0,1\}$, $s \neq t$ in (1.39), we get (1.41). $\square$

Remark 1.8 *Since the function* $\xi \mapsto \xi^{t-s}$ *is invertible, then from (1.41) we have*

$$m_1 \leq \left\{ \frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} x_i^t - \sum_{i=1}^{n} y_i^t}{\sum_{i=1}^{n} x_i^s - \sum_{i=1}^{n} y_i^s} \right\}^{\frac{1}{t-s}} \leq M_1. \tag{1.42}$$

In fact, similar result can also be given for (1.39). Namely, suppose that ϕ''/φ'' has an inverse function. Then from (1.39), we have

$$\xi = \left(\frac{\phi''}{\varphi''}\right)^{-1} \left(\frac{\sum_{i=1}^{n} \phi(x_i) - \sum_{i=1}^{n} \phi(y_i)}{\sum_{i=1}^{n} \varphi(x_i) - \sum_{i=1}^{n} \varphi(y_i)}\right). \tag{1.43}$$

So, we have that the expression on the right hand side of (1.43) is also a mean.

Theorem 1.27 ([26]) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two decreasing n-tuples,* $\mathbf{w}$ *be a real n-tuple such that conditions* (1.19) *and* (1.20) *are satisfied and* $\phi \in C^2(I_1)$*. Then there exists* $\xi \in I_1$ *such that*

$$\sum_{i=1}^{n} w_i \phi(x_i) - \sum_{i=1}^{n} w_i \phi(y_i) = \frac{\phi''(\xi)}{2} \left\{ \sum_{i=1}^{n} w_i x_i^2 - \sum_{i=1}^{n} w_i y_i^2 \right\}. \tag{1.44}$$

Theorem 1.28 ([26]) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two decreasing n-tuples,* $\mathbf{w}$ *be a real n-tuple such that conditions* (1.19) *and* (1.20) *are satisfied and* $\phi, \varphi \in C^2(I_1)$*. Then there exists* $\xi \in I_1$ *such that*

$$\frac{\sum_{i=1}^{n} w_i \phi(x_i) - \sum_{i=1}^{n} w_i \phi(y_i)}{\sum_{i=1}^{n} w_i \varphi(x_i) - \sum_{i=1}^{n} w_i \varphi(y_i)} = \frac{\phi''(\xi)}{\varphi''(\xi)} \tag{1.45}$$

provided that the denominators are non-zero.

Corollary 1.2 ([26]) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two decreasing n-tuples,* $\mathbf{w}$ *be a real n-tuple such that conditions* (1.19) *and* (1.20) *are satisfied, then for distinct* $s,t \in \mathbb{R}\backslash\{0,1\}$ *there exists* $\xi \in I_1$ *such that*

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} w_i x_i^t - \sum_{i=1}^{n} w_i y_i^t}{\sum_{i=1}^{n} w_i x_i^s - \sum_{i=1}^{n} w_i y_i^s}. \tag{1.46}$$

Remark 1.9 *Since the function* $\xi \mapsto \xi^{t-s}$ *is invertible, then from (1.46) we have*

$$m_1 \leq \left\{ \frac{s(s-1)}{t(t-1)} \frac{\sum_{i=1}^{n} w_i x_i^t - \sum_{i=1}^{n} w_i y_i^t}{\sum_{i=1}^{n} w_i x_i^s - \sum_{i=1}^{n} w_i y_i^s} \right\}^{\frac{1}{t-s}} \leq M_1. \tag{1.47}$$

In fact, similar result can also be given for (1.45). Namely, suppose that ϕ''/φ'' has inverse function. Then from (1.45), we have

$$\xi = \left(\frac{\phi''}{\varphi''}\right)^{-1}\left(\frac{\sum_{i=1}^{n} w_i\,\phi\,(x_i) - \sum_{i=1}^{n} w_i\,\phi\,(y_i)}{\sum_{i=1}^{n} w_i\,\varphi\,(x_i) - \sum_{i=1}^{n} w_i\,\varphi\,(y_i)}\right). \tag{1.48}$$

So, we have that the expression on the right hand side of (1.48) is also a mean.

Theorem 1.29 ([108]) *Let $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{y}$ be an positive n-tuples, $\psi \in C^2([0,\infty))$ and $\varphi \in C^2(I_1)$ such that conditions (1.23) and (1.24) are satisfied. Let $\mathbf{y}$ be a decreasing n-tuple and $\psi'(y) > 0$ for $y \in I_1$, then there exists $\xi \in I_1$ such that*

$$\sum_{i=1}^{n} w_i\,\varphi\,(x_i) - \sum_{i=1}^{n} w_i\,\varphi\,(y_i) = \frac{\psi'(\xi)\,\varphi''(\xi) - \varphi'(\xi)\,\psi''(\xi)}{2\,(\psi'(\xi))^3}\left[\sum_{i=1}^{n} w_i\,\psi^2(x_i) - \sum_{i=1}^{n} w_i\,\psi^2(y_i)\right]. \tag{1.49}$$

Proof. Set $m = \min_{y\in I_1} \Psi(y)$ and $M = \max_{y\in I_1} \Psi(y)$, where

$$\Psi(y) = \frac{\psi'(y)\,\varphi''(y) - \varphi'(y)\,\psi''(y)}{(\psi'(y))^3}.$$

Consider the function ϕ_1 and ϕ_2 defined on I_1 as
$\phi_1(x) = \frac{1}{2}M\,\psi^2(x) - \varphi(x)$ and $\phi_2(x) = \varphi(x) - \frac{1}{2}m\,\psi^2(x)$ for $x \in I_1$.
It is easily seen that for

$$G(x) = \phi_1\left[\psi^{-1}(x)\right] = \frac{1}{2}Mx^2 - \varphi\left[\psi^{-1}(x)\right]$$

we have

$$G''(x) = M - \frac{\psi'\left[\psi^{-1}(x)\right]\,\varphi''\left[\psi^{-1}(x)\right] - \varphi'\left[\psi^{-1}(x)\right]\,\psi''\left[\psi^{-1}(x)\right]}{(\psi'\left[\psi^{-1}(x)\right])^3}.$$

Similarly,

$$H(x) = \phi_2\left[\psi^{-1}(x)\right] = \varphi\left[\psi^{-1}(x)\right] - \frac{1}{2}mx^2.$$

We have

$$H''(x) = \frac{\psi'\left[\psi^{-1}(x)\right]\,\varphi''\left[\psi^{-1}(x)\right] - \varphi'\left[\psi^{-1}(x)\right]\,\psi''\left[\psi^{-1}(x)\right]}{(\psi'\left[\psi^{-1}(x)\right])^3} - m.$$

This shows that ϕ_1 and ϕ_1 are convex functions with respect to ψ.
Applying (1.21) for ϕ_1 and ϕ_2, we have

$$\sum_{i=1}^{n} w_i\,\phi_1(x_i) \geq \sum_{i=1}^{n} w_i\,\phi_1(y_i)$$

and

$$\sum_{i=1}^{n} w_i\, \phi_2\left(x_i\right) \geq \sum_{i=1}^{n} w_i\, \phi_2\left(y_i\right),$$

that is,

$$\frac{M}{2}\left[\sum_{i=1}^{n} w_i\, \psi^2\left(x_i\right) - \sum_{i=1}^{n} w_i\, \psi^2\left(y_i\right)\right] \geq \sum_{i=1}^{n} w_i\, \varphi\left(x_i\right) - \sum_{i=1}^{n} w_i\, \varphi\left(y_i\right) \tag{1.50}$$

and

$$\sum_{i=1}^{n} w_i\, \varphi\left(x_i\right) - \sum_{i=1}^{n} w_i\, \varphi\left(y_i\right) \geq \frac{m}{2}\left[\sum_{i=1}^{n} w_i\, \psi^2\left(x_i\right) - \sum_{i=1}^{n} w_i\, \psi^2\left(y_i\right)\right]. \tag{1.51}$$

By combining (1.50) and (1.51), (1.49) follows from continuity of Ψ. □

Theorem 1.30 ([108]) *Let* $\mathbf{w}$, $\mathbf{x}$ *and* $\mathbf{y}$ *be positive n-tuples,* $\psi \in C^2([0,\infty))$ *and* $\varphi_1, \varphi_2 \in C^2(I_1)$ *such that conditions (1.23) and (1.24) are satisfied. Let* $\mathbf{y}$ *be a decreasing n-tuple and* $\psi'(y) > 0$ *for* $y \in I_1$*, then there exists* $\xi \in I_1$ *such that*

$$\frac{\psi'(\xi)\, \varphi_1''(\xi) - \varphi_1'(\xi)\, \psi''(\xi)}{\psi'(\xi)\, \varphi_2''(\xi) - \varphi_2'(\xi)\, \psi''(\xi)} = \frac{\sum_{i=1}^{n} w_i\, \varphi_1\left(x_i\right) - \sum_{i=1}^{n} w_i\, \varphi_1\left(y_i\right)}{\sum_{i=1}^{n} w_i\, \varphi_2\left(x_i\right) - \sum_{i=1}^{n} w_i\, \varphi_2\left(y_i\right)} \tag{1.52}$$

provided that $\psi'(y)\, \varphi_2''(y) - \varphi_2'(y)\, \psi''(y) \neq 0$ *for every* $y \in I_1$.

Proof. Define the functional $\Theta : C^2(I_1) \to \mathbb{R}$ by

$$\Theta(\varphi) = \sum_{i=1}^{n} w_i\, \varphi\left(x_i\right) - \sum_{i=1}^{n} w_i\, \varphi\left(y_i\right)$$

and set $\varphi_0 = \Theta(\varphi_2)\, \varphi_1 - \Theta(\varphi_1)\, \varphi_2$. Obviously $\Theta(\varphi_0) = 0$. Using Theorem 1.29, there exists $\xi \in I_1$ such that

$$\Theta(\varphi_0) = \frac{\psi'(\xi)\, \varphi_0''(\xi) - \varphi_0'(\xi)\, \psi''(\xi)}{2\left(\psi'(\xi)\right)^3}\left[\sum_{i=1}^{n} w_i\, \psi^2\left(x_i\right) - \sum_{i=1}^{n} w_i\, \psi^2\left(y_i\right)\right]. \tag{1.53}$$

We give a proof that the expression in square brackets in (1.53) is non-zero due to $\boldsymbol{x} \neq \boldsymbol{y}$. Suppose that the expression in square brackets in (1.53) is equal to zero, i.e.,

$$0 = \sum_{i=1}^{n} w_i\, \psi^2\left(x_i\right) - \sum_{i=1}^{n} w_i\, \psi^2\left(y_i\right). \tag{1.54}$$

By using (1.54), (1.23) and (1.24), we have

$$0 = \sum_{i=1}^{n} w_i\, \psi^2\left(x_i\right) - \sum_{i=1}^{n} w_i \psi^2\left(y_i\right) \geq \sum_{i=1}^{n} w_i\left(2\, \psi\left(y_i\right)\right)\left[\psi\left(x_i\right) - \psi\left(y_i\right)\right] \geq 0.$$

This implies

$$\sum_{i=1}^{n} w_i\, \psi^2(x_i) - \sum_{i=1}^{n} w_i\, \psi^2(y_i) = \sum_{i=1}^{n} w_i\, (2\,\psi(y_i))\, [\psi(x_i) - \psi(y_i)]$$

or equivalently

$$\sum_{i=1}^{n} w_i\, (\psi(x_i) - \psi(y_i))^2 = 0.$$

Which obviously implies that $\boldsymbol{x} \neq \boldsymbol{y}$.
Since $\boldsymbol{x} \neq \boldsymbol{y}$, the expression in square brackets in (1.53) is non-zero which implies that $\psi'(\xi)\,\varphi_0''(\xi) - \varphi_0'(\xi)\,\psi''(\xi) = 0$, and this gives (1.52). Notice that Theorem 1.29 for $\varphi = \varphi_2$ implies that the denominator of the right-hand side of (1.52) is non-zero. □

Corollary 1.3 ([108]) *Let* $\mathbf{w}$*,* $\mathbf{x}$ *and* $\mathbf{y}$ *be an positive n-tuples such that conditions* (1.23) *and* (1.24) *are satisfied. Also let* $\mathbf{y}$ *be a decreasing n-tuple, then for* $s,t \in \mathbb{R}\backslash\{0,q\}$, $s \neq t$, *there exists* $\xi \in I_1$ *such that*

$$\xi^{t-s} = \frac{s(s-q)}{t(t-q)} \frac{\sum_{i=1}^{n} w_i x_i^t - \sum_{i=1}^{n} w_i y_i^t}{\sum_{i=1}^{n} w_i x_i^s - \sum_{i=1}^{n} w_i y_i^s}. \tag{1.55}$$

Remark 1.10 *Since the function* $\xi \to \xi^{t-s}$ *is an invertible, therefore from (1.55) we have*

$$m_1 \leq \left(\frac{s(s-q)}{t(t-q)} \frac{\sum_{i=1}^{n} w_i x_i^t - \sum_{i=1}^{n} w_i y_i^t}{\sum_{i=1}^{n} w_i x_i^s - \sum_{i=1}^{n} w_i y_i^s} \right)^{\frac{1}{t-s}} \leq M_1. \tag{1.56}$$

In fact, similar result can also be given for (1.52). Namely, suppose that $\Lambda(y) = (\psi'(y)\,\varphi_1''(y) - \varphi_1'(y)\,\psi''(y)) \,/\, (\psi'(y)\,\varphi_2''(y) - \varphi_2'(y)\,\psi''(y))$ has an inverse function. Then from (1.52), we have

$$\xi = \Lambda^{-1} \left(\frac{\sum_{i=1}^{n} w_i\, \varphi_1(x_i) - \sum_{i=1}^{n} w_i\, \varphi_1(y_i)}{\sum_{i=1}^{n} w_i\, \varphi_2(x_i) - \sum_{i=1}^{n} w_i\, \varphi_2(y_i)} \right). \tag{1.57}$$

So, the expression on the right hand side of (1.57) is a mean.

Throughout this section I_2 denotes the interval defined by

$$I_2 = [m_2, M_2], \text{ where } m_2 = \min\{m_{x(t)}, m_{y(t)}\} \text{ and } M_2 = \max\{M_{x(t)}, M_{y(t)}\}. \tag{1.58}$$

In the above expression, $m_{x(t)}$ and $m_{y(t)}$ are the minimums of $x(t)$ and $y(t)$ respectively. Similarly, $M_{x(t)}$ and $M_{y(t)}$ are the maximums of $x(t)$ and $y(t)$ respectively.

Theorem 1.31 ([26]) *Let* $x(t)$ *and* $y(t)$ *be two functions on* $[a,b]$ *such that* $x(t) \succ y(t)$ *and* $\phi \in C^2(I_2)$. *Then there exists* $\xi \in I_2$ *such that*

$$\int_a^b \phi(x(t))\, dt - \int_a^b \phi(y(t))\, dt = \frac{\phi''(\xi)}{2} \left\{ \int_a^b x^2(t)\, dt - \int_a^b y^2(t)\, dt \right\}.$$

Proof. As in the proof of Theorem 1.25, we use Theorem 1.17 instead of Theorem 1.12. □

Theorem 1.32 ([26]) *Let $x(t)$ and $y(t)$ be two functions in $[a,b]$ such that $x(t) \succ y(t)$ and $\phi, \varphi \in C^2(I_2)$. Then there exists $\xi \in I_2$ such that*

$$\frac{\int_a^b \phi\,(x(t))\,dt - \int_a^b \phi\,(y(t))\,dt}{\int_a^b g\,(x(t))\,dt - \int_a^b g\,(y(t))\,dt} = \frac{\phi''\,(\xi)}{\varphi''\,(\xi)}, \tag{1.59}$$

provided that the denominators are non zero.

Corollary 1.4 ([26]) *Let $x(t)$ and $y(t)$ be two functions in $[a,b]$ such that $x(t) \succ y(t)$, then for distinct $s,r \in \mathbb{R}\backslash\{0,1\}$, there exists $\xi \in I_2$ such that*

$$\xi^{r-s} = \frac{s(s-1)}{r(r-1)} \frac{\int_a^b x^r(t)\,dt - \int_a^b y^r(t)\,dt}{\int_a^b x^s(t)\,dt - \int_a^b y^s(t)\,dt}. \tag{1.60}$$

Remark 1.11 *Since the function $\xi \mapsto \xi^{r-s}$ is invertible, therefore from (1.60) we have*

$$m_2 \leq \left\{ \frac{s(s-1)}{r(r-1)} \frac{\int_a^b x^r(t)\,dt - \int_a^b y^r(t)\,dt}{\int_a^b x^s(t)\,dt - \int_a^b y^s(t)\,dt} \right\}^{\frac{1}{r-s}} \leq M_2. \tag{1.61}$$

In fact, similar result can also be given for (1.59). Namely, suppose that ϕ''/φ'' has an inverse function. Then from (1.59), we have

$$\xi = \left(\frac{\phi''}{\varphi''}\right)^{-1} \left(\frac{\int_a^b \phi\,(x(t))\,dt - \int_a^b \phi\,(y(t))\,dt}{\int_a^b \varphi\,(x(t))\,dt - \int_a^b \varphi\,(y(t))\,dt} \right). \tag{1.62}$$

So, we have that the expression on the right hand side of (1.62) is also a mean.

Theorem 1.33 ([26]) *Let $x(t)$ and $y(t)$ be decreasing real valued functions defined on $[a,b]$ such that conditions* (1.27) *and* (1.28) *are satisfied and $\phi \in C^2(I_2)$ and $\mu : [a,b] \to \mathbb{R}$ be a function of bounded variation, then there exists $\xi \in I_2$ such that*

$$\int_a^b \phi\,(x(t))\,d\mu(t) - \int_a^b \phi\,(y(t))\,d\mu(t) = \frac{\phi''\,(\xi)}{2} \left\{ \int_a^b x^2(t)\,d\mu(t) - \int_a^b y^2(t)\,d\mu(t) \right\}.$$

Theorem 1.34 ([26]) *Let $x(t)$ and $y(t)$ be decreasing real valued functions defined on $[a,b]$ such that conditions* (1.27) *and* (1.28) *are satisfied, $\mu : [a,b] \to \mathbb{R}$ be a function of bounded variation and $\phi, \varphi \in C^2(I_2)$. Then there exists $\xi \in I_2$ such that*

$$\frac{\int_a^b \phi(x(t))\, d\mu(t) - \int_a^b \phi(y(t))\, d\mu(t)}{\int_a^b \varphi(x(t))\, d\mu(t) - \int_a^b \varphi(y(t))\, d\mu(t)} = \frac{\phi''(\xi)}{\varphi''(\xi)}, \tag{1.63}$$

provided that the denominators are non zero.

Corollary 1.5 ([26]) *Let $x(t)$ and $y(t)$ be positive decreasing functions defined on $[a,b]$ such that conditions* (1.27) *and* (1.28) *be satisfied and $\mu : [a,b] \to \mathbb{R}$ be a function of bounded variation, then for $r,s \in \mathbb{R}\backslash\{0,1\}$, $s \neq r$, there exists $\xi \in I_2$ such that*

$$\xi^{r-s} = \frac{s(s-1)}{r(r-1)} \frac{\int_a^b x^r(t)\, d\mu(t) - \int_a^b y^r(t)\, d\mu(t)}{\int_a^b x^s(t)\, d\mu(t) - \int_a^b y^s(t)\, d\mu(t)}. \tag{1.64}$$

Remark 1.12 *Since the function $\xi \mapsto \xi^{r-s}$ is invertible, therefore from (1.64) we have*

$$m_2 \leq \left\{ \frac{s(s-1)}{r(r-1)} \frac{\int_a^b x^r(t)\, d\mu(t) - \int_a^b y^r(t)\, d\mu(t)}{\int_a^b x^s(t)\, d\mu(t) - \int_a^b y^s(t)\, d\mu(t)} \right\}^{\frac{1}{t-s}} \leq M_2. \tag{1.65}$$

In fact, similar result can also be given for (1.63). Namely, suppose that ϕ''/φ'' has an inverse function. Then from (1.63), we have

$$\xi = \left(\frac{\phi''}{\varphi''}\right)^{-1} \left(\frac{\int_a^b \phi(x(t))\, d\mu(t) - \int_a^b \phi(y(t))\, d\mu(t)}{\int_a^b \varphi(x(t))\, d\mu(t) - \int_a^b \varphi(y(t))\, d\mu(t)} \right). \tag{1.66}$$

So, we have that the expression on the right hand side of (1.66) is also a mean.

Theorem 1.35 ([102]) *Let $\overline{F}(\tau)$ and $\overline{G}(\tau)$ be defined in (1.30) such that $\overline{F}(\tau) \succ \overline{G}(\tau)$ and $\phi \in C^2[0,+\infty)$, then there exists $\xi \in [0,+\infty)$ such that*

$$\int_0^\infty \phi(\tau)\, dF(\tau) - \int_0^\infty \phi(\tau)\, dG(\tau) = \frac{\phi''(\xi)}{2} \left\{ \int_0^\infty \tau^2\, dF(\tau) - \int_0^\infty \tau^2\, dG(\tau) \right\}.$$

Proof. As in the proof of Theorem 1.25, we use Theorem 1.19 instead of Theorem 1.12. □

Theorem 1.36 ([102]) *Let $\overline{F}(\tau)$ and $\overline{G}(\tau)$ be defined in (1.30) such that $\overline{F}(\tau) \succ \overline{G}(\tau)$ and $\phi, \varphi \in C^2[0,+\infty)$. Then there exists $\xi \in [0,+\infty)$ such that*

$$\frac{\int_0^\infty \phi(\tau)\, dF(\tau) - \int_0^\infty \phi(\tau)\, dG(\tau)}{\int_0^\infty \varphi(\tau)\, dF(\tau) - \int_0^\infty \varphi(\tau)\, dG(\tau)} = \frac{\phi''(\xi)}{\varphi''(\xi)}, \tag{1.67}$$

provided that the denominators are non zero.

Corollary 1.6 ([102]) *Let $\overline{F}(\tau)$ and $\overline{G}(\tau)$ be defined in (1.30) such that $\overline{F}(\tau) \succ \overline{G}(\tau)$, then for $r,s \in \mathbb{R}\backslash\{0,1\}$, $s \neq r$, there exists $\xi \in [0,+\infty)$ such that*

$$\xi^{r-s} = \frac{s(s-1)}{r(r-1)} \frac{\int_0^\infty \tau^r\, dF(\tau) - \int_0^\infty \tau^r\, dG(\tau)}{\int_0^\infty \tau^s\, dF(\tau) - \int_0^\infty \tau^s\, dG(\tau)}. \tag{1.68}$$

Remark 1.13 *Since the function $\xi \mapsto \xi^{r-s}$ is an invertible, therefore from (1.68) we have*

$$0 \leq \left\{ \frac{s(s-1)}{r(r-1)} \frac{\int_0^\infty \tau^r\, dF(\tau) - \int_0^\infty \tau^r(t)\, dG(\tau)}{\int_0^\infty \tau^s\, dF(\tau) - \int_a^b \tau^s\, dG(\tau)} \right\}^{\frac{1}{r-s}} < \infty. \tag{1.69}$$

In fact, similar result can also be given for (1.67). Namely, suppose that ϕ''/φ'' has an inverse function. Then from (1.67), we have

$$\xi = \left(\frac{\phi''}{\varphi''}\right)^{-1} \left(\frac{\int_0^\infty \phi(\tau)\, dF(\tau) - \int_0^\infty \phi(\tau)\, dG(\tau)}{\int_0^\infty \varphi(\tau)\, dF(\tau) - \int_0^\infty \varphi(\tau)\, dG(\tau)}\right). \tag{1.70}$$

So, we have that the expression on the right hand side of (1.70) is also a mean.

Theorem 1.37 ([109]) *Let w be a weight function on $[a,b]$, $x(t)$ and $y(t)$ be two positive functions on $[a,b]$ such that conditions* (1.34) *and* (1.35) *are satisfied, $\psi \in C^2([0,\infty))$ and $\phi \in C^2(I_2)$. Also let $x(t)$ be a decreasing function on $[a,b]$ and $\psi'(y) > 0$ for $y \in I_2$. Then there exists $\xi \in I_2$ such that*

$$\begin{aligned} &\int_a^b \phi[x(t)]\, w(t)\, dt - \int_a^b \phi[y(t)]\, w(t)\, dt \\ &= \frac{\psi'(\xi)\, \phi''(\xi) - \phi'(\xi)\, \psi''(\xi)}{2\,(\psi'(\xi))^3} \Big[\int_a^b \psi^2[x(t)]\, w(t)\, dt \\ &- \int_a^b \psi^2[y(t)]\, w(t)\, dt \Big]. \end{aligned} \tag{1.71}$$

Theorem 1.38 ([109]) *Let w be a weight function on $[a,b]$, $x(t)$ and $y(t)$ be two positive functions on $[a,b]$ such that conditions* (1.34) *and* (1.35) *are satisfied, $\psi \in C^2([0,\infty))$ and $\phi_1, \phi_2 \in C^2(I_2)$. Also let $x(t)$ be a decreasing function on $[a,b]$, $\psi'(y) > 0$ for $y \in I_2$ and $x(t) \neq y(t)$ (a.e.). Then there exists $\xi \in I_2$ such that*

$$\frac{\psi'(\xi)\, \phi_1''(\xi) - \phi_1'(\xi)\, \psi''(\xi)}{\psi'(\xi)\, \phi_2''(\xi) - \phi_2'(\xi)\, \psi''(\xi)} = \frac{\int_a^b \phi_1[x(t)]\, w(t)\, dt - \int_a^b \phi_1[y(t)]\, w(t)\, dt}{\int_a^b \phi_2[x(t)]\, w(t)\, dt - \int_a^b \phi_2[y(t)]\, w(t)\, dt} \tag{1.72}$$

provided that $\psi'(y)\, \phi_2''(y) - \phi_2'(y)\, \psi''(y) \neq 0$ for every $y \in I_2$.

Proof. Define the functional $\Theta : C^2(I_2) \to \mathbb{R}$ by

$$\Theta(\phi) = \int_a^b \phi\,[x(t)]\,w(t)\,dt - \int_a^b \phi\,[y(t)]\,w(t)\,dt$$

and set $\varphi_0 = \Theta(\phi_2)\,\phi_1 - \Theta(\phi_1)\,\phi_2$. Obviously $\Theta(\phi_0) = 0$. Using Theorem 1.37, there exists $\xi \in I_2$ such that

$$\Theta(\phi_0) = \frac{\psi'(\xi)\,\phi_0''(\xi) - \phi_0'(\xi)\,\psi''(\xi)}{2\,(\psi'(\xi))^3}\left[\int_a^b \psi^2\,[x(t)]\,w(t)\,dt - \int_a^b \psi^2[y(t)]\,w(t)\,dt\right]. \tag{1.73}$$

We give a proof that the expression in square brackets in (1.73) is non-zero due to $x(t) \neq y(t)\,(a.e.)$. Suppose that the expression in square brackets in (1.73) is equal to zero, i.e.,

$$0 = \int_a^b \left[\psi^2\,[x(t)] - \psi^2\,[y(t)]\right] w(t)\,dt. \tag{1.74}$$

In Theorem 1.21, we have that

$$\int_a^x \psi\,[y(t)]\,w(t)\,dt \le \int_a^x \psi\,[x(t)]\,w(t)\,dt,\ \ x \in [a,b].$$

Set

$$F(x) = \int_a^x [\psi\,[x(t)] - \psi\,[y(t)]]\,w(t)\,dt.$$

Obviously $F(x) \geq 0$, $F(a) = F(b) = 0$. By (1.74), obvious estimations and integration by parts, we have

$$\begin{aligned} 0 &= \int_a^b \left[\psi^2\,[x(t)] - \psi^2\,[y(t)]\right] w(t)\,dt \geq \int_a^b 2\,\psi\,[y(t)]\,[\psi\,[x(t)] - \psi\,[y(t)]]\,w(t)\,dt. \\ &= \int_a^b 2\,\psi\,[y(t)]\,dF(t) = -\int_a^b F(t)\,d\,[2\,\psi\,[y(t)]] \geq 0. \end{aligned}$$

This implies

$$\int_a^b \left[\psi^2\,[x(t)] - \psi^2\,[y(t)]\right] w(t)\,dt = \int_a^b 2\,\psi\,[y(t)]\,[\psi\,[x(t)] - \psi\,[y(t)]]\,w(t)\,dt.$$

or equivalently

$$\int_a^b (\psi\,[x(t)] - \psi\,[y(t)])^2\,w(t)\,dt = 0.$$

Which obviously implies that $x(t) = y(t)(a.e.)$.
Since $x(t) \neq y(t)\,(a.e.)$, the expression in square brackets in (1.73) is non-zero which implies that $\psi'(\xi)\,\phi_0''(\xi) - \phi_0'(\xi)\,\psi''(\xi) = 0$, and this gives (1.72). Notice that Theorem 1.37 for $\phi = \phi_2$ implies that the denominator of the right-hand side of (1.72) is non-zero. □

Corollary 1.7 ([109]) *Let w be a weight function on $[a,b]$ and let $x(t)$ and $y(t)$ be two positive functions on $[a,b]$ such that conditions* (1.34) *and* (1.35) *are satisfied. Also let $x(t)$ be a decreasing function on $[a,b]$ and $x(t) \neq y(t)$ (a.e.), then for $s,t \in \mathbb{R}\backslash\{0,q\}$, $s \neq t$, there exists $\xi \in I_2$ such that*

$$\xi^{t-s} = \frac{s(s-q)}{t(t-q)} \frac{\int_a^b x^t(r)w(r)\,dr - \int_a^b y^t(r)w(r)\,dr}{\int_a^b x^s(r)w(r)\,dr - \int_a^b y^s(r)w(r)\,dr}. \tag{1.75}$$

Proof. Set $\phi_1(x) = x^t$, $\phi_2(x) = x^s$ and $\psi(x) = x^q$, in (1.72), we get (1.75). □

Remark 1.14 *Since the function $\xi \to \xi^{t-s}$ is invertible, therefore from (1.75) we have*

$$m_2 \leq \left(\frac{s(s-q)}{t(t-q)} \frac{\int_a^b x^t(r)w(r)\,dr - \int_a^b y^t(r)w(r)\,dr}{\int_a^b x^s(r)w(r)\,dr - \int_a^b y^s(r)w(r)\,dr} \right)^{\frac{1}{t-s}} \leq M_2. \tag{1.76}$$

In fact, similar result can also be given for (1.72). Namely, suppose that $\Lambda(y) = (\psi'(y)\,\phi_1''(y) - \phi_1'(y)\,\psi''(y)) \,/\, (\psi'(y)\,\phi_2''(y) - \phi_2'(y)\,\psi''(y))$ has an inverse function. Then from (1.72), we have

$$\xi = \Lambda^{-1} \left(\frac{\int_a^b \phi_1[x(r)]\,w(r)\,dr - \int_a^b \phi_1[y(r)]\,w(r)\,dr}{\int_a^b \phi_2[x(r)]\,w(r)\,dr - \int_a^b \phi_2[y(r)]\,w(r)\,dr} \right). \tag{1.77}$$

So, we have that the expression on the right hand side of (1.77) is also a mean.

1.5 n-Exponential Convexity

Here I denotes an open interval in $\mathbb{R}$.

Definition 1.13 ([144, p. 2]) *A function $\phi : I \to \mathbb{R}$ is convex on an interval I if*

$$\phi(x_1)(x_3 - x_2) + \phi(x_2)(x_1 - x_3) + \phi(x_3)(x_2 - x_1) \geq 0, \tag{1.78}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Now, let us recall some definitions and facts about exponentially convex functions:

Definition 1.14 ([142]) *For a fixed $n \in \mathbb{N}$, a function $\phi : I \to \mathbb{R}$ is* ***n-exponentially convex in the Jensen sense*** *on I if*

$$\sum_{k,l=1}^{n} \alpha_k \alpha_l \phi\left(\frac{x_k + x_l}{2}\right) \geq 0$$

holds for all $\alpha_k \in \mathbb{R}$ and $x_k \in I$, $k = 1, 2, \ldots, n$.

Definition 1.15 ([142]) *A function $\phi : I \to \mathbb{R}$ is* ***n-exponentially convex*** *on I if it is* n-exponentially convex *in the Jensen sense and continuous on I.*

Remark 1.15 *From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}, m \leq n$.*

Proposition 1.1 *If $\phi : I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^m$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leq n$. Particularly,*

$$\det\left[\phi\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^m \geq 0$$

for all $m \in \mathbb{N}$, $m = 1,2,\ldots,n$.

Definition 1.16 *A function $\phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

Definition 1.17 *A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.*

Remark 1.16 *It is easy to show that $\phi : I \to \mathbb{R}^+$ is* ***log-convex in the Jensen sense*** *if and only if*

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex *if and only if it is 2-exponentially convex.*

Corollary 1.8 *If $\phi : I \to (0,\infty)$ is an exponentially convex function, then ϕ is a* ***log-convex function*** *that is*

$$\phi(\lambda x + (1-\lambda)y) \leq \phi^{\lambda}(x)\phi^{1-\lambda}(y), \; for\ all\ x, y \in I, \; \lambda \in [0,1].$$

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

Definition 1.18 *The second order* ***divided difference*** *of a function $\phi : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by*

$$\begin{aligned} [y_i;\phi] &= \phi(y_i), \quad i = 0,1,2, \\ [y_i, y_{i+1};\phi] &= \frac{\phi(y_{i+1}) - \phi(y_i)}{y_{i+1} - y_i}, \quad i = 0,1, \\ [y_0, y_1, y_2;\phi] &= \frac{[y_1, y_2;\phi] - [y_0, y_1;\phi]}{y_2 - y_0}. \end{aligned} \tag{1.79}$$

Remark 1.17 *The value* $[y_0,y_1,y_2;\phi]$ *is independent of the order of the points* y_0,y_1, *and* y_2. *By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows:* $\forall y_0,\, y_1,\, y_2 \in I$

$$\lim_{y_1\to y_0}[y_0,y_1,y_2;\phi] = [y_0,y_0,y_2;\phi] = \frac{\phi(y_2)-\phi(y_0)-\phi'(y_0)(y_2-y_0)}{(y_2-y_0)^2},\;\; y_2 \neq y_0,$$

provided that ϕ' *exists, and furthermore, taking the limits* $y_i \to y_0 (i=1,2)$ *in (1.79), we get*

$$[y_0,y_0,y_0;\phi] \; = \; \lim_{y_i\to y_0}[y_0,y_1,y_2;\phi] = \frac{\phi''(y_0)}{2} \text{ for } i=1,2$$

provided that ϕ'' *exists.*
Assuming that $\phi^{(j-1)}(x)$ *exists, we define*

$$[\underbrace{x,\ldots,x}_{j-times};\phi] = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \tag{1.80}$$

The notion of n-convexity goes back to Popoviciu [148]. We follow the definition given by Karlin [90]:

Definition 1.19 *A function* $\phi : [a,b] \to \mathbb{R}$ *is said to be* ***n-convex*** *on* $[a,b]$, $n \geq 0$ *if for all choices of* $(n+1)$ *distinct points in* $[a,b]$, *the n-th order divided difference of* ϕ *satisfies*

$$[x_0,\ldots,x_n;\phi] \geq 0.$$

Under the assumptions of Theorems 1.12 consider the functionals

$$F_1(\boldsymbol{x},\boldsymbol{y},\phi) = \sum_{i=1}^{n}\phi(x_i) - \sum_{i=1}^{n}\phi(y_i). \tag{1.81}$$

Under the assumptions of Theorem 1.14 or Theorem 1.15(i) consider the functional

$$F_2(\boldsymbol{x},\boldsymbol{y},\mathbf{w},\phi) = \sum_{i=1}^{n} w_i\phi(x_i) - \sum_{i=1}^{n} w_i\phi(y_i). \tag{1.82}$$

Under the assumptions of Theorem 1.18 consider the functional

$$F_3(x,y,\phi) = \int_a^b \phi(x(\tau))\, d\mu(\tau) - \int_a^b \phi(y(\tau))\, d\mu(\tau). \tag{1.83}$$

Under the assumptions of Theorem 1.19 consider the functional

$$F_4(F,G,\phi) = \int_0^\infty \phi(\tau)\,dG(\tau) - \int_0^\infty \phi(\tau)\,dF(\tau). \tag{1.84}$$

Under the assumptions of Theorem 1.20 (i) consider the functional

$$F_5(x,y,\phi) = \int_a^b \phi(x(t))\, w(t)\,dt - \int_a^b \phi(y(t))\, w(t)\,dt. \tag{1.85}$$

Under the assumptions of Theorem 1.16(i) consider the functional

$$\tilde{F}_1(\boldsymbol{x},\boldsymbol{y},\mathbf{w},\phi\circ\psi^{-1}) = \sum_{i=1}^{n} w_i\phi\,(x_i) - \sum_{i=1}^{n} w_i\phi\,(y_i)\,. \tag{1.86}$$

Under the assumptions of Theorem 1.21 (i) consider the functional

$$\tilde{F}_2(x,y,w,\phi\circ\psi^{-1}) = \int_a^b \phi\,(x(t))\,w(t)\,dt - \int_a^b \phi\,(y(t))\,w(t)\,dt. \tag{1.87}$$

Theorem 1.39 *Let $F_j (j = 1,2,..,5)$ be linear functionals as defined in (1.81),(1.82), (1.83), (1.84) and (1.85). Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on interval I such that the function $t \to [y_0,y_1,y_2;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Then the following statements hold.*

(i) The function $t \to F_j(.,.,\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[F_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1,..,t_m \in J$. Particularly,

$$\det[F_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m} \ge 0 \textit{ for all } m = 1,2,\ldots,n.$$

(ii) If the function $t \to F_j(.,.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. Fix $j = 1,2,\ldots,5$.
(i) Let us define the function

$$\omega(y) = \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}(y),$$

where $t_{kl} = \frac{t_k+t_l}{2}$, $t_k \in J, b_k \in \mathbb{R}$, $k = 1,2,\ldots,n$.
Since the function $t \to [y_0,y_1,y_2;\phi_t]$ is n-exponentially convex in the Jensen sense on J by assumption, it follows that

$$[y_0,y_1,y_2;\omega] = \sum_{k,l=1}^{n} b_k b_l [y_0,y_1,y_2;\phi_{t_{kl}}] \ge 0,$$

which implies that ω is convex on J. Hence $F_j(.,.,\omega) \ge 0$, which is equivalent to

$$\sum_{k,l=1}^{n} b_k b_l F_j(.,.,\phi_{t_{kl}}) \ge 0,$$

and so we conclude that the function $t \to F_j(.,.,\phi_t)$ is n-exponentially convex function in the Jensen sense on J.
The remaining part follows from Proposition 1.1.

(ii) If the function $t \to F_j(.,.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. □

The following corollary is an immediate consequence of the above theorem.

Corollary 1.9 *Let $\digamma_j (j = 1,2,..,5)$ be linear functionals as defined in (1.81),(1.82), (1.83), (1.84) and (1.85). Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on inteval I such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold.*

(i) The function $t \to \digamma_j(.,.,\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\digamma_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\digamma_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m \geq 0 \text{ for all } m = 1,2,\ldots,n.$$

(ii) If the function $t \to \digamma_j(.,.,\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 1.10 *Let $\digamma_j (j = 1,2,..,5)$ be linear functionals as defined in (1.81),(1.82), (1.83), (1.84) and (1.85). Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on interval interval I such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Further, assume that $\digamma_j(.,.,\phi_t)$ is strictly positive for $\phi_t \in \Omega$. Then the following statements hold.*

(i) If the function $t \to \digamma_j(.,.,\phi_t)$ is continuous on J, then it is 2-exponentially convex on J and so it log convex on J and for $r,s,t \in J$ such that $r < s < t$, we have

$$(\digamma_j(.,.,\phi_s))^{t-r} \leq (\digamma_j(.,.,\phi_r))^{t-s}(\digamma_j(.,.,\phi_t))^{s-r}. \tag{1.88}$$

(ii) If the function $t \to \digamma_j(.,.,\phi_t)$ is differentiable on J, then for every $s,t,u,v \in J$, such that $s \leq u$ and $t \leq v$, we have

$$\mathfrak{B}_{s,t}(.,.,\digamma_j,\Omega) \leq \mathfrak{B}_{u,v}(.,.,\digamma_j,\Omega) \tag{1.89}$$

where

$$\mathfrak{B}^j_{s,t}(\Omega) = \mathfrak{B}_{s,t}(.,.,\digamma_j,\Omega) = \begin{cases} \left(\frac{\digamma_j(.,.,\phi_s)}{\digamma_j(.,.,\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\digamma_j(.,.,\phi_s)}{\digamma_j(.,.,\phi_s)}\right), & s = t, \end{cases} \tag{1.90}$$

for $\phi_s, \phi_t \in \Omega$.

Proof.

(i) By Remark 1.16 and Theorem 1.39, we have log-convexity of $\digamma_j(.,.,\phi_t)$ and by using $\phi(x) = \log \digamma_j(.,.,\phi_x)$ in (1.78), we get (1.88).

(ii) For a convex function ϕ, the following inequality holds

$$\frac{\phi(s) - \phi(t)}{s - t} \leq \frac{\phi(u) - \phi(v)}{u - v}, \tag{1.91}$$

for all $s,t,u,v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$ (see [144, p.2]).
Since by (i), the function $\digamma_j(.,.,\phi_s)$ is log-convex, by setting $\phi(s) = \log \digamma_j(.,.,\phi_s)$ in (1.91), we have

$$\frac{\log \digamma_j(.,.,\phi_s) - \log \digamma_j(.,d,\phi_t)}{s-t} \leq \frac{\log \digamma_j(.,.,\phi_u) - \log \digamma_j(.,.,\phi_v)}{u-v}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$; which is equivalent to (1.89).
The cases $s = t$ and $u = v$ can be treated similarly.

□The inequality (1.88) is known as **Lyapunov's inequality** (see [79, p. 27]).

Moreover, several applications of majorization are obtained by using following important example

$$\Big(\sum_{i=1}^{n} x_i, 0, \ldots, 0\Big) \succ (x_1, \ldots, x_n). \tag{1.92}$$

We also give applications of additive and multiplicative majorizations.

Corollary 1.11 *Let $\boldsymbol{x}$ be a real n-tuple and*

$$\hat{\digamma}_1(.,\phi_t) := \phi_t\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} \phi_t(x_i), \tag{1.93}$$

be a linear functional. Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on interval I such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold.

(i) The function $t \to \hat{\digamma}_1(.,\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\hat{\digamma}_1(.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\hat{\digamma}_t(.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m} \geq 0 \text{ for all } m = 1,2,\ldots,n.$$

(ii) If the function $t \to \hat{\digamma}_t(.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. Set $\boldsymbol{x} = (\sum_{i=1}^{n} x_i, 0, \ldots, 0)$ and $\boldsymbol{y} = (x_1, \ldots, x_n)$ from Theorem 1.39 in linear functional $\digamma_1(.,.,\phi_t)$ define in (1.81), we get our required results. □

Here, we define $\log \boldsymbol{x}$ in this way:

$$\log \boldsymbol{x} = (\log x_1, \ldots, \log x_n).$$

Corollary 1.12 *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two positive n-tuples, $\boldsymbol{x} \prec_{\times} \boldsymbol{y}$,*

$$\check{\digamma}_1(\boldsymbol{x},\boldsymbol{y},\phi_t) := \sum_{i=1}^{n} \phi_t(y_i) - \sum_{i=1}^{n} \phi_t(x_i),$$

be linear functional. Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on interval I such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold.

(i) *The function $t \to \check{\mathcal{F}}_1(.,\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\check{\mathcal{F}}_1(.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1,..,t_m \in J$. Particularly,*

$$\det[\check{\mathcal{F}}_t(.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m} \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1,2,\ldots,n.$$

(ii) *If the function $t \to \check{\mathcal{F}}_t(.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Set $\boldsymbol{x} = \log \boldsymbol{x}$ and $\boldsymbol{y} = \log \boldsymbol{y}$ in Theorem 1.39 by using linear functional $\mathcal{F}_1(.,.,\phi_t)$ define in (1.81), we get our required results. □

Remark 1.18 *As Corollary 1.12, we can give Corollary 1.9 and Corollary 1.10 in a similar fashion.*

Theorem 1.40 *Let $\tilde{\mathcal{F}}_j, (j = 1,2)$ be linear functionals as defined in (1.86) and (1.87). Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on interval I such that the function $t \to [y_0,y_1,y_2;\phi_t \circ \psi^{-1}]$, where the function ψ is strictly increasing, is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Then the following statements hold.*

(i) *The function $t \to \tilde{\mathcal{F}}_j(.,.,\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\tilde{\mathcal{F}}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1,..,t_m \in J$. Particularly,*

$$\det[\tilde{\mathcal{F}}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m} \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1,2,\ldots,n.$$

(ii) *If the function $t \to \tilde{\mathcal{F}}_j(.,.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Fix $j = 1,2$.
(i) Let us define the function ω for $t_l \in J, b_l \in \mathbb{R}$, $l \in \{1,2,..,n\}$ as follows

$$\omega(z) = \sum_{l,m=1}^{n} b_l b_m \phi_{\frac{t_l+t_m}{2}}(z),$$

which implies that

$$\omega \circ \psi^{-1}(z) = \sum_{l,m=1}^{n} b_l b_m \phi_{\frac{t_l+t_m}{2}} \circ \psi^{-1}(z),$$

Since the function $t \to [z_0,z_1,z_2;\phi_t \circ \psi^{-1}]$ is n-exponentially convex in the Jensen sense, we have

$$[z_0,z_1,z_2;\omega \circ \psi^{-1}] = \sum_{l,m=1}^{n} b_l b_m [z_0,z_1,z_2;\phi_{\frac{t_l+t_m}{2}} \circ \psi^{-1}] \geq 0,$$

which implies that $\omega \circ \psi^{-1}$ is convex function on J and therefore we have
$\tilde{F}_j(.,.,\omega \circ \psi^{-1}) \geq 0$.
Hence

$$\sum_{l,m=1}^{n} b_k b_l \tilde{F}_j(.,.,\phi_{\frac{t_l+t_m}{2}} \circ \psi^{-1}) \geq 0.$$

We conclude that the function $t \to \tilde{F}_j$ is an n-exponentially convex function on J in the Jensen sense.
(ii) This part is easily followed by definition of n-exponentially convex function. □

As a consequence of the above theorem we give the following corollaries:

Corollary 1.13 *Let $\tilde{F}_j, (j = 1,2)$ be linear functionals as defined in (1.86) and (1.87). Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on interval I such that the function $t \to [y_0,y_1,y_2;\phi_t \circ \psi^{-1}]$, where the function ψ is strictly increasing, is exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Then the following statements hold.*

(i) The function $t \to \tilde{F}_j(.,.,\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\tilde{F}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\tilde{F}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m} \geq 0 \text{ for all } m = 1,2,\ldots,n.$$

(ii) If the function $t \to \tilde{F}_j(.,.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Corollary 1.14 *Let $\tilde{F}_j, (j = 1,2)$ be linear functionals as defined in (1.86) and (1.87). Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on interval I such that the function $t \to [y_0,y_1,y_2;\phi_t \circ \psi^{-1}]$, where the function ψ is strictly increasing, is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Further, assume that $\tilde{F}_j$ is strictly positive for $\phi_t \in \Omega$. Then the following statements hold.*

(i) If the function $t \to \tilde{F}_j(.,.,\phi_t)$ is continuous on J, then it is 2-exponentially convex on J and so it log convex on J and for $r,s,t \in J$ such that $r < s < t$, we have

$$(\tilde{F}_j(.,.,\phi_s))^{t-r} \leq (\tilde{F}_j(.,.,\phi_r))^{t-s} (\tilde{F}_j(.,.,\phi_t))^{s-r}. \tag{1.94}$$

(ii) If the function $t \to \tilde{F}_j(.,.,\phi_t)$ is differentiable on J, then for every $s,t,u,v \in J$, such that $s \leq u$ and $t \leq v$, we have

$$\mathfrak{B}_{s,t}(.,.,\tilde{F}_j,\Omega) \leq \mathfrak{B}_{u,v}(.,.,\tilde{F}_j,\Omega) \tag{1.95}$$

where

$$\mathfrak{B}^j_{s,t}(\Omega) = \mathfrak{B}_{s,t}(.,.,\tilde{F}_j,\Omega) = \begin{cases} \left(\frac{\tilde{F}_j(.,.,\phi_s)}{\tilde{F}_j(.,.,\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\tilde{F}_j(.,.,\phi_s)}{\tilde{F}_j(.,.,\phi_s)}\right), & s = t, \end{cases} \tag{1.96}$$

for $\phi_s, \phi_t \in \Omega$.

Proof. The proof is similar to the proof of Corollorry 1.10. □

Remark 1.19 *Note that the above results still hold when two of the points $y_0, y_1, y_2 \in [a,b]$ coincide, say $y_1 = y_0$, for a family of differentiable functions ϕ_t such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense on J); and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.17 and by using suitable characterizations of convexity.*

1.6 Examples of Exponentially Convex Functions and Cauchy Type Means

In this section we will vary on choice of a family $D = \{\phi_t : t \in J\}$ in order to construct different examples of exponentially convex functions and construct some means.

Example 1.1 *Let*

$$\widetilde{D_1} = \{\psi_t : \mathbb{R} \to [0,\infty) : t \in \mathbb{R}\}$$

be a family of functions defined by

$$\psi_t(x) = \begin{cases} \frac{1}{t^2} e^{tx}, \ t \neq 0; \\ \\ \frac{1}{2} x^2, \quad t = 0. \end{cases}$$

Here we observe that ψ_t is convex with respect to $\psi(x) = x$ which is strictly increasing and continuous. Since, $\psi_t(x)$ is a convex function on $\mathbb{R}$ and $t \to \frac{d^2}{dx^2}\psi_t(x)$ is exponentially convex function [142]. Using analogous arguing as in the proof of Theorems 1.39 and 1.40, we have that $t \mapsto [y_0, y_1, y_2; \psi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 1.9 and 1.13 we conclude that $t \to F_k(.,.,\psi_t)$; $k = 1,\ldots,5$ and $t \to \tilde{F}_j(.,.,\psi_t)$; $j = 1,2$, are exponentially convex in the Jensen sense. It is easy to see that these mappings are continuous, so they are exponentially convex.
Assume that $t \to F_k(.,.,\psi_t) > 0$; $k = 1,\ldots,5$ and $t \to \tilde{F}_j(.,.,\psi_t) > 0$; $j = 1,2$. By using convex functions ψ_t in (1.43) we obtain the following means:
For $k = 1,2,\ldots,5$

$$\mathfrak{M}_{s,t}(.,.,F_k,\widetilde{D_1}) = \begin{cases} \frac{1}{s-t}\log\left(\frac{F_k(.,.,\psi_s)}{F_k(.,.,\psi_t)}\right), & s \neq t; \\ \frac{F_k(.,.,id.\psi_s)}{F_k(.,.,\psi_s)} - \frac{2}{s}, & s = t \neq 0; \\ \frac{F_k(.,.,id.\psi_0)}{3F_k(.,.,\psi_0)}, & s = t = 0. \end{cases}$$

In particular for $k=2$ we have

$$\mathfrak{M}_{s,t}(.,.,\digamma_3,\widetilde{D_1}) = \frac{1}{s-t}\log\left(\frac{t^2}{s^2}\frac{\sum_{i=1}^n p_i e^{sx_i}-\sum_{i=1}^n p_i e^{sy_i}}{\sum_{i=1}^n p_i e^{tx_i}-\sum_{i=1}^n p_i e^{ty_i}}\right);\ s\neq t; s,t\neq 0;$$
$$\mathfrak{M}_{s,s}(.,.,\digamma_3,\widetilde{D_1}) = \frac{\sum_{i=1}^n p_i x_i e^{sx_i}-\sum_{i=1}^n p_i y_i e^{sy_i}}{\sum_{i=1}^n p_i e^{sx_i}-\sum_{i=1}^n p_i e^{sy_i}} - \frac{2}{s};\qquad s\neq 0;$$
$$\mathfrak{M}_{0,0}(.,.,\digamma_3,\widetilde{D_1}) = \frac{\sum_{i=1}^n p_i x_i^3-\sum_{i=1}^n p_i y_i^3}{3\left(\sum_{i=1}^n p_i x_i^2-\sum_{i=1}^n p_i y_i^2\right)}.$$

Since $\mathfrak{M}_{s,t}(.,.,\digamma_k,\widetilde{D_1}) = \log\mathfrak{B}_{s,t}(.,.,\digamma_k,\widetilde{D_1})$ $(k=1,2,\ldots,5)$, so by (1.89) these means are monotonic.
Similar results can also be obtained for $\tilde{\digamma}_j(.,.,\phi_t)$ $(j=1,2)$.

The following two corollaries are the applications of Example 1 given in [102].

Corollary 1.15 *Let $\mathbf{x}$ and $\mathbf{y}$ be two positive n-tuples, $\mathbf{y}\prec_\times \mathbf{x}$,*

$$\bar{\digamma}_1(\mathbf{x},\mathbf{y},\psi_t) = \digamma_1(\log\mathbf{x},\log\mathbf{y},\psi_t) := \begin{cases} \frac{1}{t^2}\left(\sum_{i=1}^n x_i^t - \sum_{i=1}^n y_i^t\right), & t\neq 0;\\ \frac{1}{2}\left(\sum_{i=1}^n \log^2 x_i - \sum_{i=1}^n \log^2 y_i\right), & t=0,\end{cases}$$

and all $x_{[i]}$'s and $y_{[i]}$'s are not equal.
Then the following statements are valid:

(a) *For every $n\in\mathbb{N}$ and $s_1,\ldots,s_n\in\mathbb{R}$, the matrix $\left[\bar{\digamma}_1(.,.,\psi_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^n$ is a positive semi-definite matrix. Particularly*

$$\det\left[\bar{\digamma}_1(.,.,\psi_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^k \geq 0 \tag{1.97}$$

for $k=1,\ldots,n$.

(b) *The function $s\to\bar{\digamma}_1(.,.,\psi_s)$ is exponentially convex.*

(c) *The function $s\to\bar{\digamma}_1(.,.,\psi_s)$ is a* log-convex *on $\mathbb{R}$ and the following inequality holds for $-\infty<r<s<t<\infty$:*

$$\bar{\digamma}_1(.,.,\psi_s)^{t-r} \leq \bar{\digamma}_1(.,.,\psi_r)^{t-s}\,\bar{\digamma}_1(.,.,\psi_t)^{s-r}. \tag{1.98}$$

Corollary 1.16 *Let $\mathbf{x}$ and $\mathbf{y}$ be two positive decreasing n-tuples, $\mathbf{p}=(p_1,\ldots,p_n)$ be a real n-tuple and let*

$$\bar{\digamma}_2(\mathbf{x},\mathbf{y},\psi_t) = \digamma_2(\log\mathbf{x},\log\mathbf{y},\psi_t) := \begin{cases} \frac{1}{t^2}\left(\sum_{i=1}^n p_i x_i^t - \sum_{i=1}^n p_i y_i^t\right), & t\neq 0;\\ \frac{1}{2}\left(\sum_{i=1}^n p_i\log^2 x_i - \sum_{i=1}^n p_i\log^2 y_i\right), & t=0,\end{cases}$$

such that conditions (1.19) and (1.20) are satisfied and $\bar{\digamma}_2(.,.,\psi_t)$ is positive.
Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \mathbb{R}$, the matrix* $\left[\bar{F}_2(.,.,\psi_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\bar{F}_2(.,.,\psi_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^{k} \geq 0 \tag{1.99}$$

for $k = 1, \ldots, n$.

(b) *The function $s \to \bar{F}_2(.,.,\psi_s)$ is exponentially convex.*

(c) *The function $s \to \bar{F}_2(.,.,\psi_s)$ is a $\log$-convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\bar{F}_2(.,.,\psi_s)^{t-r} \leq \bar{F}_2(.,.,\psi_r)^{t-s}\,\bar{F}_2(.,.,\psi_t)^{s-r}. \tag{1.100}$$

Example 1.2 *Let*

$$\widetilde{D_2} = \{\varphi_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

be a family of functions defined by

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}; & t \neq 0,1; \\ -\log x; & t = 0; \\ x\log x, & t = 1. \end{cases} \tag{1.101}$$

Since $\varphi_t(x)$ is a convex function for $x \in \mathbb{R}^+$ and $t \to \frac{d^2}{dx^2}\varphi_t(x)$ is exponentially convex, so by the same arguments given in previous example we conclude that $t \to F_k(.,.,\varphi_t)$; $k = 1, \ldots, 5$ and $t \to \tilde{F}(.,.,\varphi_t)$ are exponentially convex. We assume that $t \to F_k(.,.,\varphi_t) > 0$; $k = 1, \ldots, 5$ and $t \to \tilde{F}(.,.,\varphi_t) > 0$.
For this family of convex functions we can give the following means:
for $k = 1, 2, \ldots, 5$

$$\mathfrak{M}_{s,t}(.,.,F_k,\widetilde{D_2}) = \begin{cases} \left(\frac{F_k(.,.,\varphi_s)}{F_k(.,.,\varphi_t)}\right)^{\frac{1}{s-t}}; & s \neq t; \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{F_k(.,.,\varphi_0\varphi_s)}{F_k(.,.,\varphi_s)}\right); & s = t \neq 0,1; \\ \exp\left(1 - \frac{F_k(.,.,\varphi_0{}^2)}{2F_k(.,.,\varphi_0)}\right); & s = t = 0; \\ \exp\left(-1 - \frac{F_k(.,.,\varphi_0\varphi_1)}{2F_k(.,.,\varphi_1)}\right); & s = t = 1. \end{cases}$$

In particular for $k = 2$ we have

$$\mathfrak{M}_{s,t}(.,.,F_3,\widetilde{D_2}) = \left(\frac{t(t-1)}{s(s-1)}\frac{\sum_{i=1}^n p_i x_i{}^s - \sum_{i=1}^n p_i y_i{}^s}{\sum_{i=1}^n p_i x_i{}^t - \sum_{i=1}^n p_i y_i{}^t}\right)^{\frac{1}{s-t}}; \qquad s \neq t; s,t \neq 0;$$

$$\mathfrak{M}_{s,s}(.,.,F_3,\widetilde{D_2}) = \exp\left(\frac{\sum_{i=1}^n p_i x_i{}^s \log x_i - \sum_{i=1}^n p_i y_i{}^s \log y_i}{\sum_{i=1}^n p_i x_i{}^s - \sum_{i=1}^n p_i y_i{}^s} - \frac{2s-1}{s(s-1)}\right); \qquad s \neq 0,1;$$

$$\mathfrak{M}_{0,0}(.,.,F_3,\widetilde{D_2}) = \exp\left(\frac{\sum_{i=1}^n p_i \log^2 x_i - \sum_{i=1}^n p_i \log^2 y_i}{2\left(\sum_{i=1}^n p_i \log x_i - \sum_{i=1}^n p_i \log y_i\right)} + 1\right);$$

$$\mathfrak{M}_{1,1}(.,.,F_3,\widetilde{D_2}) = \exp\left(\frac{\sum_{i=1}^n p_i x_i \log^2 x_i - \sum_{i=1}^n p_i y_i \log^2 y_i}{2\left(\sum_{i=1}^n p_i x_i \log x_i - \sum_{i=1}^n p_i y_i \log y_i\right)} - 1\right);$$

Since $\mathfrak{M}_{s,t}(.,.,\digamma_k,\widetilde{D_2}) = \mathfrak{B}_{s,t}(.,.,\digamma_k,\widetilde{D_2})$ $(k = 1,2,\ldots,5)$*, so by (1.89) these means are monotonic.*
Similar results can also be obtained for $\tilde{\digamma}_j(.,.,\phi_t)$ $(j = 1,2)$.

Marshall, Olkin and Arnold (2011) give our results about log-convexity in ([123], p.666-667). They use our results in statistical theory. The following corollary is given in ([123], p.373) which is in fact application of our result in Example 2.

Corollary 1.17 *If* W *is a positive random variable for which the expectation exists and* $\alpha \geq \beta$*, then the function*

$$g(t) = \begin{cases} \frac{EW^{\alpha t} - (EW^{\beta t})(EW^{\alpha}/EW^{\beta})^t}{t(t-1)}; \; t \neq 0,1; \\ (\log EW^{\alpha} - E\log W^{\alpha}) - (\log EW^{\beta} - E\log W^{\beta}); \quad t=0; \\ E(W^{\alpha}\log W^{\alpha}) - (EW^{\alpha})(\log EW^{\beta}) \\ -E(W^{\beta}\log W^{\beta}) - (EW^{\beta})(\log EW^{\beta})(EW^{\alpha}/EW^{\beta}), \quad t=1. \end{cases} \tag{1.102}$$

is log convex.

The following remark is given in [26].

Remark 1.20 *Let* $x = \frac{\sum_{i=1}^n p_i y_i}{\sum_{i=1}^n p_i}$ *be such that* $p_i > 0$ *and* $\sum_{i=1}^n p_i = 1$*. If we substitute in Theorem 1.10* $(x_1;x_2;\ldots;x_n) = (x;x;\ldots;x)$ *for* $\digamma_2(\mathbf{x},\mathbf{y},\boldsymbol{w},\varphi_t)$*, we get Lypunov's inequality given in [27]. In fact in such results we have that y is monotonic n-tuple. But since the weights are positive, our results are also valid for arbitrary y.*

Example 1.3 *Let*

$$\widetilde{D_3} = \{\vartheta_t : [0,\infty) \to \mathbb{R} : t \in \mathbb{R}^+\}$$

be a family of functions defined by

$$\vartheta_t(x) = \begin{cases} \frac{x^t}{t(t-1)}; \; t \neq 0,1; \\ x\log x, \quad t=1. \end{cases} \tag{1.103}$$

In our results we use the notation $0\log 0 = 0$.
We can give the similar result as in Example 1.3, as exponential convexity and means.

We give applications of Example 3 which is in fact corollaries in [102].

Corollary 1.18 *Let* $\mathbf{x}$ *be non-negative n-tuple and* $\hat{\digamma}_1$ *is defined in (1.93). Then the following statements are valid:*

(a) *For every* $n \in \mathbb{N}$ *and* $s_1,\ldots,s_n \in \mathbb{R}^+$*, the matrix* $\left[\hat{\digamma}_1(.,.,\vartheta_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^n$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\hat{\digamma}_1(.,.,\vartheta_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^k \geq 0 \tag{1.104}$$

for $k = 1,\ldots,n$.

(*b*) *The function* $s \to \hat{F}_1(.,.,\vartheta_s)$ *is exponentially convex.*

(*c*) *The function* $s \to \hat{F}_1(.,.,\vartheta_s)$ *is a* log*-convex on* $\mathbb{R}^+$ *and the following inequality holds for* $0 < r < s < t < \infty$:

$$\hat{F}_1(.,.,\vartheta_s)^{t-r} \leq \hat{F}_1(.,.,\vartheta_r)^{t-s}\hat{F}_1(.,.,\vartheta_t)^{s-r}. \tag{1.105}$$

Corollary 1.19 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two positive n-tuples,* $\mathbf{y} \prec_{\times} \mathbf{x}$,

$$\check{F}_1(\mathbf{x},\mathbf{y},\psi_t) = \hat{F}_1(\log\mathbf{x},\log\mathbf{y},\psi_t) := \begin{cases} \frac{1}{t^2}\left(\sum_{i=1}^n x_i^t - \sum_{i=1}^n y_i^t\right), & t \neq 0; \\ \frac{1}{2}\left(\sum_{i=1}^n \log^2 x_i - \sum_{i=1}^n \log^2 y_i\right), & t = 0, \end{cases}$$

and all $x_{[i]}$*'s and* $y_{[i]}$*'s are not equal.*
Then the following statements are valid:

(*a*) *For every* $n \in \mathbb{N}$ *and* $s_1,\ldots,s_n \in \mathbb{R}$*, the matrix* $\left[\hat{F}_1(.,.,\psi_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^n$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\check{F}_1(.,.,\psi_{\frac{s_i+s_j}{2}})\right]_{i,j=1}^k \geq 0 \tag{1.106}$$

for $k = 1,\ldots,n$.

(*b*) *The function* $s \to \check{F}_1(.,.,\psi_s)$ *is exponentially convex.*

(*c*) *The function* $s \to \check{F}_1(.,.,\psi_s)$ *is a* log*-convex on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$:

$$\check{F}_1(.,.,\psi_s)^{t-r} \leq \check{F}_1(.,.,\psi_r)^{t-s}\check{F}_1(.,.,\psi_t)^{s-r}. \tag{1.107}$$

We define the following means of Cauchy type for Corollary 1.18 which is also given in [102].

$$\overline{\mathfrak{M}}_{t,r}(.,\hat{F}_1,\widetilde{D_3}) = \left(\frac{\hat{F}_t(.,\vartheta_t)}{\hat{F}_r(.,\vartheta_r)}\right)^{\frac{1}{t-r}}, \quad t,r \in \mathbb{R}^+,\ r \neq t. \tag{1.108}$$

$$\overline{\mathfrak{M}}_{r,r}(.,\hat{F}_1,\widetilde{D_3}) = \exp\left(\frac{\left(\sum_{i=1}^n x_i\right)^r \log\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n x_i{}^r \log x_i}{\left(\sum_{i=1}^n x_i\right)^r - \sum_{i=1}^n x_i{}^r} - \frac{2r-1}{r(r-1)}\right),\ r \neq 1.$$

$$\overline{\mathfrak{M}}_{1,1}(.,\hat{F}_1,\widetilde{D_3}) = \exp\left(\frac{\left(\sum_{i=1}^n x_i\right)\left(\log\left(\sum_{i=1}^n x_i\right)\right)^2 - \sum_{i=1}^n x_i\left(\log x_i\right)^2}{2\left(\left(\sum_{i=1}^n x_i\right)\left(\log\left(\sum_{i=1}^n x_i\right)\right) - \sum_{i=1}^n x_i \log x_i\right)} - 1\right).$$

Similarly as in Example 2, these means are monotonic.

We define the following Cauchy means in [102], which are similar to [146] for $p_i = 1$, $i = 1, \ldots, n$.

$$\widetilde{\mathfrak{M}}_s^{t,r}(.,\hat{\mathcal{F}}_1,\widetilde{D_3}) = \left(\frac{r(r-s)}{t(t-s)}.\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n x_i^t}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n x_i^r}\right)^{\frac{1}{t-r}}, \ t, r, s \in \mathbb{R}^+, t \neq r, t \neq s, r \neq s. \tag{1.109}$$

$$\widetilde{\mathfrak{M}}_s^{s,r}(.,\hat{\mathcal{F}}_1,\widetilde{D_3}) = \left(\frac{r(r-s)}{s^2}.\frac{\sum_{i=1}^n x_i^s \log\left(\sum_{i=1}^n x_i^s\right) - s\sum_{i=1}^n x_i^s \log x_i}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n x_i^r}\right)^{\frac{1}{s-r}}, \ r \neq s.$$

$$\widetilde{\mathfrak{M}}_s^{r,r}(.,\hat{\mathcal{F}}_1,\widetilde{D_3}) = \exp\left(\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} \log\left(\sum_{i=1}^n x_i^s\right) - s\sum_{i=1}^n {x_i}^r \log x_i}{s\left(\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n {x_i}^r\right)} - \frac{2r-s}{r(r-s)}\right), \ r \neq s.$$

$$\widetilde{\mathfrak{M}}_s^{s,s}(.,\hat{\mathcal{F}}_1,\widetilde{D_3}) = \exp\left(\frac{\left(\sum_{i=1}^n x_i^s\right)\left(\log\left(\sum_{i=1}^n x_i^s\right)\right)^2 - s^2\sum_{i=1}^n {x_i}^s (\log x_i)^2}{2s\left(\left(\sum_{i=1}^n x_i^s\right)\log\left(\sum_{i=1}^n x_i^s\right) - s\sum_{i=1}^n {x_i}^s \log x_i\right)} - \frac{1}{s}\right).$$

Similarly as in Example 2, these means are monotonic.
The following Corollary is given in [102].

Corollary 1.20 *Let $t, r, u, v \in \mathbb{R}^+$ such that $t \leq u$, $r \leq v$, then the following inequality is valid*

$$\widetilde{\mathfrak{M}}_s^{t,r}(.,\hat{\mathcal{F}}_1,\widetilde{D_3}) \leq \widetilde{\mathfrak{M}}_s^{u,v}(.,\hat{\mathcal{F}}_1,\widetilde{D_3}). \tag{1.110}$$

Proof. Let

$$\hat{\mathcal{F}}_1(.,\vartheta_t) := \begin{cases} \frac{1}{t(t-1))}\left(\left(\sum_{i=1}^n x_i\right)^t - \sum_{i=1}^n x_i^t\right), & t \neq 1; \\ \sum_{i=1}^n x_i \log\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n x_i \log x_i, & t = 1. \end{cases} \tag{1.111}$$

Using monotonicity of the means, we have

$$\left(\frac{r(r-1)}{t(t-1)}.\frac{\left(\sum_{i=1}^n x_i\right)^t - \sum_{i=1}^n x_i^t}{\left(\sum_{i=1}^n x_i\right)^r - \sum_{i=1}^n x_i^r}\right)^{\frac{1}{t-r}} \leq \left(\frac{u(u-1)}{v(v-1)}.\frac{\left(\sum_{i=1}^n x_i\right)^v - \sum_{i=1}^n x_i^v}{\left(\sum_{i=1}^n x_i\right)^u - \sum_{i=1}^n x_i^u}\right)^{\frac{1}{v-u}}.$$

Since $s > 0$ by substituting $x_i = x_i^s$, $t = \frac{t}{s}$, $r = \frac{r}{s}$, $u = \frac{u}{s}$ and $v = \frac{v}{s}$ in above inequality, we get

$$\left(\frac{r(r-s)}{t(t-s)}.\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n x_i^t}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n x_i^r}\right)^{\frac{s}{t-r}} \leq \left(\frac{u(u-s)}{v(v-s)}.\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{v}{s}} - \sum_{i=1}^n x_i^v}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{u}{s}} - \sum_{i=1}^n x_i^u}\right)^{\frac{s}{v-u}}.$$

By raising power $\frac{1}{s}$, we get (1.110). □

Remark 1.21 *Let us note that in [145], the following function $\phi_t = t\hat{\mathcal{F}}_t$ was considered. It was proved that*

$$\phi_s^{t-r} \leq \phi_r^{t-s}\,\phi_t^{s-r}. \tag{1.112}$$

In [146], it was proved that this implies

$$F_s^{t-r} \leq \frac{s^{t-r}}{r^{t-s}t^{s-r}} F_r^{t-s} F_t^{s-r}.$$

Since $\frac{s^{t-r}}{r^{t-s}t^{s-r}} < 1$, *we have that (1.112) is better than (1.105).*

Example 1.4 *Let*

$$\widetilde{D_4} = \{\theta_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be family of functions defined by

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

Since $t \to \frac{d^2}{dx^2}\theta_t(x) = e^{-x\sqrt{t}}$ *is exponentially convex, being the Laplace transform of a non-negative function [172]. So by same argument given in Example 1.1 we conclude that* $t \to F_k(.,.,\theta_t)$; $k = 1,\ldots,5$ *and* $t \to \tilde{F}_j(.,.,\theta_t)$; $j = 1,2$, *are exponentially convex. We assume that* $t \to F_k(.,.,\theta_t) > 0$; $k = 1,\ldots,5$ *and* $t \to \tilde{F}_j(.,.,\theta_t) > 0$; $j = 1,2$.
For this family of functions we have the following possible cases of $\mu_{s,t}(.,.,F_k,\widetilde{D_4})$:
for $k = 1,2,\ldots,5$

$$\mu_{s,t}(.,.,F_k,\widetilde{D_4}) = \begin{cases} \left(\frac{F_k(.,.,\theta_s)}{F_k(.,.,\theta_t)}\right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(-\frac{(F_k(.,.,id.\theta_s))}{2\sqrt{s}(F_k(.,.,\theta_s))} - \frac{1}{s}\right), & s = t. \end{cases}$$

In particular for $k = 2$ *we have*

$$\mu_{s,t}(F_3,\widetilde{D_4}) = \left(\frac{t}{s}\frac{\sum_{i=1}^n p_i e^{-x_i\sqrt{s}} - \sum_{i=1}^n p_i e^{-y_i\sqrt{s}}}{\sum_{i=1}^n p_i e^{-x_i\sqrt{t}} - \sum_{i=1}^n p_i e^{-y_i\sqrt{t}}}\right)^{\frac{1}{s-t}}, \qquad s \neq t;$$

$$\mu_{s,s}(F_3,\widetilde{D_4}) = \exp\left(-\frac{1}{2\sqrt{s}}\frac{\sum_{i=1}^n p_i x_i e^{-x_i\sqrt{s}} - \sum_{i=1}^n p_i y_i e^{-y_i\sqrt{s}}}{\sum_{i=1}^n p_i e^{-x_i\sqrt{s}} - \sum_{i=1}^n p_i e^{-y_i\sqrt{s}}} - \frac{1}{s}\right).$$

Monotonicity of $\mathfrak{B}_{s,t}(.,.,F_k,\widetilde{D_3})$ *is followed by (1.89). By (1.43)*

$$\mathfrak{M}_{s,t}(.,.,F_k,\widetilde{D_3}) = -(\sqrt{s}+\sqrt{t})\log\mathfrak{B}_{s,t}(.,.,F_k,\widetilde{D_4}) \quad (k = 1,2,\ldots,5)$$

defines a class of means.
Similar results can also be obtained for $\tilde{F}_j(.,.,\phi_t)$ *for* $j = 1,2$.

Example 1.5 *Let*

$$\widetilde{D_5} = \{\phi_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be family of functions defined by

$$\phi_t(x) = \begin{cases} \frac{t^{-x}}{(\log t)^2}, & t \neq 1; \\ \frac{x^2}{2}, & t = 1. \end{cases}$$

Since $\frac{d^2}{dx^2}\phi_t(x) = t^{-x} = e^{-x\ln t} > 0$, *for* $x > 0$, *so by same argument given in Example 1.1 we conclude that* $t \to \digamma_k(.,.,\phi_t)$; $k = 1,\ldots,5$ *and* $t \to \tilde{\digamma}(.,.,\phi_t)$ *are exponentially convex.*
We assume that $t \to \digamma_k(.,.,\phi_t) > 0$; $k = 1,\ldots,5$ *and* $t \to \tilde{\digamma}(.,.,\phi_t) > 0$.
For this family of functions we have the following possible cases of $\mu_{s,t}(.,.,\digamma_k,\widetilde{D_4})$*:*
for $k = 1,2,\ldots,5$

$$\mu_{s,t}(.,.,\digamma_k,\widetilde{D_5}) = \begin{cases} \left(\frac{\digamma_k(.,.,\phi_s)}{\digamma_k(.,.,\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(-\frac{\digamma_k(.,.,id.\phi_s)}{s\digamma_k(.,.,\phi_s)} - \frac{2}{s\log s}\right), & s = t \neq 1; \\ \exp\left(-\frac{1}{3}\frac{\digamma_k(.,.,id.\phi_1)}{\digamma_k(.,.,\phi_1)}\right), & s = t = 1. \end{cases}$$

In particular for $k = 3$ *we have*

$$\mu_{s,t}(.,.,\digamma_3,\widetilde{D_5}) = \left(\frac{(\ln t)^2}{(\ln s)^2}\frac{\sum_{i=1}^n p_i s^{-x_i} - \sum_{i=1}^n p_i s^{-y_i}}{\sum_{i=1}^n p_i t^{-x_i} - \sum_{i=1}^n p_i t^{-y_i}}\right)^{\frac{1}{s-t}}; \qquad s \neq t;\ s,t \neq 1;$$
$$\mu_{s,s}(.,.,\digamma_3,\widetilde{D_5}) = \exp\left(-\frac{1}{s}\frac{\sum_{i=1}^n p_i x_i s^{-x_i} - \sum_{i=1}^n p_i y_i s^{-y_i}}{\sum_{i=1}^n p_i s^{-x_i} - \sum_{i=1}^n p_i s^{-y_i}} - \frac{2}{s\log s}\right); \qquad s \neq 1,$$

Monotonicity of $\mu_{s,t}(.,.,\digamma_k,\widetilde{D_5})$ *is followed by (1.89). By (1.43)*

$$\mathfrak{M}_{s,t}(.,.,\digamma_k,\widetilde{D_5}) = -L(s,t)\log\mu_{s,t}(.,.,\digamma_k,\widetilde{D_5}) \quad (k = 1,2,\ldots,5)$$

defines a class of means, where $L(s,t)$ *is Logarithmic mean defined as:*

$$L(s,t) = \begin{cases} \frac{s-t}{\log s - \log t}, & s \neq t; \\ s, & s=t. \end{cases}$$

Similar results can also be obtained for $\tilde{\digamma}(.,.,\phi_t)$.

Example 1.6 *Let*

$$\widetilde{D_6} = \{\delta_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

be a family of functions defined by

$$\delta_t(x) = \begin{cases} \frac{q^2}{t(t-q)}x^t; & t \neq 0,1; \\ -q\log x; & t=0; \\ qx^q\log x, & t=q. \end{cases} \tag{1.113}$$

Here we observe that δ_t *is convex with respect to* $\psi(x) = x^q$, $q > 0$ *which is strictly increasing and continuous. Since,* $\delta_t(x)$ *is a convex function on* $\mathbb{R}^+$ *and* $t \to \frac{d^2}{dx^2}\delta_t(x)$ *is exponentially convex function [142]. Using analogous arguing as in the proof of Theorem 1.40, we have that* $t \mapsto [y_0,y_1,y_2;\delta_t]$ *is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 1.13 and 1.14 we conclude that* $t \to \tilde{\digamma}(.,.,\delta_t)$ *is exponentially convex in the Jensen sense. It is easy to see that these mappings are continuous, so they are exponentially convex.*
Assume that $t \to \tilde{\digamma}(.,.,\delta_t) > 0$. *By using convex functions* δ_t *in (1.113) we obtain the following means:*
for $k = 3$ *is given in [109],*

$$\mu_{t,s}(.,.,\widetilde{\digamma}_3,\widetilde{D_6}) := \left(\frac{s(s-q)}{t(t-q)}\frac{\int_a^b x^t(r)w(r)\,dr - \int_a^b y^t(r)w(r)\,dr}{\int_a^b x^s(r)w(r)\,dr - \int_a^b y^s(r)w(r)\,dr}\right)^{\frac{1}{t-s}}, \tag{1.114}$$

for $s,t \in \mathbb{R}\setminus\{0,q\}, s \neq t$, as means in broader sense.

$$\log \mu_{s,s}(.,.,\widetilde{F}_3,\widetilde{D_6}) = \frac{\int_a^b x^s(r)\log g(r)w(r)\,dr - \int_a^b y^s(r)\log f(r)w(r)\,dr}{\int_a^b x^s(r)w(r)\,dr - \int_a^b y^s(r)w(r)\,dr} - \frac{2s-q}{s(s-q)}, \ s \neq 0,q.$$

$$\log \mu_{q,q}(.,.,\widetilde{F}_3,\widetilde{D_6}) = \frac{\int_a^b x^q(r)\log^2 x(r)w(r)\,dr - \int_a^b y^q(r)\log^2 y(r)w(r)\,dr}{2\left[\int_a^b x^q(r)\log x(r)w(r)\,dr - \int_a^b y^q(r)\log y(r)w(r)\,dr\right]} - \frac{1}{q}.$$

$$\log \mu_{0,0}(.,.,\widetilde{F}_3,\widetilde{D_6}) = \frac{\int_a^b \log^2 x(r)w(r)\,dr - \int_a^b \log^2 y(r)w(r)\,dr}{2\left[\int_a^b \log x(r)w(r)\,dr - \int_a^b \log y(r)w(r)\,dr\right]} + \frac{1}{q}.$$

By using (1.95), we can prove the monotonicity of these means.
Similar results can be obtained for $\tilde{F}(.,.,\delta_t)$, given in [109] and [108].

1.7 Further Results on Majorization

Theorem 1.41 ([70]) *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on an interval I, $x_i, y_i \in I$ $(i = 1,2,\ldots,n)$, $w_i \geq 0$ $(i = 1,2,\ldots,n)$ with $W_n = \sum_{i=1}^n w_i > 0$. If $(x_i - y_i)_{(i=\overline{1,n})}$ is nondecreasing (nonincreasing), $(y_i)_{(i=\overline{1,n})}$ is nondecreasing (nonincreasing) and satisfying* (1.20)*, then* (1.21) *holds.*

Now we give further generalization of Theorem 1.41. For this use some notations and definitions from [134].

We define the inner product on $\mathbb{R}^n$ by

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle = \sum_{k=1}^{n} x_k y_k w_k \quad \text{for } \boldsymbol{x} = (x_1,\ldots,x_n) \text{ and } \boldsymbol{y} = (y_1,\ldots,y_n), \tag{1.115}$$

where $w_1,\ldots,w_n$ are positive numbers.

We assume that $e = \{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n\}$ is a basis in $\mathbb{R}^n$, and $d = \{\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n\}$ is the dual basis of e, that is $\langle \boldsymbol{e}_i, \boldsymbol{d}_j\rangle = \delta_{ij}$ (Kronecker delta).

We say that a vector $\boldsymbol{v} \in \mathbb{R}^n$ is *e-positive*, if $\langle \boldsymbol{e}_i, \boldsymbol{v}\rangle > 0$ for all $i = 1,\ldots,n$.

We denote $J = \{1,\ldots,n\}$. Let J_1 and J_2 be two sets of indices such that $J_1 \cup J_2 = J$.

Let $\boldsymbol{v} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. A vector $\boldsymbol{z} \in \mathbb{R}^n$ is said to be $\mu,\boldsymbol{v}$*-separable* on J_1 and J_2 (with respect to the basis e), if

$$\langle \boldsymbol{e}_i, \boldsymbol{z} - \mu\boldsymbol{v}\rangle \geq 0 \ \text{ for } i \in J_1, \ \text{ and } \ \langle \boldsymbol{e}_j, \boldsymbol{z} - \mu\boldsymbol{v}\rangle \leq 0 \ \text{ for } j \in J_2 \tag{1.116}$$

(see [134]).

If $\boldsymbol{v}$ is e-positive, then $\boldsymbol{z}$ is $\mu,\boldsymbol{v}$-separable on J_1 and J_2 with respect to e if and only if

$$\max_{j\in J_2}\frac{\langle \boldsymbol{e}_j,\boldsymbol{z}\rangle}{\langle \boldsymbol{e}_j,\boldsymbol{v}\rangle}\le\mu\le\min_{i\in J_1}\frac{\langle \boldsymbol{e}_i,\boldsymbol{z}\rangle}{\langle \boldsymbol{e}_i,\boldsymbol{v}\rangle}. \tag{1.117}$$

A vector $\boldsymbol{z}\in\mathbb{R}^n$ is said to be $\boldsymbol{v}$-*separable* on J_1 and J_2 (with respect to e), if $\boldsymbol{z}$ is μ, $\boldsymbol{v}$-separable on J_1 and J_2 for some μ. By (1.117), $\boldsymbol{z}$ is $\boldsymbol{v}$-separable on J_1 and J_2 with respect to e if and only if

$$\max_{j\in J_2}\frac{\langle \boldsymbol{e}_j,\boldsymbol{z}\rangle}{\langle \boldsymbol{e}_j,\boldsymbol{v}\rangle}\le\min_{i\in J_1}\frac{\langle \boldsymbol{e}_i,\boldsymbol{z}\rangle}{\langle \boldsymbol{e}_i,\boldsymbol{v}\rangle}\quad\text{(provided } \boldsymbol{v} \text{ is } e\text{-positive).} \tag{1.118}$$

We say that a function $\phi: I\subset\mathbb{R}\to\mathbb{R}$ *preserves* $\boldsymbol{v}$-*separability* on J_1 and J_2 with respect to e, if $(\varphi(z_1),\varphi(z_2),\ldots,\varphi(z_n))$ is $\boldsymbol{v}$-separable on J_1 and J_2 with respect to e for each $\boldsymbol{z}=(z_1,z_2,\ldots,z_n)\in I^n$ such that $\boldsymbol{z}$ is $\boldsymbol{v}$-separable on J_1 and J_2 with respect to e.

Theorem 1.42 ([135]) *Let $\phi: I\to\mathbb{R}$ be a continuous convex function on an interval I. Assume $\varphi\in\partial\phi$, where $\partial\phi$ is the subdifferential of ϕ. Let $\mathbf{x}=(x_1,x_2,\ldots,x_m)$, $\mathbf{y}=(y_1,y_1,\ldots,y_m)$ and $\mathbf{w}=(w_1,w_1,\ldots,w_m)$, where $x_i,y_i\in I$, $w_i>0$ for $i\in J=\{1,2,\ldots,m\}$, and let $\mathbf{u}$, $\mathbf{v}\in\mathbb{R}^m$ with $\langle\boldsymbol{u},\boldsymbol{v}\rangle>0$. If there exist index sets J_1 and J_2 with $J_1\cup J_2=J$ such that*

(i) $\mathbf{y}$ *is* $\mathbf{v}$*-separable on J_1 and J_2 with respect to* e*,*

(ii) $\mathbf{x}$-$\mathbf{y}$ *is λ,* $\mathbf{u}$*-separable on J_1 and J_2 with respect to* d*, where $\lambda=\langle\boldsymbol{x}-\boldsymbol{y},\boldsymbol{v}\rangle/\langle\boldsymbol{u},\boldsymbol{v}\rangle$,*

(iii) $\langle\boldsymbol{x}-\boldsymbol{y},\boldsymbol{v}\rangle=0$*, or* $\langle\boldsymbol{x}-\boldsymbol{y},\boldsymbol{v}\rangle\langle\boldsymbol{z},\boldsymbol{u}\rangle\ge 0$*, where* $\boldsymbol{z}=(\varphi(y_1),\varphi(y_2),\ldots,\varphi(y_m))$*,*

(iv) φ *preserves* $\boldsymbol{v}$*-separability on J_1 and J_2 with respect to e,*

then (1.21) *holds.*

Remark 1.22 Theorem 1.42 remains valid for arbitrary interval $I\subset\mathbb{R}$ whenever ϕ and φ are continuous on I (e.g., $\phi\in C^1(I)$).

Remark 1.23 It is not hard to check that the quadratic function $\phi(t):=t^2$, $t\in I$, satisfies condition (iv). So, it follows from Theorem 1.42 that

$$\sum_{k=1}^{n} w_k y_k^2\le\sum_{k=1}^{n} w_k x_k^2, \tag{1.119}$$

provided $\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}$ satisfy the above conditions (i)-(ii) and (iii) for $\boldsymbol{z}=2\boldsymbol{y}$.

Remark 1.24 For some bases e and d and vectors $\boldsymbol{w}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ (see Corollaries 1.21 and 1.22), condition (iv) is satisfied automatically, since $\varphi\in\partial\phi$ is nondecreasing by the convexity of ϕ. In such cases, (iv) can be dropped from Theorem 1.42.

We present refinement of Theorem 1.42 for twice differentiable functions (not necessarily convex).

Theorem 1.43 ([14]) *Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on open interval I. Assume that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with the property that*

$$\gamma \le \phi''(t) \le \Gamma \quad \text{for all } t \in I. \tag{1.120}$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$, where $x_i, y_i \in I$, $w_i > 0$ for $i \in J = \{1, \ldots, n\}$, and let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ with $\langle \boldsymbol{u}, \boldsymbol{v} \rangle > 0$.

If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that

(i) $\boldsymbol{y}$ *is* $\boldsymbol{v}$*-separable on J_1 and J_2 with respect to e,*

(ii) $\boldsymbol{x} - \boldsymbol{y}$ *is* $\lambda, \boldsymbol{u}$*-separable on J_1 and J_2 with respect to d, where $\lambda = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle / \langle \boldsymbol{u}, \boldsymbol{v} \rangle$,*

(iii') $\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle = 0$, *or* $\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle \langle \boldsymbol{z}, \boldsymbol{u} \rangle \ge 0$ *for*

$$\boldsymbol{z} = (\varphi_\gamma(y_1), \ldots, \varphi_\gamma(y_n)) \quad \text{and} \quad \boldsymbol{z} = (\varphi_\Gamma(y_1), \ldots, \varphi_\Gamma(y_n)), \tag{1.121}$$

where

$$\varphi_\gamma(t) := \phi'(t) - \gamma t \quad \text{and} \quad \varphi_\Gamma(t) := \Gamma t - \phi'(t), \quad t \in I, \tag{1.122}$$

(iv') φ_γ *and* φ_Γ *preserve* $\boldsymbol{v}$*-separability on J_1 and J_2 with respect to e,*

then

$$\frac{1}{2}\gamma \sum_{k=1}^{n} w_k (x_k^2 - y_k^2) \le \sum_{k=1}^{n} w_k \phi(x_k) - \sum_{k=1}^{n} w_k \phi(y_k) \le \frac{1}{2}\Gamma \sum_{k=1}^{n} w_k (x_k^2 - y_k^2). \tag{1.123}$$

Proof. Similarly as in the proof of [36, Proposition 1], it is sufficient to apply Theorem 1.42 to the convex functions $\phi_\gamma(t) := \phi(t) - \frac{1}{2}\gamma t^2$ and $\phi_\Gamma(t) := \frac{1}{2}\Gamma t^2 - \phi(t)$, $t \in I$. □

Remark 1.25 *Theorem 1.43 remains valid for arbitrary interval I whenever ϕ and ϕ' are defined and continuous on I.*

Remark 1.26 *For some bases e and d and vectors* **u** *and* **v** *(see Corollaries 1.21 and 1.22), condition (iv') holds automatically, since the functions φ_γ and φ_Γ are nondecreasing by (1.120).*

In the rest of this section, we demonstrate special cases of Theorem 1.43 for various vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and bases e and d in $\mathbb{R}^n$. This leads to generalizations of [135, Corollaries 2.3, 2.6, 2.10, 2.11].

Corollary 1.21 ([14]) *Under the assumptions of Theorem 1.43, let $\boldsymbol{u} = \boldsymbol{v} = (1, \ldots, 1)$ and let $e = d$ be the basis in $\mathbb{R}^n$ (orthonormal with respect to inner product (1.115)) given by*

$$\boldsymbol{e}_i = \boldsymbol{d}_i = \frac{1}{\sqrt{w_i}}(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0), \quad i = 1, \ldots, n. \tag{1.124}$$

Denote

$$\lambda = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle / \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{W_n} \sum_{k=1}^{n} (x_k - y_k) w_k, \quad \text{where } W_n = \sum_{k=1}^{n} w_k. \tag{1.125}$$

If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that *(i)* $\boldsymbol{y}$ *is* $\boldsymbol{v}$*-separable on* J_1 *and* J_2 *with respect to* e*, i.e.,*

$$y_j \le y_i \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{1.126}$$

(ii) $\boldsymbol{x}-\boldsymbol{y}$ *is* $\lambda, \boldsymbol{u}$*-separable on* J_1 *and* J_2 *with respect to* $d = e$*, i.e.,*

$$x_j - y_j \le \lambda \le x_i - y_i \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{1.127}$$

(iii') $\langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{v}\rangle = 0$*, or* $\langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{v}\rangle \langle \boldsymbol{z}, \boldsymbol{v}\rangle \ge 0$ *where* $\boldsymbol{z}$ *and* φ_γ *and* φ_Γ *are defined by (1.121)-(1.122),*

then (1.123) holds.

Proof. It is sufficient to show that condition (iv') in Theorem 1.43 is fulfilled.

Since $\phi_\gamma(t) := \phi(t) - \frac{1}{2}\gamma t^2$, $t \in I$, is a convex function (see (1.120)),

$\varphi_\gamma(t) = \phi_\gamma'(t)$ is a nondecreasing function. If $\mathbf{a} = (a_1, \ldots, a_n)$ is a $\mathbf{v}$-separable vector on J_1 and J_2 with respect to e, then $a_j \le a_i$ for $i \in J_1$ and $j \in J_2$ (see (1.118), (1.115) and (1.124)). Consequently,

$$\varphi_\gamma(a_j) \le \varphi_\gamma(a_i) \quad \text{for } i \in J_1 \text{ and } j \in J_2.$$

Therefore the vector $(\varphi_\gamma(a_1), \ldots, \varphi_\gamma(a_n))$ is $\boldsymbol{v}$-separable on J_1 and J_2 with respect to e. Thus φ_γ preserves $\boldsymbol{v}$-separability on J_1 and J_2.

In a similar way it can proved that φ_Γ preserves $\boldsymbol{v}$-separability on J_1 and J_2 with respect to e.

In summary, condition (iv') is satisfied, as required. □

Observe that conditions (1.126)-(1.127) are satisfied for

$$J_1 = \{1, 2, \ldots, m\} \text{ and } J_2 = \{m+1, \ldots, n\}$$

for some $m \in J$, if both $\boldsymbol{y}$ and $\boldsymbol{x}-\boldsymbol{y}$ are monotonic nonincreasing vectors, i.e.,

$$y_1 \ge \ldots \ge y_n \quad \text{and} \quad x_1 - y_1 \ge \ldots \ge x_n - y_n.$$

Corollary 1.22 ([14]) *Under the assumptions of Theorem 1.43, let* $\boldsymbol{u} = \boldsymbol{v} = (1, \ldots, 1)$ *and let* λ *be as in (1.125). Suppose that* e *is the basis in* $\mathbb{R}^n$ *consisting of the vectors*

$$\boldsymbol{e}_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, \frac{1}{w_i}, -\frac{1}{w_{i+1}}, 0, \ldots, 0), \quad i = 1, \ldots, n-1, \text{ and} \tag{1.128}$$

$$\boldsymbol{e}_n = (0, \ldots, 0, \frac{1}{w_n}). \tag{1.129}$$

Let d *be the dual basis of* e*, that is*

$$\boldsymbol{d}_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0), \quad i = 1, \ldots, n. \tag{1.130}$$

If there exist index sets J_1 *and* J_2 *with* $J_1 \cup J_2 = J$ *such that* *(i)* $\boldsymbol{y}$ *is* $\boldsymbol{v}$*-separable on* J_1 *and* J_2 *with respect to* e*, i.e., there exists* $\mu \in \mathbb{R}$ *satisfying*

$$y_j - y_{j+1} \le 0 \le y_i - y_{i+1} \quad \text{for } i \in J_1 \text{ and } j \in J_2 \tag{1.131}$$

with the convention $y_{n+1} = \mu$,

(ii) $\boldsymbol{x} - \boldsymbol{y}$ *is* $\lambda, \boldsymbol{u}$*-separable on* J_1 *and* J_2 *with respect to* d*, i.e.,*

$$\frac{1}{W_j} \sum_{k=1}^{j} (x_k - y_k) w_k \le \lambda \le \frac{1}{W_i} \sum_{k=1}^{i} (x_k - y_k) w_k \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{1.132}$$

where $W_l = \sum_{k=1}^{l} w_k$ *for* $l = 1, 2, \ldots, n$,

(iii') $\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle = 0$, *or* $\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle \langle \boldsymbol{z}, \boldsymbol{v} \rangle \ge 0$ *where* $\boldsymbol{z}$ *and* φ_γ *and* φ_Γ *are defined by (1.121)-(1.122),*

then (1.123) holds.

Proof. It is not hard to check that condition (iv') of Theorem 1.43 is met (see the proof of Corollary 1.21). Now, Corollary 1.22 follows from Theorem 1.43. □

If $\boldsymbol{y}$ is monotonic nondecreasing, i.e., $y_1 \le y_2 \le \ldots \le y_n$, and $\boldsymbol{x} - \boldsymbol{y}$ is *monotonic nondecreasing in P-mean* [167, p. 318], i.e.,

$$\frac{1}{W_l} \sum_{k=1}^{l} (x_k - y_k) w_k \le \frac{1}{W_{l+1}} \sum_{k=1}^{l+1} (x_k - y_k) w_k \ , \quad l = 1, 2, \ldots, n-1, \tag{1.133}$$

then conditions (1.131)-(1.132) are satisfied for

$$J_1 = \{n\} \text{ and } J_2 = \{1, 2, \ldots, n-1\}.$$

Moreover, (1.133) can be replaced by

$$\frac{1}{W_l} \sum_{k=1}^{l} (x_k - y_k) w_k \le \frac{1}{W_n} \sum_{k=1}^{n} (x_k - y_k) w_k \ , \quad l = 1, 2, \ldots, n-1.$$

Corollary 1.23 ([14]) *Under the assumptions of Theorem 1.43, let* $\boldsymbol{u} = \boldsymbol{v} = (1, 2, \ldots, n)$ *and let* $e = d$ *be the basis in* $\mathbb{R}^n$ *given by (1.124). Denote*

$$\lambda = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle / \langle \boldsymbol{u}, \boldsymbol{v} \rangle \text{ where } \tilde{W}_n = \sum_{k=1}^{n} k^2 w_k. \tag{1.134}$$

If there exist index sets J_1 *and* J_2 *with* $J_1 \cup J_2 = J$ *such that*

(i) $\boldsymbol{y}$ *is* $\boldsymbol{v}$*-separable on* J_1 *and* J_2 *with respect to* e*, i.e.,*

$$\frac{y_j}{j} \le \frac{y_i}{i} \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{1.135}$$

(ii) $\boldsymbol{x}-\boldsymbol{y}$ *is* λ, w*-separable on* J_1 *and* J_2 *with respect to* $d = e$*, i.e.,*

$$\frac{x_j - y_j}{j} \le \lambda \le \frac{x_i - y_i}{i} \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{1.136}$$

(iii') $\langle\langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{v}\rangle = 0$*, or* $\langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{v}\rangle\langle z, \boldsymbol{v}\rangle \ge 0$
where
z *and* φ_γ *and* φ_Γ *are defined by (1.121)-(1.122),*

(iv') φ_γ *and* φ_Γ *preserve* $\boldsymbol{v}$*-separability on* J_1 *and* J_2 *with respect to* e*, i.e., (1.135) implies*

$$\frac{\varphi_\gamma(y_j)}{j} \le \frac{\varphi_\gamma(y_i)}{i} \quad \text{and} \quad \frac{\varphi_\Gamma(y_j)}{j} \le \frac{\varphi_\Gamma(y_i)}{i} \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{1.137}$$

then (1.123) holds.

Proof. Apply Theorem 1.43. □

A vector $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is said to be *star-shaped* [167, p. 318], if

$$\frac{y_l}{l} \le \frac{y_{l+1}}{l+1} \quad \text{for } l = 1, 2, \ldots, n-1. \tag{1.138}$$

A function $\varphi : I \to \mathbb{R}$, $t \in I$, where $I \subset \mathbb{R}^+$, is said to be *star-shaped*, if the function $t \to \frac{\varphi(t)}{t}$ is nondecreasing.

It has been proved in [135] that if $\varphi : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable nondecreasing convex and star-shaped function on open interval I, then φ preserves star-shapeness of vectors, i.e., (1.138) implies

$$\frac{\varphi(y_l)}{l} \le \frac{\varphi(y_{l+1})}{l+1} \quad \text{for } l = 1, 2, \ldots, n-1. \tag{1.139}$$

If $\boldsymbol{y}$ and $\boldsymbol{x}-\boldsymbol{y}$ are star-shaped vectors, and φ_γ and φ_Γ preserve star-shaped vectors, then conditions (1.135)-(1.137) are satisfied for the index sets

$$J_1 = \{m+1, \ldots, n\} \quad \text{and} \quad J_2 = \{1, 2, \ldots, m\}$$

for some m.

Corollary 1.24 ([14]) *Under the assumptions of Theorem 1.43, let* $\boldsymbol{u} = \boldsymbol{v} = (1, 2, \ldots, n)$ *and let* λ *be as in (1.134). Assume that* e *and* d *are the bases in* $\mathbb{R}^n$ *defined by (1.128)-(1.130).*

If there exist index sets J_1 *and* J_2 *with* $J_1 \cup J_2 = J$ *such that*

(i) $\boldsymbol{y}$ *is* $\boldsymbol{v}$*-separable on* J_1 *and* J_2 *with respect to* e*, i.e., there exists* $\mu \in \mathbb{R}$ *satisfying*

$$y_{j+1} - y_j \ge \mu \ge y_{i+1} - y_i \quad \text{for } i \in J_1 \text{ and } j \in J_2 \tag{1.140}$$

with the convention $y_{n+1} = \mu(n+1)$*,* *(ii)* $\boldsymbol{x}-\boldsymbol{y}$ *is* $\lambda,\boldsymbol{u}$*-separable on* J_1 *and* J_2 *with respect to d, i.e.,*

$$\frac{1}{\widehat{W}_j}\sum_{k=1}^{j}(x_k-y_k)w_k \le \lambda \le \frac{1}{\widehat{W}_i}\sum_{k=1}^{i}(x_k-y_k)w_k \quad \text{for } i\in J_1 \text{ and } j\in J_2, \tag{1.141}$$

where $\widehat{W}_l=\sum\limits_{k=1}^{l} kw_k$, $l=1,\ldots,n$,

(iii') $\langle \boldsymbol{x}-\boldsymbol{y},\boldsymbol{v}\rangle=0$, *or* $\langle \boldsymbol{x}-\boldsymbol{y},\boldsymbol{v}\rangle\langle \boldsymbol{z},\boldsymbol{v}\rangle\ge 0$ *where* $\boldsymbol{z}$ *and* φ_γ *and* φ_Γ *are defined by (1.121)-(1.122),*

(iv') φ_γ *and* φ_Γ *preserve* $\boldsymbol{v}$*-separability on* J_1 *and* J_2 *with respect to e, i.e., (1.140) implies that there exist* $\nu,\rho\in\mathbb{R}$ *satisfying*

$$\varphi_\gamma(y_{j+1})-\varphi_\gamma(y_j)\ge \nu \ge \varphi_\gamma(y_{i+1})-\varphi_\gamma(y_i) \quad \text{for } i\in J_1 \text{ and } j\in J_2, \tag{1.142}$$

$$\varphi_\Gamma(y_{j+1})-\varphi_\Gamma(y_j)\ge \rho \ge \varphi_\Gamma(y_{i+1})-\varphi_\Gamma(y_i) \quad \text{for } i\in J_1 \text{ and } j\in J_2 \tag{1.143}$$

with the convention $\varphi_\gamma(y_{n+1})=\nu(n+1)$ *and* $\varphi_\Gamma(y_{n+1})=\rho(n+1)$,

then (1.123) holds.

Proof. Use Theorem 1.43. □

A vector $\boldsymbol{y}=(y_1,\ldots,y_n)$ is said to be *convex* [167, p. 318], if

$$y_2-y_1\le y_3-y_2\le\ldots\le y_n-y_{n-1}. \tag{1.144}$$

Equivalently, (1.144) says that

$$y_l\le\frac{y_{l-1}+y_{l+1}}{2} \quad \text{for } l=2,\ldots,n-1. \tag{1.145}$$

In consequence, a function $\varphi: I\to\mathbb{R}$ preserves convex vectors if (1.145) implies

$$\varphi(y_l)\le\frac{\varphi(y_{l-1})+\varphi(y_{l+1})}{2} \quad \text{for } l=2,\ldots,n-1. \tag{1.146}$$

For instance, if φ is nondecreasing and convex, then (1.146) is met.

Conditions (1.140)-(1.143) are fulfilled for the index sets

$$J_1=\{1,2,\ldots,m\} \text{ and } J_2=\{m+1,\ldots,n\}$$

for some m depending on λ, whenever φ_γ and φ_Γ are nondecreasing convex functions with $\varphi_\gamma(0)=0$ and $\varphi_\Gamma(0)=0$, and $\boldsymbol{x}-\mathbf{y}$ is *monotonic nonincreasing in* $\widehat{P}$*-mean*, i.e.,

$$\frac{1}{\widehat{W}_l}\sum_{k=1}^{l}(x_k-y_k)w_k\ge\frac{1}{\widehat{W}_{l+1}}\sum_{k=1}^{l+1}(x_k-y_k)w_k \quad \text{for } l=1,2,\ldots,n-1,$$

and, in addition, $y=(y_1,\ldots,y_n)$ is a decreasing convex vector such that $y_1\le n(y_2-y_1)$ (e.g., $y=-(n+1,n+2,\ldots,2n)$).

In [36] the following majorization type theorem for the Stieltjes integral and its refinement have been proved (cf. [74, p. 11], [144, pp. 324-325]).

Theorem 1.44 ([36]) *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function on interval I and let $x, y, p, \mu : [a,b] \to I$ be real functions such that:*

(i) x, y, w, μ are continuous on $[a,b]$ with $w(t) \geq 0$ for any $t \in [a,b]$;

(ii) μ is monotonic non-decreasing on $[a,b]$;

(iii) w is of bounded variation on $[a,b]$;

(iv) y is monotonic non-decreasing (non-increasing) and $x-y$ is monotonic non-decreasing (non-increasing) on $[a,b]$ and

$$\int_a^b w(t)x(t)\,d\mu(t) = \int_a^b w(t)y(t)\,d\mu(t).$$

Then

$$\int_a^b w(t)\phi(y(t))\,d\mu(t) \leq \int_a^b w(t)\phi(x(t))\,d\mu(t). \tag{1.147}$$

We present extension of Theorem 1.44 by using generalization of N. A. Sapogov's result.

Lemma 1.3 ([13]) *Let $w, \mu, v, x, y, z : [a,b] \to \mathbb{R}$ be continuous functions on $[a,b]$ with μ be increasing and $w(t), x(t), v(t) > 0$ for all $t \in [a,b]$. Denote $\lambda = \frac{\int_a^b w(t)z(t)v(t)d\mu(t)}{\int_a^b w(t)x(t)v(t)d\mu(t)}$.*

Suppose that there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a,b]$ such that

(i) $\dfrac{y(t_2)}{v(t_2)} \leq \dfrac{y(t_1)}{v(t_1)}$ *for $t_1 \in I_1, t_2 \in I_2$,*

(ii) $\dfrac{z(t_2)}{x(t_2)} \leq \lambda \leq \dfrac{z(t_1)}{x(t_1)}$ *for $t_1 \in I_1, t_2 \in I_2$.*

Then the following inequality holds

$$\begin{aligned}\int_a^b w(t)x(t)y(t)d\mu(t) &\int_a^b w(t)z(t)v(t)d\mu(t) \\ &\leq \int_a^b w(t)z(t)y(t)d\mu(t) \int_a^b w(t)x(t)v(t)d\mu(t).\end{aligned} \tag{1.148}$$

Proof. From (i) we can say that there exists some $\alpha \in \mathbb{R}$ such that

$$\frac{y(t_2)}{v(t_2)} \leq \alpha \leq \frac{y(t_1)}{v(t_1)}, \qquad \text{for } t_1 \in I_1, t_2 \in I_2. \tag{1.149}$$

Let us consider $g(t) = y(t) - \alpha v(t)$ and
$h(t) = z(t)\left(\int_a^b w(t)x(t)v(t)d\mu(t)\right) - x(t)\left(\int_a^b w(t)z(t)v(t)d\mu(t)\right)$, $t \in [a,b]$.

Now from (1.149) we may write

$$g(t_1) \geq 0, \; g(t_2) \leq 0 \;\text{ for } t_1 \in I_1, \; t_2 \in I_2, \tag{1.150}$$

and similarly from (ii) we may write

$$z(t_1) - \lambda x(t_1) \geq 0, \; z(t_2) - \lambda x(t_2) \leq 0 \text{ for } t_1 \in I_1, t_2 \in I_2. \tag{1.151}$$

Since $\int_a^b w(t)x(t)v(t)d\mu(t) > 0$, so multiplying this with (1.151) we obtain

$$h(t_1) \geq 0, \text{ and } h(t_2) \leq 0 \text{ for } t_1 \in I_1, t_2 \in I_2. \tag{1.152}$$

By using (1.150) and (1.152) we have $g(t)h(t) \geq 0$ for all $t \in [a,b]$, so we may write

$$\int_a^b w(t)g(t)h(t)d\mu(t) \geq 0. \tag{1.153}$$

From (1.153) we obtain (1.148). □

Remark 1.27 *In [138] Z. Otachel proved inequality (1.148) using the relation of synchronicity between vectors with respect to dual bases in Banach spaces V and its dual V^*.*

Remark 1.28 *If we set in Lemma 1.3: $v(t) = x(t) = 1$ for every $t \in [a,b]$ we will get Čebyšev's result. On the other hand if we set in the corresponding Čebyšev's result: $z(t) \to \frac{z(t)}{v(t)}$ and $y(t) \to \frac{y(t)}{x(t)}$, we will get Lemma 1.3.*

In the following theorem we prove majorization type inequality by using Lemma 1.3.

Theorem 1.45 ([13]) *Let $\phi : I \to R$ be a continuous convex function on the interval I. If $\varphi \in \partial\phi$ ($\partial\phi$ is the subdifferential of ϕ) and $u,v,w,x,y,\mu : [a,b] \to \mathbb{R}$ are continuous functions such that μ is increasing, $w(t),u(t),v(t) > 0$ and $x(t),y(t) \in I$ for all $t \in [a,b]$. Denote $\lambda = \frac{\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t)}{\int_a^b w(t)u(t)v(t)d\mu(t)}$.*

Suppose that there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a,b]$ such that

$$(i) \; \frac{\varphi(y(t_2))}{v(t_2)} \leq \frac{\varphi(y(t_1))}{v(t_1)} \; \textit{for } t_1 \in I_1, t_2 \in I_2, \tag{1.154}$$

$$(ii) \; \frac{x(t_2) - y(t_2)}{u(t_2)} \leq \lambda \leq \frac{x(t_1) - y(t_1)}{u(t_1)} \; \textit{for } t_1 \in I_1, t_2 \in I_2. \tag{1.155}$$

Under the above assumptions, the following assertions hold.

$$(A) \textit{ If } \int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) = 0, \quad \textit{then (1.147) holds.} \tag{1.156}$$

(B) If $\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) \int_a^b w(t)\varphi(y(t))u(t)d\mu(t) \geq 0$, then (1.147) holds.

Proof. It follows from [36, Theorem 5] that

$$\int_a^b w(t)(\phi(x(t)) - \phi(y(t))d\mu(t)) \geq \int_a^b w(t)(x(t)-y(t))\varphi(y(t))d\mu(t). \tag{1.157}$$

Utilizing Lemma 1.3, we get

$$\begin{aligned}&\int_a^b w(t)(x(t)-y(t))\varphi(y(t))d\mu(t)\\&\geq \frac{\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t)\int_a^b w(t)\varphi(y(t))u(t)d\mu(t)}{\int_a^b w(t)u(t)v(t)d\mu(t)},\end{aligned}\tag{1.158}$$

since $\int_a^b w(t)u(t)v(t)d\mu(t) > 0$. So, if $\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) = 0$ then (refc00) follows from (1.157) and (1.158).

Similarly, if the condition $\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t)\int_a^b w(t)\varphi(y(t))u(t)d\mu(t) \geq 0$ is fulfilled, then (refc00) holds by virtue of (1.157) and (1.158). This completes the proof. □

In fact in the following corollary we prove majorization type inequality by using N. A Sapogov's result.

Corollary 1.25 ([13]) *Under the assumptions of Theorem 1.45, let $u(t) = v(t) = 1$ for all $t \in [a,b]$. Denote $\lambda = \frac{1}{W}\int_a^b w(t)(x(t)-y(t))d\mu(t)$, where $W = \int_a^b w(t)d\mu(t) > 0$.*

If there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a,b]$ such that

$$(i)\ y(t_2) \leq y(t_1)\ \text{for}\ t_1 \in I_1,\ t_2 \in I_2,\tag{1.159}$$

$$(ii)\ x(t_2)-y(t_2) \leq \lambda \leq x(t_1)-y(t_1)\ \text{for}\ t_1 \in I_1, t_2 \in I_2,\tag{1.160}$$

then assertions (A) and (B) of Theorem 1.45 hold.

Proof. It is sufficient to show that condition (i) of Theorem 1.45 is satisfied for $v(t) = 1$, $t \in [a,b]$. Since ϕ is convex function and $\varphi \in \partial\phi$, φ is nondecreasing function, so (1.159) implies (1.154), for $v(t) = 1, t \in [a,b]$. □

Conditions (1.159) and (1.160) are fulfilled for $I_1 = [a,c], I_2 = [c,b]$ where $a < c < b$, if both y and $x-y$ are monotonic nonincreasing functions.
Likewise, if both y and $x-y$ are monotonic nondecreasing functions, then (1.159) and (1.160) hold for $I_1 = [c,b]$ and $I_2 = [a,c]$.
In these cases, Corollary 1.25, assertion (A) of Theorem 1.45, reduces to a result [13, Theorem 6].

Corollary 1.26 ([13]) *Under the assumptions of Theorem 1.45, let $u(t) = v(t) = t$ for all $t \in [a,b] \subset \mathbb{R}^+$. Denote $\lambda = \frac{1}{\tilde{W}}\int_a^b tw(t)(x(t)-y(t))d\mu(t)$, $\tilde{W} = \int_a^b t^2 w(t)d\mu(t) > 0$.*

If there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a,b]$ such that

$$(i)\ \frac{\varphi(y(t_2))}{t_2} \leq \frac{\varphi(y(t_1))}{t_1}\ \text{for}\ t_1 \in I_1,\ t_2 \in I_2,\tag{1.161}$$

$$(ii)\ \frac{x(t_2)-y(t_2)}{t_2} \leq \lambda \leq \frac{x(t_1)-y(t_1)}{t_1}\ \text{for}\ t_1 \in I_1, t_2 \in I_2,\tag{1.162}$$

then assertions (A) and (B) of Theorem 1.45 hold.

Proof. Apply Theorem 1.45. □

Remark 1.29 *For related discrete version of Lemma 1.3, Theorem 1.45, Corollary 1.25 and Corollary 1.26 see [134] and [135].*

We give refinement of (1.147).

Proposition 1.2 ([36]) *Let $\mu : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interior I° of interval I and such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with the property that $\gamma \leq \frac{d^2\mu(z)}{dz^2} \leq \Gamma$ for any $z \in I^\circ$, and let $x, y, p, u : [a,b] \to I$ be real functions such that the conditions $(i) - (iv)$ of Theorem 1.44 are satisfied.*

Then

$$\begin{aligned} \frac{1}{2}\Gamma \int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t) &\geq \int_a^b p(t)\mu(x(t))\,du(t) - \int_a^b p(t)\mu(y(t))\,du(t) \\ &\geq \frac{1}{2}\gamma \int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t). \end{aligned} \tag{1.163}$$

Remark 1.30 It was proved in [36, Remark 4] that if statements $(i) - (iv)$ of Theorem 1.44 are valid, then we have

$$\int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t) \geq 0.$$

By using equality condition for Čebyšev inequality [144, p.197], we have that

$$\int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t) = 0 \text{ iff } x(t) - y(t) \text{ or } x(t) + y(t) \text{ is constant.}$$

Theorem 1.46 ([14]) *Let $\phi \in C^2(I)$, where I is compact interval in $\mathbb{R}$, and let $\mathbf{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$, where $x_i, y_i \in I$, $w_i > 0$ for $i \in J = \{1, \ldots, n\}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ with $\langle \boldsymbol{u}, \boldsymbol{v} \rangle > 0$.*

Suppose that $\mathbf{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}$ satisfy conditions (i)- (iii) from Theorem 1.42, where $\boldsymbol{z} = 2\boldsymbol{y}$ and conditions (iii')-(iv') from Theorem 1.43, where $\gamma := \min_{t \in I} \phi''(t)$ and $\Gamma := \max_{t \in I} \phi''(t)$. Then there exists $\xi \in I$ such that

$$\sum_{k=1}^n w_k\left[\phi(x_k) - \phi(y_k)\right] = \frac{\phi''(\xi)}{2}\sum_{k=1}^n w_k\left(x_k^2 - y_k^2\right). \tag{1.164}$$

Theorem 1.47 ([14]) *Let $\phi, \psi \in C^2(I)$, where I is compact interval in $\mathbb{R}$, and let $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$, where $x_i, y_i \in I$, $w_i > 0$ for $i \in J = \{1, \ldots, n\}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ with $\langle \boldsymbol{u}, \boldsymbol{v} \rangle > 0$, $\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v} \rangle = 0$.*

Suppose that $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}$ satisfy conditions (i)- (ii) from Theorem 1.43 for some index sets J_1 and J_2 ($J_1 \cup J_2 = J$), and all nondecreasing functions defined on I preserve $\boldsymbol{v}$-separability on J_1 and J_2 with respect to e. Then there exists $\xi \in I$ such that

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\sum_{k=1}^n w_k\left[\phi(x_k) - \phi(y_k)\right]}{\sum_{k=1}^n w_k\left[\psi(x_k) - \psi(y_k)\right]}, \tag{1.165}$$

provided that the denominators are non-zero.

Proof. The proof is anologous to the proof of Theorem 1.26. □

Corollary 1.27 ([14]) *Under the assumptions of Theorem 1.47, set $\phi(x) = x^a$ and $\psi(x) = x^b$, for $a,b \in \mathbb{R}\setminus\{0,1\}$, $a \neq b$, with $I \subset \mathbb{R}^+$.*
Then there exists $\xi \in I$ such that

$$\xi^{a-b} = \frac{b(b-1)\sum_{k=1}^{n} w_k\left(x_k^a - y_k^a\right)}{a(a-1)\sum_{k=1}^{n} w_k\left(x_k^b - y_k^b\right)}. \tag{1.166}$$

Proof. Apply Theorem 1.47. □

Remark 1.31 *Since the function $\xi \to \xi^{a-b}$, $a,b \in \mathbb{R}\setminus\{0,1\}$, $a \neq b$, is invertible, then from (1.166) we have*

$$m_1 \leq \left\{\frac{b(b-1)\sum_{k=1}^{n} w_k\left(x_k^a - y_k^a\right)}{a(a-1)\sum_{k=1}^{n} w_k\left(x_k^b - y_k^b\right)}\right\}^{\frac{1}{a-b}} \leq M_1 \tag{1.167}$$

In fact, similar result can also be given for (1.165). Namely, suppose that $\frac{\phi''}{\psi''}$ has inverse function. Then from (1.165) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\sum_{k=1}^{n} w_k\left[\phi(x_k) - \phi(y_k)\right]}{\sum_{k=1}^{n} w_k\left[\psi(x_k) - \psi(y_k)\right]}\right). \tag{1.168}$$

So, the expression on the right hand side of (1.168) is a mean.

Theorem 1.48 ([15]) *Let $\phi \in C^2(I)$, where I is a compact interval in $\mathbb{R}$. If x,y,w,u satisfy conditions (i)-(iv) from Proposition 1.2, then there exists $\xi \in I_I$ such that*

$$\int_a^b w(t)\left[\phi(x(t)) - \phi(y(t))\right] du(t) = \frac{\phi''(\xi)}{2}\left\{\int_a^b w(t)\left[x^2(t) - y^2(t)\right] du(t)\right\}. \tag{1.169}$$

Proof. Analogous to the proof of Theorem 1.25 but use Proposition 1.2. □

Remark 1.32 *In the proof of Theorem 1.48 if $\gamma > 0$ and $x(t) - y(t)$ and $x(t) + y(t)$ are non-constants, then*

$$\int_a^b w(t)\left[\phi(x(t)) - \phi(y(t))\right] du(t) > 0.$$

Theorem 1.49 ([15]) *Let $\phi, \psi \in C^2(I)$, where I is a compact interval in $\mathbb{R}$. If x,y,w,u satisfy conditions (i)-(iv) from Proposition 1.2 and $x(t) - y(t)$ and $x(t) + y(t)$ are non-constants, then there exists $\xi \in I$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\int_a^b w(t)\left[\phi(x(t)) - \phi(y(t))\right] du(t)}{\int_a^b w(t)\left[\psi(x(t)) - \psi(y(t))\right] du(t)}, \tag{1.170}$$

provided that the denominators are non-zero.

Proof. Analogous to the proof of Theorem 1.26. □

Corollary 1.28 ([15]) *Under the assumptions of Theorem 1.49, set $\phi(z) = z^q$ and $\psi(z) = z^r$, for $q, r \in \mathbb{R} \setminus \{0, 1\}$, $q \neq r$, with $I \subset \mathbb{R}^+$. Then there exists $\xi \in I$ such that*

$$\xi^{q-r} = \frac{r(r-1)\int_a^b w(t)\,[x^q(t) - y^q(t)]\,du(t)}{q(q-1)\int_a^b w(t)\,[x^r(t) - y^r(t)]\,du(t)}. \tag{1.171}$$

Proof. Apply Theorem 1.49. □

Remark 1.33 Since the function $\xi \to \xi^{q-r}$ with $q \neq r$ is invertible, then from (1.171) we have

$$m_2 \leq \left\{ \frac{r(r-1)\int_a^b w(t)\,[x^q(t) - y^q(t)]\,du(t)}{q(q-1)\int_a^b w(t)\,[x^r(t) - y^r(t)]\,du(t)} \right\}^{\frac{1}{q-r}} \leq M_2. \tag{1.172}$$

In fact, similar result can also be given for (1.170). Namely, suppose that $\frac{\phi''}{\psi''}$ has inverse function. Then from (1.170) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{\int_a^b w(t)\,[\phi(x(t)) - \phi(y(t))]\,du(t)}{\int_a^b w(t)\,[\psi(x(t)) - \psi(y(t))]\,du(t)} \right). \tag{1.173}$$

So, the expression on the right hand side of (1.173) is also a mean of $x(t)$ and $y(t)$.

Remark 1.34 *We can obtain n-exponential convex functions for Theorem 1.41, Theorem 1.42, Theorem 1.43 and Theorem 1.44 as in Section 3.*

1.8 Majorization Inequalities for Double Integrals

We give majorization inequalities for double integrals.

Theorem 1.50 ([13]) *(a) Let $\phi : I \to \mathbb{R}$ be a continuous convex function on the interval I and $w, x, y : [a,b] \times [c,d] \to \mathbb{R}$ be continuous functions such that $x(t,s), y(t,s)$ be decreasing in $t \in [a,b]$ and let $\mu : [a,b] \to \mathbb{R}$ be a function of bounded variation, $u : [c,d] \to \mathbb{R}$ be increasing function.*

(a_1) If for each $s \in [c,d]$

$$\int_a^\nu w(t,s)y(t,s)\,d\mu(t) \leq \int_a^\nu w(t,s)x(t,s)\,d\mu(t), \quad \nu \in [a,b] \tag{1.174}$$

and

$$\int_a^b w(t,s)x(t,s)\,d\mu(t) = \int_a^b w(t,s)y(t,s)\,d\mu(t) \tag{1.175}$$

hold, then

$$\int_c^d \int_a^b w(t,s)\phi\,(y(t,s))d\mu(t)du(s) \leq \int_c^d \int_a^b w(t,s)\phi\,(x(t,s))d\mu(t)du(s) \tag{1.176}$$

(a_2) *If for each $s \in [c,d]$, (1.174) holds, then for continuous increasing convex function $\phi : I \to \mathbb{R}$, (1.176) holds.*

(b) Suppose that $\phi : [0,\infty) \to \mathbb{R}$ is a convex function and $w,x,y : [a,b]\times[c,d] \to \mathbb{R}^+$ are integrable functions. Let $\mu : [a,b] \to \mathbb{R}$, $u : [c,d] \to \mathbb{R}$ be increasing and satisfying conditions (1.174) and (1.175).

(b_1) *If for each $s \in [c,d]$, $y(t,s)$ is a decreasing function in $t \in [a,b]$, then (1.176) holds.*

(b_2) *If for each $s \in [c,d]$, $x(t,s)$ is an increasing function in $t \in [a,b]$, then the reverse inequality in (1.176) holds.*

(c) Let $\phi : I \to \mathbb{R}$ be continuous convex function on the interval I, $w,x,y : [a,b]\times[c,d]$ $\to I$ be continuous functions with $w(t,s) > 0$ be a function of bounded variation and let $\mu : [a,b] \to \mathbb{R}$, $u : [c,d] \to \mathbb{R}$ be increasing functions. If $y(t,s)$ and $x(t,s) - y(t,s)$ are increasing(decreasing) in $t \in [a,b]$ and satisfying condition (1.175), then (1.176) holds.

(d) Let $\phi : I \to \mathbb{R}$ be a continuous convex function on the interval I, $\varphi \in \partial\phi$ ($\partial\phi$ is the subdifferential of ϕ), $w,x,y,g,h : [a,b]\times[c,d] \to \mathbb{R}$ be continuous functions with $x(t,s), y(t,s) \in I$, $w(t,s), g(t,s), h(t,s) > 0$ and $\mu : [a,b] \to \mathbb{R}$, $u : [c,d] \to \mathbb{R}$ be increasing functions. Denote $\lambda = \frac{\int_a^b w(t,s)(x(t,s)-y(t,s))d\mu(t)}{\int_a^b w(t,s)g(t,s)h(t,s)d\mu(t)}$.
Suppose that there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a,b]$ such that for each $s \in [c,d]$

$$(i)\quad \frac{\varphi(y(t_2,s))}{h(t_2,s)} \leq \frac{\varphi(y(t_1,s))}{h(t_1,s)} \quad \text{for } t_1 \in I_1, t_2 \in I_2, \tag{1.177}$$

$$(ii)\quad \frac{x(t_2,s)-y(t_2,s)}{g(t_2,s)} \leq \lambda \leq \frac{x(t_1,s)-y(t_1,s)}{g(t_1,s)} \quad \text{for } t_1 \in I_1, t_2 \in I_2. \tag{1.178}$$

If $\int_a^b w(t,s)(x(t,s) - y(t,s))h(t,s)d\mu(t) \int_a^b w(t,s)\varphi(y(t,s))w(t,s)d\mu(t) \geq 0$, then (1.176) holds.

Proof. (a) By using Theorem 1.18 we may write

$$\int_a^b w(t,s)\phi\,(y(t,s))\,d\mu(t) \leq \int_a^b w(t,s)\phi\,(x(t,s))\,d\mu(t), \text{ for each } s \in [c,d]. \tag{1.179}$$

Integrating both hand sides with respect to $u(s)$, we deduce the desired result (1.176).

In a similar way we can prove (b),(c) and (d). □

Now, we give a majorization type result by using Green's function.
Consider G defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha\leq s\leq t;\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t\leq s\leq\beta.\end{cases} \tag{1.180}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $\phi:[\alpha,\beta]\to\mathbb{R}$, $\phi\in C^2([\alpha,\beta])$, we can easily show by integrating by parts that the following is valid

$$\phi(x)=\frac{\beta-x}{\beta-\alpha}\phi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\phi(\beta)+\int_\alpha^\beta G(x,s)\phi''(s)ds, \tag{1.181}$$

where the function G is defined as above in (1.180) ([171]).

Theorem 1.51 ([13]) *Let $w,x,y:[a,b]\times[c,d]\to\mathbb{R}$, $\mu:[a,b]\to\mathbb{R}$ and $u:[c,d]\to\mathbb{R}$ be continuous functions and $[\alpha,\beta]$interval such that $x(t,s),y(t,s)\in[\alpha,\beta]$ for $(t,s)\in[a,b]\times[c,d]$. Also let (1.175) holds.*

Then the following are equivalent.

(i) For every continuous convex function $\phi:[\alpha,\beta]\to\mathbb{R}$, (1.176) holds.

(ii) For all $\tau\in[\alpha,\beta]$ holds

$$\int_c^d\int_a^b w(t,s)G(y(t,s),\tau)d\mu(t)du(s)\leq\int_c^d\int_a^b w(t,s)G(x(t,s),\tau)d\mu(t)du(s). \tag{1.182}$$

Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both inequalities, in (1.176) and in (1.182).

Proof. (i)⇒(ii): Let (i) holds. As the function $G(\cdot,\tau)$ $(\tau\in[\alpha,\beta])$ is also continuous and convex, it follows that also for this function (1.176) holds, i.e., (1.182) holds .
(ii)⇒(i): Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be a convex function and without loss of generality we can assume that $\phi\in C^2([\alpha,\beta])$. Also let (ii) holds. Then, we can represent the function ϕ in the form (1.181), where the function G is defined in (1.180). By easy calculation, using (1.181), we can easily get that

$$\begin{aligned}&\int_c^d\int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s)-\int_c^d\int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s)\\ &=\int_\alpha^\beta\left[\int_c^d\int_a^b w(t,s)G(x(t,s),\tau)d\mu(t)du(s)\right.\\ &\left.-\int_c^d\int_a^b w(t,s)G(y(t,s),\tau)d\mu(t)du(s)\right]\phi''(\tau)d\tau.\end{aligned}$$

Since ϕ is a convex function, then $\phi''(\tau)\geq0$ for all $\tau\in[\alpha,\beta]$. So, if for every $\tau\in[\alpha,\beta]$ the inequality (1.182) holds , then it follows that for every convex function $\phi:[\alpha,\beta]\to\mathbb{R}$,

with $\phi \in C^2([\alpha,\beta])$, inequality (1.176) holds.
At the end, note that it is not necessary to demand the existence of the second derivative of the function ϕ ([144, p.172]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials. □

Remark 1.35 *Under the assumptions of Theorem 1.51, if for all $\tau \in [\alpha,\beta]$, the inequality* (1.182) *holds then by setting $\phi(x) = x^2, x \in [\alpha,\beta]$, in (1.176) we get*

$$\int_c^d \int_a^b w(t,s)y^2(t,s)d\mu(t)du(s) \leq \int_c^d w(t,s)x^2(t,s)d\mu(t)du(s). \tag{1.183}$$

Theorem 1.52 ([13]) *Let $\phi \in C^2([\alpha,\beta])$ and $w,x,y : [a,b]\times[c,d] \to \mathbb{R}$, $\mu : [a,b] \to \mathbb{R}$, $u : [a,b] \to \mathbb{R}$ be continuous functions such that $x(t,s), y(t,s) \in [\alpha,\beta]$ for $(t,s) \in [a,b]\times[c,d]$. Let (1.175) holds. If for all $\tau \in [\alpha,\beta]$, the inequality (1.182) holds or if for all $\tau \in [\alpha,\beta]$, the reverse inequality in (1.182) holds, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\int_c^d \int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s) =$$
$$\frac{\psi''(\xi)}{2}\left(\int_c^d\int_a^b w(t,s)x^2(t,s)d\mu(t)du(s) - \int_c^d\int_a^b w(t,s)y^2(t,s)d\mu(t)du(s)\right). \tag{1.184}$$

Proof. The idea of the proof is the same as the proof of Theorem 1.25. □

Theorem 1.53 ([13]) *Let $\phi, \psi \in C^2([\alpha,\beta])$ and w,x,y,u,μ be defined as in Theorem 1.52. Also let (1.175) holds. If for all $\tau \in [\alpha,\beta]$, the inequality (1.182) holds or if for all $\tau \in [\alpha,\beta]$, the reverse inequality in (1.182) holds, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\int_c^d \int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s)}{\int_c^d \int_a^b w(t,s)\psi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\psi(y(t,s))d\mu(t)du(s)}, \tag{1.185}$$

provided that the denominators are non zero.

Proof. The idea of the proof is the same as the proof of Theorem 1.26. □

Corollary 1.29 ([13]) *Under the assumptions of Theorem 1.53, set $\phi(x) = x^l$ and $\psi(x) = x^m$, for $l,m \in \mathbb{R}\setminus\{0,1\}$, $l \neq m$ with $[\alpha,\beta] \subset \mathbb{R}^+$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\xi^{l-m} = \frac{m(m-1)\int_c^d \int_a^b w(t,s)x^l(t,s)\,d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)y^l(t,s)\,d\mu(t)du(s)}{l(l-1)\int_c^d \int_a^b w(t,s)x^m(t,s)\,d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)y^m(t,s)\,d\mu(t)u(s)}, \tag{1.186}$$

provided that the denominator is non zero.

Proof. Theorem 1.53 can be applied. □

Now we are able to introduce generalized Cauchy means from (1.185). Namely, suppose that $\frac{\phi''}{\psi''}$ has inverse function, then from (1.185) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\int_c^d\int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s) - \int_c^d\int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s)}{\int_c^d\int_a^b w(t,s)\psi(x(t,s))d\mu(t)du(s) - \int_c^d\int_a^b w(t,s)\psi(y(t,s))d\mu(t)du(s)}\right). \tag{1.187}$$

Remark 1.36 *Since the function $\xi \to \xi^{l-m}$ with $l \neq m$ is invertible, then from (1.186) we have*

$$\alpha \leq \left\{\frac{m(m-1)\int_c^d\int_a^b w(t,s)x^l(t,s)d\mu(t)du(s) - \int_c^d\int_a^b w(t,s)y^l(t,s)d\mu(t)du(s)}{l(l-1)\int_c^d\int_a^b w(t,s)x^m(t,s)d\mu(t)du(s) - \int_c^d\int_a^b w(t,s)y^m(t,s)d\mu(t)u(s)}\right\}^{\frac{1}{l-m}} \leq \beta. \tag{1.188}$$

We shall say that the expression in the middle defines a class of means.

1.9 On Majorization for Matrices

Matrix majorization: The notion of majorization concerns a partial ordering of the diversity of the components of two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x},\mathbf{y} \in \mathbb{R}^m$. A natural problem of interest is the extension of this notion from m-tuples (vectors) to $n \times m$ matrices. For example, let

$$X = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)' \text{ and } Y = (\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n)'$$

be two $n \times m$ real matrices, where $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$; $\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n$ are the corresponding row vectors.

Definition 1.20 ([144]) *Let X,Y be two $n \times m$ real matrices for $n \geq 2$, $m \geq 2$. X is said to row-wise majorize Y $(X \succ^r Y)$ if $\mathbf{x}_i \succ \mathbf{y}_i$ holds for $i = 1,2,\ldots,n$.*

Theorem 1.54 ([12]) *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on an interval I and $X = [x_{ij}]$, $Y = [y_{ij}]$ and $W = [w_{ij}]$ be matrices, where $x_{ij}, y_{ij} \in I$ and $w_{ij} \in \mathbb{R}$ $(i = 1,2,\ldots,n$, $j = 1,2,\ldots,m)$.*

(a) If $X \succ^r Y$, the following inequality holds

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(y_{ij}) \leq \sum_{i=1}^{n}\sum_{j=1}^{m}\phi(x_{ij}) \tag{1.189}$$

If ϕ is strictly convex on I, then the strict inequality holds in (1.189) if and only if $X \neq Y$.

(b) If $(x_{ij})_{j=\overline{1,m}}$, $(y_{ij})_{j=\overline{1,m}}$ $(i=1,2,..,n)$ *are decreasing and satisfy the following conditions,*

$$\sum_{j=1}^{k} w_{ij}y_{ij} \le \sum_{j=1}^{k} w_{ij}x_{ij}, \quad k=1,2,\ldots,m-1, \tag{1.190}$$

for $i=1,2,\ldots,n$ *and*

$$\sum_{j=1}^{m} w_{ij}y_{ij} = \sum_{j=1}^{m} w_{ij}x_{ij} \tag{1.191}$$

for $i=1,2,\ldots,n$.

Then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}) \le \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}). \tag{1.192}$$

(c) (c_1) *If* $(y_{ij})_{j=\overline{1,m}}$ $(i=1,2,..,n)$ *is decreasing with* $w_{ij}>0$ $(i=1,2,\ldots,n,$ $j=1,2,\ldots,m)$ *and satisfying conditions (1.190) and (1.191), then (1.192) holds. If* ϕ *is strictly convex on* I*, then the strict inequality holds in (1.192) if and only if* $X \neq Y$.

(c_2) *If* $(x_{ij})_{j=\overline{1,m}}$ $(i=1,2,..,n)$ *is increasing with* $w_{ij}>0$ $(i=1,2,\ldots,n,$ $j=1,2,\ldots,m)$ *and satisfying conditions (1.190) and (1.191), then reverse inequality in (1.192) holds. If* ϕ *is strictly convex on* I*, then the reverse strict inequality holds in (1.192) if and only if* $X \neq Y$.

(d) If $(x_{ij}-y_{ij})_{j=\overline{1,m}}$ *and* $(y_{ij})_{j=\overline{1,m}}$ $(i=1,2,..,n)$ *are increasing (decreasing) with* $w_{ij}\ge 0$ $(i=1,2,\ldots,n,$ $j=1,2,\ldots,m)$ *and satisfying condition (1.191), then (1.192) holds. If* ϕ *is strictly convex on* I *and* $w_{ij}>0$*, then the strict inequality holds in (1.192) if and only if* $X \neq Y$.

(e) Let $w_{ij}>0$ $(i=1,2,\ldots,n, j=1,2,..,m)$ *and* $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^m$ *with* $\langle \boldsymbol{u},\boldsymbol{v}\rangle>0$*. If there exist index sets* J_1 *and* J_2 *with* $J_1\cup J_2=J$ *such that for each* $i=1,2,..,n$

- *(i)* $(y_{ij})_{j=\overline{1,m}}$ *is* $\boldsymbol{v}$*-separable on* J_1 *and* J_2 *with respect to e,*
- *(ii)* $(x_{ij}-y_{ij})_{j=\overline{1,m}}$ *is* $\lambda,\boldsymbol{u}$*-separable on* J_1 *and* J_2 *with respect to d, where* $\lambda=\langle (x_{ij}-y_{ij})_{j=\overline{1,m}},\ \boldsymbol{v}\rangle/\langle \boldsymbol{u},\boldsymbol{v}\rangle$,
- *(iii)* $\langle (x_{ij}-y_{ij})_{j=\overline{1,m}},\boldsymbol{v}\rangle = 0$, *or* $\langle (x_{ij}-y_{ij})_{j=\overline{1,m}},\boldsymbol{v}\rangle\langle (z_{ij})_{j=\overline{1,m}},\boldsymbol{u}\rangle \ge 0$, *where* $(z_{ij})_{j=\overline{1,m}}=(\varphi(y_{i1}),\ldots,\varphi(y_{im}))$,
- *(iv)* φ *preserves* $\boldsymbol{v}$*-separability on* J_1 *and* J_2 *with respect to e,*

then (1.192) holds.

Proof. (a) By using Theorem 1.12, we can write

$$\sum_{j=1}^{m} \phi\left(y_{ij}\right) \le \sum_{j=1}^{m} \phi\left(x_{ij}\right), \quad \text{for } i=1,2,\ldots,n. \tag{1.193}$$

Summing (1.12) over i from 1 to n, we get (1.189).
In a similar way, we can prove (b), (c), (d) and (e). □

Theorem 1.55 ([12]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be a continuous convex function on the interval $[\alpha,\beta]$ and $X = [x_{ij}]$, $Y = [y_{ij}]$ and $W = [w_{ij}]$ be matrices, where $x_{ij}, y_{ij} \in [\alpha,\beta]$ and $w_{ij} \in \mathbb{R}$ $(i = 1,2,\ldots,n,\ j = 1,2,\ldots,m)$ such that condition (1.191) is satisfied.*

Then the following two statements are equivalent.

(i) For every continuous convex function $\phi : [\alpha,\beta] \to \mathbb{R}$, (1.192) holds.

(ii) For all $\tau \in [\alpha,\beta]$ holds

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij} G(y_{ij},\tau) \le \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij} G(x_{ij},\tau). \tag{1.194}$$

Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both inequalities, in (1.192) and in (1.194).

Proof. The proof is similar to the proof of Theorem 1.51. □

We give mean value theorems.

Theorem 1.56 ([12]) *Let X, Y and W be matrices as in Theorem 1.54 such that condition (1.191) is satisfied. Let also $\phi \in C^2([\alpha,\beta])$. If for all $\tau \in [\alpha,\beta]$, the inequality (1.194) holds or if for all $\tau \in [\alpha,\beta]$, the reverse inequality in (1.194) holds, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij}) = \frac{\phi''(\xi)}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^2\right). \tag{1.195}$$

Proof. The idea of the proof is the same as the proof of Theorem 1.25. □

Theorem 1.57 ([12]) *Let X, Y and W be matrices as in Theorem 1.54 such that condition (1.191) is satisfied. Let also $\phi, \psi \in C^2([\alpha,\beta])$. If for all $\tau \in [\alpha,\beta]$, the inequality (1.194) holds or if for all $\tau \in [\alpha,\beta]$, the reverse inequality in (1.194) holds, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\phi(y_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(x_{ij}) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\psi(y_{ij})}, \tag{1.196}$$

provided that the denominators are non zero.

Proof. The idea of the proof is the same as the proof of Theorem 1.26. □

Corollary 1.30 ([12]) *Let X, Y and W be matrices as in Theorem 1.54 such that condition (1.191) is satisfied. If for all $\tau \in [\alpha,\beta]$, the inequality (1.194) holds or if for all $\tau \in [\alpha,\beta]$ the reverse inequality in (1.194) holds and $[\alpha,\beta]$ is closed interval in $\mathbb{R}^+$, then for $u,v \in \mathbb{R}\setminus\{0,1\}$, $u \ne v$, there exists $\xi \in [\alpha,\beta]$ such that*

$$\xi^{u-v} = \frac{v(v-1)}{u(u-1)} \cdot \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^u - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^u}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}x_{ij}^v - \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}y_{ij}^v}, \tag{1.197}$$

provided that the denominator is non zero.

Proof. Set $\phi(x) = x^u$ and $\psi(x) = x^v$ in Theorem 1.57, we get (1.197). □

Now we are able to introduce generalized Cauchy means from (1.196). Namely, suppose that $\frac{\phi''}{\psi''}$ has inverse function, then from (1.196) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\sum_{i=1}^n \sum_{j=1}^m w_{ij}\phi(x_{ij}) - \sum_{i=1}^n \sum_{j=1}^m w_{ij}\phi(y_{ij})}{\sum_{i=1}^n \sum_{j=1}^m w_{ij}\psi(x_{ij}) - \sum_{i=1}^n \sum_{j=1}^m w_{ij}\psi(y_{ij})}\right). \tag{1.198}$$

Remark 1.37 *Note that we can consider the interval $[\alpha,\beta] = [m_{x,y}, M_{x,y}]$, where $m_{x,y} = min\{\min_{ij} x_{ij}, \min_{ij} y_{ij}\}$, $M_{x,y} = max\{\max_{ij} x_{ij}, \max_{ij} y_{ij}\}$.*
Since the function $\xi \to \xi^{u-v}$, $u \neq v$ is invertible, then from (1.197) we have

$$m_{x,y} \leq \left\{\frac{v(v-1)}{u(u-1)}\cdot\frac{\sum_{i=1}^n \sum_{j=1}^m w_{ij}x_{ij}^u - \sum_{i=1}^n \sum_{j=1}^m w_{ij}y_{ij}^u}{\sum_{i=1}^n \sum_{j=1}^m w_{ij}x_{ij}^v - \sum_{i=1}^n \sum_{j=1}^m w_{ij}y_{ij}^v}\right\}^{\frac{1}{u-v}} \leq M_{x,y}. \tag{1.199}$$

We shall say that the expression in the middle defines a class of means of x_{ij} and y_{ij}.

Let X, Y, W and ϕ be defined as in Theorem 1.54. We define the functional $\Lambda_1(X,Y,W;\phi)$ by

$$\Lambda_1(X,Y,W;\phi) = \sum_{i=1}^n \sum_{j=1}^m w_{ij}\phi(x_{ij}) - \sum_{i=1}^n \sum_{j=1}^m w_{ij}\phi(y_{ij}).$$

Let w, x, y, u, μ, ϕ be defined as in Theorem 1.51. We define the functional $\Lambda_2(x,y,w;\phi)$ by

$$\Lambda_2(x,y,w;\phi) = \int_c^d \int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s).$$

Theorem 1.58 *Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$ be a family of functions defined on I such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\overline{F}_1 = \Lambda_1(X,Y,W;\phi_t)$ and $\overline{F}_2 = \Lambda_2(x,y,w;\phi_t)$, if (1.194), (1.182) hold for every $\tau \in [\alpha,\beta]$ and $\overline{F}_3 = -\Lambda_1(X,Y,W;\phi_t)$ and $\overline{F}_4 = -\Lambda_2(x,y,w;\phi_t)$, if (1.194), (1.182) hold in the opposite direction for every $\tau \in [\alpha,\beta]$. Then for the linear functionals $\overline{F}_j(.,.,\phi_t)$ $(j = 1,2,3,4)$ the following statements hold:*

(i) The function $t \to \overline{F}_j(.,.,\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\overline{F}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det[\overline{F}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m \geq 0 \textit{ for all } m \in \mathbb{N},\ m = 1,2,\ldots,n.$$

(ii) If the function $t \to \overline{F}_j(.,.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. The proof is similar to the proof of Theorem 1.39. □

As a consequence of the above theorem we can give the following corollary.

Corollary 1.31 *Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$ be a family of functions defined on I such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\overline{F}_1 = \Lambda_1(X,Y,W;\phi_t)$ and $\overline{F}_2 = \Lambda_2(x,y,w;\phi_t)$, if (1.194), (1.182) hold for every $\tau \in [\alpha,\beta]$ and $\overline{F}_3 = -\Lambda_1(X,Y,W;\phi_t)$ and $\overline{F}_4 = -\Lambda_2(x,y,w;\phi_t)$, if (1.194), (1.182) hold in the opposite direction for every $\tau \in [\alpha,\beta]$. Then for the linear functionals $\overline{F}_j(.,.,\phi_t)$ $(j = 1,2,3,4)$ the following statements hold:*

(i) The function $t \to \overline{F}_j(.,.,\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\overline{F}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m$ is a positive semi-definite, for all $m \in \mathbb{N}$, $t_1,..,t_m \in J$. Particularly,

$$\det[\overline{F}_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m \geq 0.$$

(ii) If the function $t \to \overline{F}_j(.,.,\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 1.32 *Let $\Omega = \{\phi_t : t \in J \subseteq \mathbb{R}\}$ be a family of functions defined on I such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\overline{F}_1 = \Lambda_1(X,Y,W;\phi_t)$ and $\overline{F}_2 = \Lambda_2(x,y,w;\phi_t)$, if (1.194), (1.182) hold for every $\tau \in [\alpha,\beta]$ and $\overline{F}_3 = -\Lambda_1(X,Y,W;\phi_t)$ and $\overline{F}_4 = -\Lambda_2(x,y,w;\phi_t)$, if (1.194), (1.182) hold in the opposite direction for every $\tau \in [\alpha,\beta]$ and also suppose that $\overline{F}_j(.,.,\phi_t)$ $(j = 1,2,3,4)$ is strictly positive for $\phi_t \in \Omega$. Then for the linear functionals $\overline{F}_j(.,.,\phi_t)$ $(j = 1,2,3,4)$ the following statements hold:*

(i) If the function $t \to \overline{F}_j(.,.,\phi_t)$ is continuous on J, then it is log convex on J and for $r,s,t \in J$ such that $r < s < t$, we have

$$(\overline{F}_j(.,.,\phi_s))^{t-r} \leq (\overline{F}_j(.,.,\phi_r))^{t-s}(\overline{F}_j(.,.,\phi_t))^{s-r}. \tag{1.200}$$

If $r < t < s$ or $s < r < t$, then (1.88) holds in the reverse direction.

(ii) If the function $t \to \overline{F}_j(.,.,\phi_t)$ is differentiable on J, then for every $s,t,u,v \in J$, such that $s \leq u$ and $t \leq v$, we have

$$\overline{\mathfrak{B}}_{s,t}(.,.,\overline{F}_j,\Omega) \leq \overline{\mathfrak{B}}_{u,v}(.,.,\overline{F}_j,\Omega) \tag{1.201}$$

where

$$\overline{\mathfrak{B}}^j_{s,t}(\Omega) = \overline{\mathfrak{B}}_{s,t}(.,.,\overline{F}_j,\Omega) = \begin{cases} \left(\frac{\overline{F}_j(.,.,\phi_s)}{\overline{F}_j(.,.,\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\overline{F}_j(.,.,\phi_s)}{\overline{F}_j(.,.,\phi_s)}\right), & s = t, \end{cases} \tag{1.202}$$

for $\phi_s, \phi_t \in \Omega$.

Proof. The proof is similar to the proof of Corollary 1.10. □

Remark 1.38 *Similar examples can be discussed as given in Section 1.6.*

Chapter 2

Majorization and n-convex Functions

2.1 Majorization and Lidstone Interpolation Polynomial

In mathematics, a Lidstone series, named after George James Lidstone, were introduced to offer series representation of infinitely times continuously differentiable functions, using the even derivatives, in the neighborhood of two points instead of one point representation given by Taylor series (see Section 2.3). Such series have been studied by G. J. Lidstone (1929), H. Poritsky (1932), J. M. Wittaker (1934) and others (see [30]).

Definition 2.1 *Let $\phi \in C^{\infty}([0,1])$, then* ***Lidstone's series*** *has the form*

$$\sum_{k=0}^{\infty} \left(\phi^{(2k)}(0)\Lambda_k(1-t) + \phi^{(2k)}(1)\Lambda_k(t) \right), \tag{2.1}$$

where Λ_n is a polynomial of degree $(2n+1)$ defined by the relations

$$\begin{aligned} &\Lambda_0(t) = t, \\ &\Lambda_n''(t) = \Lambda_{n-1}(t), \\ &\Lambda_n(0) = \Lambda_n(1) = 0, \quad n \geq 1. \end{aligned} \tag{2.2}$$

Other explicit representations of **Lidstone's polynomials** are given by [16] and [170],

$$\begin{aligned}
\Lambda_n(t) &= (-1)^n \tfrac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \tfrac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t, \\
\Lambda_n(t) &= \tfrac{1}{6}\left[\tfrac{6t^{2n+1}}{(2n+1)!} - \tfrac{t^{2n-1}}{(2n-1)!}\right] \\
&\quad - \sum_{k=0}^{n-2} \tfrac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \tfrac{t^{2n-2k-3}}{(2n-2k-3)!},\ n = 1,2,\ldots, \\
\Lambda_n(t) &= \tfrac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\tfrac{1+t}{2}\right),\ n = 1,2\ldots,
\end{aligned}$$

where B_{2k+4} is the $(2k+4)$-th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is the Bernoulli polynomial.

In [171], Widder proved the following fundamental lemma.

Lemma 2.1 *If $\phi \in C^{2n}[0,1]$, then*

$$\phi(t) = \sum_{k=0}^{n-1} \left[\phi^{(2k)}(0)\Lambda_k(1-t) + \phi^{(2k)}(1)\Lambda_k(t)\right] + \int_0^1 G_n(t,s)\phi^{(2n)}(s)ds,$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s\,, & s \le t, \\ (s-1)t, & t \le s, \end{cases} \tag{2.3}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$

$$G_n(t,s) = \int_0^1 G_1(t,p)G_{n-1}(p,s)dp, \quad n \ge 2. \tag{2.4}$$

Remark 2.1 *The Lidstone polynomial can be expressed in terms of $G_n(t,s)$ as*

$$\Lambda_n(t) = \int_0^1 G_n(t,s)sds, \quad n \ge 1.$$

We arrange this section in this way. In the first subsection we give some new identities by using interpolation by Lidstone's polynomial which enable us to obtain generalized results of majorization theorems, in discrete and integral forms. Obtain generalizations hold for $(2n)$-convex functions. We give bounds for identities related to the generalizations of majorization inequalities by using Čebyšev functionals. As outcome we give the Grüss and Ostrowski type inequalities for these functionals. We present related results in the form of the mean value theorems which leads to construction of several families which are exponentially convex and as outcome we obtain some new classes of Cauchy type means. In the second subsection we using Lidstone's interpolation in combination with new Green's functions to generate analogous results. In the last subsection we consider generalization of Jensen's and the Jensen-Steffensen inequality obtained by using interpolation by Lidstone's polynomial.

2.1.1 Results Obtained by Lidstone Interpolation Polynomial

Theorem 2.1 ([7]) *Let $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \dots, x_m)$, $\boldsymbol{y} = (y_1, \dots, y_m)$ and $\boldsymbol{w} = (w_1, \dots, w_m)$ be m-tuples such that x_i, $y_i \in [a,b]$ and $w_i \in \mathbb{R}$ $(i = 1, \dots, m)$ and $\phi \in C^{2n}[a,b]$. Then*

$$\begin{aligned}
&\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \\
&\quad = \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-x_i}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-y_i}{b-a} \right) \right] \\
&\quad + \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{x_i-a}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right] \\
&\quad + (b-a)^{2n-1} \int_a^b \left[\sum_{i=1}^{m} w_i G_n \left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a} \right) - \sum_{i=1}^{m} w_i G_n \left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a} \right) \right] \phi^{(2n)}(t) dt.
\end{aligned} \tag{2.5}$$

Proof. Consider

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i). \tag{2.6}$$

We use Widder's Lemma for representation of function in the form:

$$\begin{aligned}
\phi(x) = {} & \sum_{k=0}^{n-1} (b-a)^{2k} \left[\phi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \phi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] \\
& + (b-a)^{2n-1} \int_a^b G_n \left(\frac{x-a}{b-a}, \frac{t-a}{b-a} \right) \phi^{(2n)}(t) dt,
\end{aligned} \tag{2.7}$$

where, Λ_k is a Lidstone polynomial.

Using value of $\phi(x)$ from (2.7) in (2.6), we have

$$\begin{aligned}
&\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \\
&= \sum_{i=1}^{m} w_i \left\{ \sum_{k=0}^{n-1} (b-a)^{2k} \left[\phi^{(2k)}(a) \Lambda_k \left(\frac{b-x_i}{b-a} \right) + \phi^{(2k)}(b) \Lambda_k \left(\frac{x_i-a}{b-a} \right) \right] \right\} \\
&+ \sum_{i=1}^{m} w_i \left[(b-a)^{2n-1} \int_a^b G_n \left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a} \right) \phi^{(2n)}(t) dt \right] \\
&- \sum_{i=1}^{m} w_i \left\{ \sum_{k=0}^{n-1} (b-a)^{2k} \left[\phi^{(2k)}(a) \Lambda_k \left(\frac{b-y_i}{b-a} \right) + \phi^{(2k)}(a) \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right] \right\} \\
&- \sum_{i=1}^{m} w_i \left[(b-a)^{2n-1} \int_a^b G_n \left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a} \right) \phi^{(2n)}(t) dt \right],
\end{aligned}$$

after some arrangement we get (2.5). □

Integral version of the above theorem can be stated as follows.

Theorem 2.2 ([7]) *Let $n \in \mathbb{N}$, $x, y : [\alpha, \beta] \to [a, b]$, $w : [\alpha, \beta] \to \mathbb{R}$ be continuous functions and $\phi \in C^{2n}[a, b]$. Then*

$$\begin{aligned}
&\int_\alpha^\beta w(t) \phi(x(t)) dt - \int_\alpha^\beta w(t) \phi(y(t)) dt \\
&= \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\int_\alpha^\beta w(t) \Lambda_k \left(\frac{b-x(t)}{b-a} \right) dt - \int_\alpha^\beta w(t) \Lambda_k \left(\frac{b-y(t)}{b-a} \right) dt \right] \\
&+ \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\int_\alpha^\beta w(t) \Lambda_k \left(\frac{x(t)-a}{b-a} \right) dt - \int_\alpha^\beta w(t) \Lambda_k \left(\frac{y(t)-a}{b-a} \right) dt \right] \\
&+ (b-a)^{2n-1} \int_a^b \phi^{(2n)}(s) \left[\int_\alpha^\beta w(t) G_n \left(\frac{x(t)-a}{b-a}, \frac{s-a}{b-a} \right) dt - \int_\alpha^\beta w(t) G_n \left(\frac{y(t)-a}{b-a}, \frac{s-a}{b-a} \right) dt \right] ds.
\end{aligned}$$

We give generalization of majorization theorem for $2n$-convex function.

Theorem 2.3 ([7]) *Let $n \in \mathbb{N}$, , $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ be m-tuples such that x_i, $y_i \in [a, b]$ and $w_i \in \mathbb{R}$ $(i = 1, \ldots, m)$.*
If for all $t \in [a, b]$

$$\sum_{i=1}^{m} w_i G_n \left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a} \right) \le \sum_{i=1}^{m} w_i G_n \left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a} \right) \tag{2.8}$$

then for every $(2n)$-convex function $\phi : [a, b] \to \mathbb{R}$, we have

$$\begin{aligned}
&\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i) \\
&\geq \sum_{k=0}^{n-1} (b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right] \\
&+ \sum_{k=0}^{n-1} (b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right].
\end{aligned} \tag{2.9}$$

If the reverse inequality in (2.8) holds, then also the reverse inequality in (2.9) holds.

Proof. If the function ϕ is $2n$-convex, without loss of generality we can assume that ϕ is $2n$-times differentiable therefore we have $\phi^{(2n)}(x) \geq 0$, for all $x \in [a,b]$, and by using (2.8), we get (2.9). □

Theorem 2.4 ([7]) *Let $n \in \mathbb{N}$, $x,y : [\alpha,\beta] \to [a,b]$ and $w : [\alpha,\beta] \to \mathbb{R}$ be any continuous functions. If for all $s \in [a,b]$*

$$\int_\alpha^\beta w(t)G_n\left(\frac{y(t)-a}{b-a},\frac{s-a}{b-a}\right)dt \leq \int_\alpha^\beta w(t)G_n\left(\frac{x(t)-a}{b-a},\frac{s-a}{b-a}\right)dt$$

then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$,

$$\begin{aligned}
&\int_\alpha^\beta w(t)\phi(x(t))dt \geq \int_\alpha^\beta w(t)\phi(y(t))dt \\
&+ \sum_{k=0}^{n-1} (b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\
&+ \sum_{k=0}^{n-1} (b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right].
\end{aligned} \tag{2.10}$$

If the reverse inequality in (2.10) holds, then also the reverse inequality in (2.10) holds.

The following theorem is majorization theorem for $2n$-convex function:

Theorem 2.5 ([7]) *Let $n \in \mathbb{N}$, Let $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ be two decreasing real m-tuples with x_i, $y_i \in [a,b]$ $(i = 1,\ldots,m)$, let $\mathbf{w} = (w_1,\ldots,w_m)$ be a real m-tuple such that which satisfies (1.19), (1.20) and G_n be defined in (2.4).*

(i) If n is odd, then for every $2n$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i) \\
&\geq \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right] \\
&+ \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right].
\end{aligned} \tag{2.11}$$

(ii) Let the inequality (2.11) holds and let $\vartheta : [a,b] \to \mathbb{R}$ be a function defined by

$$\vartheta(.) := \sum_{k=1}^{n-1} (b-a)^{2k} \left(\phi^{(2k)}(a)\Lambda_k \left(\frac{b-.}{b-a} \right) + \phi^{(2k)}(b)\Lambda_k \left(\frac{.-a}{b-a} \right) \right). \quad (2.12)$$

If ϑ is a convex function, then the right hand side of (2.11) is non-negative that is the following weighted majorization inequality holds

$$\sum_{i=1}^{m} w_i \phi(y_i) \leq \sum_{i=1}^{m} w_i \phi(x_i). \quad (2.13)$$

(iii) If n is even, then for every $2n$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned} &\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \\ &\leq \sum_{k=1}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-x_i}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-y_i}{b-a} \right) \right] \\ &+ \sum_{k=1}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{x_i-a}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right]. \end{aligned} \quad (2.14)$$

(iv) Let the inequality (2.14) holds and let $\vartheta : [a,b] \to \mathbb{R}$ be a function defined in (2.12). If ϑ is a concave function, then the right hand side of (2.14) is non-positive, that is, the reverse inequality in (2.13) is valid.

Proof. (i) By (2.3), $G_1(t,s) \leq 0$, for $0 \leq t,s \leq 1$.
By using (2.4), we have $G_n(t,s) \leq 0$ for odd n and $G_n(t,s) \geq 0$ for even n.
Now as G_1 is convex and G_{n-1} is positive for odd n, therefore by using (2.4), G_n is convex in first variable if n is odd. Similarly G_n is concave in first variable if n is even.
Hence if n is odd then by majorization theorem we have

$$\sum_{i=1}^{m} w_i G_n \left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a} \right) \leq \sum_{i=1}^{m} w_i G_n \left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a} \right). \quad (2.15)$$

Therefore if n is odd, then by Theorem 2.3, (2.11) holds.
(ii) We can easily get the equivalent form of the inequality (2.11) as

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \geq \sum_{i=1}^{m} w_i \vartheta(x_i) - \sum_{i=1}^{m} w_i \vartheta(y_i).$$

By using (1.19), (1.20) and the fact that ϑ is a convex function, so by applying weighted majorization inequality, we get immediately the non-negativity of the right hand side of (2.11) and we have the inequality (2.13).
Similarly using these arguments for parts (iii) and (iv), we get (2.14) and the reverse inequality in (2.13) by using function defined in (2.12). □

The following theorem is majorization theorem for $2n$-convex function in integral case.

Theorem 2.6 ([7]) *Let $n \in \mathbb{N}$, $x, y : [\alpha, \beta] \to [a, b]$ be decreasing and $w : [\alpha, \beta] \to \mathbb{R}$ be any continuous functions and G_n be defined in (2.4). Let*

$$\int_{\alpha}^{\upsilon} w(t)y(t)dt \leq \int_{\alpha}^{\upsilon} w(t)x(t)dt, \quad for\ \upsilon \in [\alpha, \beta] \tag{2.16}$$

and

$$\int_{\alpha}^{\beta} w(t)y(t)dt = \int_{\alpha}^{\beta} w(t)x(t)dt. \tag{2.17}$$

(i) If n is odd, then for every $2n$-convex function $\phi : [a, b] \to \mathbb{R}$, it holds

$$\begin{aligned} &\int_{\alpha}^{\beta} w(t)\phi(x(t))dt - \int_{\alpha}^{\beta} w(t)\phi(y(t))dt \\ &\geq \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(a)\left[\int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\ &+ \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(b)\left[\int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right]. \end{aligned} \tag{2.18}$$

(ii) Let the inequality (2.18) holds and let $\vartheta : [a, b] \to \mathbb{R}$ be a function defined in (2.12) be a convex function, then the right hand side of (2.18) is non-negative that is the following weighted majorization inequality in integral case holds

$$\int_{\alpha}^{\beta} w(t)\phi(y(t))dt \leq \int_{\alpha}^{\beta} w(t)\phi(x(t))dt. \tag{2.19}$$

(iii) If n is even, then for every $2n$-convex function $\phi : [a, b] \to \mathbb{R}$, it holds

$$\begin{aligned} &\int_{\alpha}^{\beta} w(t)\phi(x(t))dt - \int_{\alpha}^{\beta} w(t)\phi(y(t))dt \\ &\leq \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(a)\left[\int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\ &+ \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(b)\left[\int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right]. \end{aligned} \tag{2.20}$$

(iv) Let the inequality (2.20) holds and let $\vartheta : [a, b] \to \mathbb{R}$ be a function defined in (2.12). If ϑ is a concave function, then the right hand side of (2.20) is non-positive that is the reverse inequality in (2.19) is valid.

For m-tuples $\mathbf{w} = (w_1,\dots,w_m)$, $\mathbf{x} = (x_1,\dots,x_m)$ and $\mathbf{y} = (y_1,\dots,y_m)$ with $x_i, y_i \in [a,b], w_i \in \mathbb{R}$ $(i = 1,\dots,m)$ and the function G_n as defined above, denote

$$\Upsilon(t) = \sum_{i=1}^{m} w_i G_n\left(\frac{x_i - a}{b-a}, \frac{t-a}{b-a}\right) - \sum_{i=1}^{m} w_i G_n\left(\frac{y_i - a}{b-a}, \frac{t-a}{b-a}\right), \tag{2.21}$$

similarly for $x, y : [\alpha,\beta] \to [a,b]$ and $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and for all $s \in [a,b]$, denote

$$\widetilde{\Upsilon}(s) = \int_\alpha^\beta w(t) G_n\left(\frac{x(t)-a}{b-a}, \frac{s-a}{b-a}\right) dt - \int_\alpha^\beta w(t) G_n\left(\frac{y(t)-a}{b-a}, \frac{s-a}{b-a}\right) dt. \tag{2.22}$$

We have the Čebyšev functionals defined as:

$$T(\Upsilon,\Upsilon) = \frac{1}{b-a}\int_a^b \Upsilon^2(t)dt - \left(\frac{1}{b-a}\int_a^b \Upsilon(t)dt\right)^2 \tag{2.23}$$

$$T(\widetilde{\Upsilon},\widetilde{\Upsilon}) = \frac{1}{b-a}\int_a^b \widetilde{\Upsilon}^2(s)ds - \left(\frac{1}{b-a}\int_a^b \widetilde{\Upsilon}(s)ds\right)^2 \tag{2.24}$$

Theorem 2.7 ([7]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)\left[\phi^{(2n+1)}\right]^2 \in L[a,b]$, and $x_i, y_i \in [a,b]$ and $w_i \in \mathbb{R}$ $(i = 1,2,\dots,m)$ and let the functions G_n, Υ and T be defined in (2.4), (2.21) and (2.23) respectively. Then*

$$\begin{aligned}
&\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) \\
&= \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right] \\
&+ \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right] \\
&+ (b-a)^{2n-2}\left(\phi^{(2n-1)}(b) - \phi^{(2n-1)}(a)\right)\int_a^b \Upsilon(t)dt + H_n^1(\phi;a,b),
\end{aligned} \tag{2.25}$$

where the remainder $H_n^1(\phi;a,b)$ satisfies the estimation

$$\left|H_n^1(\phi;a,b)\right| \le \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}}\left[T(\Upsilon,\Upsilon)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(2n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}. \tag{2.26}$$

Proof. If we apply Theorem 1.10 for $f \to \Upsilon$ and $h \to \phi^{(2n)}$ we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \Upsilon(t)\phi^{(2n)}(t)dt - \frac{1}{b-a}\int_a^b \Upsilon(t)dt.\frac{1}{b-a}\int_a^b \phi^{(2n)}(t)dt\right| \\
&\le \frac{1}{\sqrt{2}}\left[T(\Upsilon,\Upsilon)\right]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(2n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.
\end{aligned} \tag{2.27}$$

Therefore we have

$$
\begin{aligned}
&(b-a)^{2n-1}\int_a^b \Upsilon(t)\phi^{(2n)}(t)dt \\
&= (b-a)^{2n-2}\left(\phi^{(2n-1)}(b)-\phi^{(2n-1)}(a)\right)\int_a^b \Upsilon(t)dt + H_n^1(\phi;a,b) \qquad (2.28)
\end{aligned}
$$

where the remainder $H_n^1(\phi;a,b)$ satisfies the estimation (2.26). Now from identity (2.5) we obtain (2.25). □

Integral case of the above theorem can be given as follows.

Theorem 2.8 ([7]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)\left[\phi^{(2n+1)}\right]^2 \in L[a,b]$, and $x,y : [\alpha,\beta] \to [a,b]$, $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and let the functions $G_n, \widetilde{\Upsilon}$ and T be defined in (2.4), (2.22) and (2.24) respectively. Then*

$$
\begin{aligned}
&\int_\alpha^\beta w(t)\,\phi\,(x(t))\,dt - \int_\alpha^\beta w(t)\,\phi\,(y(t))\,dt \\
&= \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\
&+ \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right] \\
&+ (b-a)^{2n-2}\left(\phi^{(2n-1)}(b)-\phi^{(2n-1)}(a)\right)\int_a^b \widetilde{\Upsilon}(s)ds + \widetilde{H}_n^1(\phi;a,b), \qquad (2.29)
\end{aligned}
$$

where the remainder $\widetilde{H}_n^1(\phi;a,b)$ satisfies the estimation

$$
\left|\widetilde{H}_n^1(\phi;a,b)\right| \le \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}}\left[T(\widetilde{\Upsilon},\widetilde{\Upsilon})\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(2n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.
$$

Using Theorem 1.11 we obtain the following Grüss type inequality.

Theorem 2.9 ([7]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ $(n \in \mathbb{N})$ and $\phi^{(2n+1)} \ge 0$ on $[a,b]$ and let the function Υ and T be defined by (2.21) and (2.23) respectively. Then we have the representation (2.25) and the remainder $H_n^1(\phi;a,b)$ satisfies the bound*

$$
\left|H_n^1(\phi;a,b)\right| \le (b-a)^{2n-1}\left\|\Upsilon'\right\|_\infty \left\{\frac{\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)}{2} - \frac{\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)}{b-a}\right\}. \qquad (2.30)
$$

Proof. Applying Theorem 1.11 for $f \to \Upsilon$ and $h \to \phi^{(2n)}$ we obtain

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \Upsilon(t)\phi^{(2n)}(t)dt - \frac{1}{b-a}\int_a^b \Upsilon(t)dt.\frac{1}{b-a}\int_a^b \phi^{(2n)}(t)dt\right| \\
&\le \frac{1}{2(b-a)}\left\|\Upsilon'\right\|_\infty \int_a^b (t-a)(b-t)\phi^{(2n+1)}(t)dt. \qquad (2.31)
\end{aligned}
$$

Since

$$\int_a^b (t-a)(b-t)\phi^{(2n+1)}(t)dt = \int_a^b [2t-(a+b)]\phi^{(2n)}(t)dt$$
$$= (b-a)\left[\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)\right] - 2\left(\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)\right),$$

using the identity (2.5) and (2.31) we deduce (2.30). □

Integral version of the above theorem can be given as follows.

Theorem 2.10 ([7]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ $(n \in \mathbb{N})$ and $\phi^{(2n+1)} \geq 0$ on $[a,b]$ and let the functions $\widetilde{\Upsilon}$ and T be defined by (2.22) and (2.24) respectively. Then we have the representation (2.29) and the remainder $\widetilde{H}_n^1(\phi;a,b)$ satisfies the bound*

$$\left|\widetilde{H}_n^1(\phi;a,b)\right| \leq (b-a)^{2n-1}\left\|\widetilde{\Upsilon}'\right\|_\infty \left\{\frac{\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)}{2} - \frac{\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)}{b-a}\right\}.$$

We give the Ostrowski-type inequality related to the generalization of majorization inequality.

Theorem 2.11 ([7]) *Suppose that all the assumptions of Theorem 2.1 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $\left|\phi^{(2n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\Bigg|\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) \\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^m w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^m w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right] \\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^m w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^m w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right]\Bigg| \\ &\leq (b-a)^{2n-1}\left\|\phi^{(2n)}\right\|_p \left(\int_a^b \left|\sum_{i=1}^m w_iG_n\left(\frac{x_i-a}{b-a},\frac{t-a}{b-a}\right) - \sum_{i=1}^m w_iG_n\left(\frac{y_i-a}{b-a},\frac{t-a}{b-a}\right)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.32}$$

The constant on the right-hand side of (2.32) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Let us denote

$$\Psi(t) = (b-a)^{2n-1}\left[\sum_{i=1}^m w_iG_n\left(\frac{x_i-a}{b-a},\frac{t-a}{b-a}\right) - \sum_{i=1}^m w_iG_n\left(\frac{y_i-a}{b-a},\frac{t-a}{b-a}\right)\right]$$

Using the identity (2.5) and applying Hölder's inequality we obtain

$$\left|\sum_{i=1}^{m} w_i\,\phi\left(x_i\right)-\sum_{i=1}^{m} w_i\,\phi\left(y_i\right)-\sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right)-\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right]\right.$$
$$\left.-\sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{x_i-a}{b-\alpha}\right)-\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right]\right|$$
$$=\left|\int_a^b \Psi(t)\phi^{(2n)}(t)dt\right|\le\left\|\phi^{(2n)}\right\|_p\left(\int_a^b|\Psi(t)|^q\,dt\right)^{\frac{1}{q}}.$$

For the proof of the sharpness of the constant $\left(\int_a^b|\Psi(t)|^q\,dt\right)^{\frac{1}{q}}$ let us find a function ϕ for which the equality in (2.32) is obtained.
For $1<p<\infty$ take ϕ to be such that

$$\phi^{(2n)}(t)=sgn\Psi(t)\,|\Psi(t)|^{\frac{1}{p-1}}.$$

For $p=\infty$ take $\phi^{(2n)}(t)=sgn\Psi(t)$.
For $p=1$ we prove that

$$\left|\int_a^b \Psi(t)\phi^{(2n)}(t)\right|\le\max_{t\in[a,b]}|\Psi(t)|\left(\int_a^b\left|\phi^{(2n)}(t)\right|dt\right) \tag{2.33}$$

is the best possible inequality. Suppose that $|\Psi(t)|$ attains its maximum at $t_0\in[a,b]$. First we assume that $\Psi(t_0)>0$. For ε small enough we define $\phi_\varepsilon(t)$ by

$$\phi_\varepsilon(t):=\begin{cases}0, & a\le t\le t_0,\\ \frac{1}{\varepsilon n!}\left(t-t_0\right)^n, & t_0\le t\le t_0+\varepsilon,\\ \frac{1}{n!}\left(t-t_0\right)^{n-1}, & t_0+\varepsilon\le t\le b.\end{cases}$$

Then for ε small enough

$$\left|\int_a^b \Psi(t)\phi^{(2n)}(t)\right|=\left|\int_{t_0}^{t_0+\varepsilon}\Psi(t)\frac{1}{\varepsilon}dt\right|=\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Psi(t)dt.$$

Now from the inequality (2.33) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Psi(t)dt\le\Psi(t_0)\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}dt=\Psi(t_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Psi(t)dt=\Psi(t_0)$$

the statement follows. In the case $\Psi(t_0)<0$, we define $\phi_\varepsilon(t)$ by

$$\phi_\varepsilon(t):=\begin{cases}\frac{1}{n!}\left(t-t_0-\varepsilon\right)^{n-1}, & a\le t\le t_0,\\ -\frac{1}{\varepsilon n!}\left(t-t_0-\varepsilon\right)^n, & t_0\le t\le t_0+\varepsilon,\\ 0, & t_0+\varepsilon\le t\le b,\end{cases}$$

and the rest of the proof is the same as above. □

Integral version of the above theorem can be stated as follows.

Theorem 2.12 ([7]) *Suppose that all the assumptions of Theorem 2.2 hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\left|\phi^{(2n)}\right|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \in \mathbb{N}$. *Then we have*

$$\begin{aligned}
&\Bigg|\int_\alpha^\beta w(t)\,\phi\,(x(t))\,dt - \int_\alpha^\beta w(t)\,\phi\,(y(t))\,dt \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right]\Bigg| \\
&\leq (b-a)^{2n-1} \\
&\left\|\phi^{(2n)}\right\|_p \left(\int_a^b \left|\int_\alpha^\beta w(t)G_n\left(\frac{x(t)-a}{b-a},\frac{s-a}{b-a}\right)dt - \int_\alpha^\beta w(t)G_n\left(\frac{y(t)-a}{b-a},\frac{s-a}{b-a}\right)dt\right|^q ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.34}$$

The constant on the right-hand side of (2.34) is sharp for $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Motivated by the inequalities (2.9) and (2.10), we define functional $\Theta_1(\phi)$ and $\Theta_2(\phi)$ by

$$\begin{aligned}
\Theta_1(\phi) &= \sum_{i=1}^m w_i\,\phi\,(x_i) - \sum_{i=1}^m w_i\,\phi\,(y_i) \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^m w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^m w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right] \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^m w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^m w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right]
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
\Theta_2(\phi) &= \int_\alpha^\beta w(t)\phi(x(t))dt - \int_\alpha^\beta w(t)\phi(y(t))dt \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right].
\end{aligned} \tag{2.36}$$

Theorem 2.13 ([7]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$. If the inequalities in (2.9) and (2.10) hold, then there exist $\xi_i \in [a,b]$ such that*

$$\Theta_i(\phi) = \phi^{(2n)}(\xi_i)\Theta_i(\eta), \;\; i = 1,2. \tag{2.37}$$

where $\eta(x) = \frac{x^{2n}}{(2n)!}$.

Proof. (See also Theorem 7 in [30]) Since $\phi^{(2n)}$ is continuous on $[a,b]$, so $m \le \phi^{(2n)}(x)$ $\le M$ for $x \in [a,b]$, where $m = \min_{x\in[a,b]} \phi^{(2n)}(x)$ and $M = \max_{x\in[a,b]} \phi^{(2n)}(x)$. Consider the function ϕ_1 and ϕ_2 defined on I as

$$\phi_1(x) = \frac{Mx^{2n}}{(2n)!} - \phi(x) \quad and \quad \phi_2(x) = \phi(x) - \frac{mx^{2n}}{(2n)!} \;\; for \;\; x \in [a,b].$$

It is easily seen that

$$\phi_1^{(2n)}(x) = M - \phi^{(2n)}(x) \quad and \quad \phi_2^{(2n)}(x) = \phi^{(2n)}(x) - m \;\; for \;\; x \in I.$$

So, ϕ_1 and ϕ_2 are $2n$-convex functions.
Now by applying ϕ_1 for ϕ in Theorem 2.3, we have

$$\begin{aligned}\sum_{i=1}^{m} w_i\,\phi_1(x_i) &\ge \sum_{i=1}^{m} w_i\,\phi_1(y_i) \\ &+ \sum_{k=0}^{n-1}(b-a)^{2k}\phi_1^{(2k)}(a)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right] \\ &+ \sum_{k=0}^{n-1}(b-a)^{2k}\phi_1^{(2k)}(b)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right]\end{aligned} \tag{2.38}$$

Hence, we get after some simplification

$$\Theta_1(\phi) \le M\Theta_1(\eta). \tag{2.39}$$

Now by applying ϕ_2 for f in Theorem 2.3 and some simplification we get

$$m\Theta_1(\eta) \le \Theta_1(\phi). \tag{2.40}$$

If $\Theta_1(\eta) = 0$, then from (2.39) and (2.40) follows that for any $\xi \in [a,b]$, (2.37) holds.
If $\Theta_1(\eta) > 0$, it follows from (2.39) and (2.40) that

$$m \le \frac{\Theta_1(\phi)}{\Theta_1(\eta)} \le M. \tag{2.41}$$

Now using the fact that for $m \le \rho \le M$ there exists $\xi_1 \in [a,b]$ such that $\phi^{(2n)}(\xi_1) = \rho$, we get (2.37) for $i = 1$. Similarly we can prove (2.37) for $i = 2$. □

Theorem 2.14 ([7]) *Let $\phi, \psi : [a,b] \to \mathbb{R}$ be such that $\phi, \psi \in C^{2n}[a,b]$. If the inequalities in (2.9) and (2.10) hold, then there exist $\xi_i \in [a,b]$ such that*

$$\frac{\Theta_i(\phi)}{\Theta_i(\varphi)} = \frac{\phi^{(2n)}(\xi_i)}{\psi^{(2n)}(\xi_i)}, \ i = 1,2. \tag{2.42}$$

provided that the denominators are not zero.

Proof. (See also Corollary 12 in [30]) We use the following standard technique: Let us define the linear functional $\Gamma \in C^{2n}[a,b]$ as $\Gamma(\chi) = \Theta_1(\chi)$. Next, we define

$$\chi(t) = \phi(t)\Gamma(\varphi) - \varphi(t)\Gamma(\phi).$$

According to Theorem 2.13, applied on χ, there exists $\xi_1 \in [a,b]$ so that

$$\Gamma(\chi) = \chi^{(2n)}(\xi_1)\Theta_1(\eta), \ \ \eta(x) = \frac{x^{2n}}{(2n)!}.$$

From $\Gamma(\chi) = 0$, it follows $\phi^{(2n)}(\xi_1)\Gamma(\varphi) - \varphi^{(2n)}(\xi)\Gamma(\phi) = 0$ and so (2.42) is proved.
Similarly we can prove (2.42) for $i = 2$. □

We use an idea from [84] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [142]):

Theorem 2.15 ([7]) *Let $\Phi = \{\phi_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s[x_0,\ldots,x_{2l}]$ is an n-exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $x_0,\ldots,x_{2l} \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (2.35) and (2.36). Then $s \mapsto \Theta_i(\phi_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is n-exponentially convex function on J.*

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39 but using linear functionals $\Theta_k (k = 1,2)$ instead of $\digamma_k (k = 1,2,..,5)$. □

The following corollaries are immediate consequence of the above theorem.

Corollary 2.1 ([7]) *Let $\Phi = \{\phi_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s[x_0,\ldots,x_{2l}]$ is an exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $x_0,\ldots,x_{2l} \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (2.35) and (2.36). Then $s \mapsto \Theta_i(\phi_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is exponentially convex function on J.*

Corollary 2.2 ([7]) *Let $\Phi = \{\phi_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s[x_0,\ldots,x_{2l}]$ is an 2-exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $x_0,\ldots,x_{2l} \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (2.35) and (2.36). Then the following statements hold:*

(i) If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Theta_i(\phi_s)$ is additionally strictly positive, then it is log-convex *on J. Furthermore, the Lypunov's inequality holds true:*

$$[\Theta_i(\phi_s)]^{t-r} \leq [\Theta_i(\phi_r)]^{t-s}\,[\Theta_i(\phi_t)]^{s-r} \tag{2.43}$$

for every choice $r,s,t \in J$ such that $r < s < t$.

(ii) If the function $s \mapsto \Theta_i(\phi_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v \in J$ such that $s \leq u$ and $q \leq v$, we have

$$\mu_{s,q}(\Theta_i,\Phi) \leq \mu_{u,v}(\Theta_i,\Phi), \tag{2.44}$$

where

$$\mu_{s,q}(\Theta_i,\Phi) = \begin{cases} \left(\frac{\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right), & s = q, \end{cases} \tag{2.45}$$

for $\phi_s, \phi_q \in \Phi$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10 but using linear functionals $\Theta_k(k = 1,2)$ instead of $F_k(k = 1,2,..,5)$. □

Remark 1.19 is also valid for these functionals.

Remark 2.2 ([7]) *Similar examples can be discussed as given in Section 1.4.*

2.1.2 Results Obtained by New Green's Functions and Lidstone Interpolation Polynomial

In this subsection we give generalized results for majorization theorems obtained by using newly defined Green's functions [114] and Lidstone's polynomial. We find new upper bounds for the Grüss and Ostrowski type inequalities. We also give further results for majorization inequality by making linear functionals constructed to convex functions $\frac{f(x)}{x}$. At the end we give some applications.

Remark 2.3 *As a special choice the Abel-Gontscharoff polynomial for 'two-point right focal' interpolating polynomial for $n = 2$ can be given as:*

$$f(z) = f(\alpha) + (z-\alpha)f'(\beta) + \int_\alpha^\beta G_{\Omega,2}(z,w)f''(w)dw, \tag{2.46}$$

where $G_{\Omega,2}(z,w)$ is the Green function for 'two-point right focal problem' given as

$$G_1(z,w) = G_{\Omega,2}(z,w) = \begin{cases} \alpha - w, & \alpha \leq w \leq z, \\ \alpha - z, & z \leq w \leq \beta. \end{cases} \tag{2.47}$$

As mentioned in [114], the complete reference about the Abel-Gontscharoff polynomial and theorem for 'two-point right focal' problem is given in [16].

Pečarić et al. (2017) [114] introduced some new types of **Green's functions** by keeping in view the Abel-Gontscharoff Green's function for 'two-point right focal problem' that are:

$$G_2(z,w) = \begin{cases} z-\beta, & \alpha \le w \le z, \\ w-\beta, & z \le w \le \beta. \end{cases} \tag{2.48}$$

$$G_3(z,w) = \begin{cases} z-\alpha, & \alpha \le w \le z, \\ w-\alpha, & z \le w \le \beta. \end{cases} \tag{2.49}$$

$$G_4(z,w) = \begin{cases} \beta-w, & \alpha \le w \le z, \\ \beta-z, & z \le w \le \beta. \end{cases} \tag{2.50}$$

They gave the following lemma, using this we obtain the new generalizations of majorization inequality.

Lemma 2.2 *Let $f : [\alpha,\beta] \to \mathbb{R}$ be a twice differentiable function and G_p, $(p = 1,2,3,4)$ be the new Green functions defined above, then along with (2.46) the following identities hold:*

$$f(z) = f(\beta) + (z-\beta)f'(\alpha) + \int_{\alpha}^{\beta} G_2(z,w)f''(w)dw, \tag{2.51}$$

$$f(z) = f(\beta) - (\beta-\alpha)f'(\alpha) + (z-\alpha)f'(\alpha) + \int_{\alpha}^{\beta} G_3(z,w)f''(w)dw, \tag{2.52}$$

$$f(z) = f(\alpha) - (\beta-\alpha)f'(\alpha) - (\beta-z)f'(\beta) + \int_{\alpha}^{\beta} G_4(z,w)f''(w)dw. \tag{2.53}$$

The following identity is the equivalent statements between classical weighted majorization inequality and the inequality constructed by newly developed Green's functions.

Theorem 2.16 *Let $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m) \in [\alpha,\beta]^m$ be two decreasing m-tuples and also $\mathbf{w} = (w_1,\ldots,w_m)$ be a real m-tuple such that satisfying (1.20) and $G_p\,(p = 1,2,3,4)$ is defined as in (2.47)-(2.50) respectively. Then the following statements are equivalent:*

(i) For every continuous convex function $f : [\alpha,\beta] \to \mathbb{R}$

$$\sum_{i=1}^{m} w_i f(y_i) \le \sum_{i=1}^{m} w_i f(x_i). \tag{2.54}$$

(ii) For $s \in [\alpha,\beta]$, the following inequality holds

$$\sum_{i=1}^{m} w_i G_p(y_i,s) \le \sum_{i=1}^{m} w_i G_p(x_i,s), \quad p = 1,2,3,4. \tag{2.55}$$

Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both inequalities, in (2.54) and (2.55).

Proof. "$(i) \Rightarrow (ii)$" Let the statement (i) holds. By fixing $p = 1,2,3,4$, and as the functions $G_p(.,s)$ $(s \in [\alpha,\beta])$ are also continuous and convex, follows that for these functions also inequality (2.54) holds for each fix p, i.e., (2.55) holds.
"$(ii) \Rightarrow (i)$" Let $f : [\alpha,\beta] \to \mathbb{R}$ be a convex function and without loss of generality we can assume that $f \in C^2([\alpha,\beta])$ and also let (ii) holds. Then we can represent the function f in the form (2.46), (2.51), (2.52) and (2.53) for the functions G_p, $p = 1,2,3,4$ respectively. By an easy calculation we get for all $s \in [\alpha,\beta]$

$$\begin{aligned}
&\sum_{i=1}^{m} w_i f(x_i) - \sum_{i=1}^{m} w_i f(y_i) \\
&= \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) f''(s)ds, \quad p = 1,2,3,4.
\end{aligned} \tag{2.56}$$

Since f is a convex function, then $f''(x) \geq 0$ for all $x \in [\alpha,\beta]$. So, if for every $s \in [\alpha,\beta]$ the inequality (2.55) holds for each $p = 1,2,3,4$, then it follows that for every convex function $f : [\alpha,\beta] \to \mathbb{R}$, with $f \in C^2[\alpha,\beta]$, inequality (2.54) holds.
At the end, note that it is not necessary to demand the existence of the second derivative of the function f ([144], p.172). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex functions by convex polynomials. $\square$

We present the majorization difference as in terms of Lidstone's interpolating polynomial and newly defined Green's functions.

Theorem 2.17 *Let $n \in \mathbb{N}$ be such that $n \geq 3$, $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ and $\mathbf{w} = (w_1,\ldots,w_m)$ be m-tuples such that x_i, $y_i \in [\alpha,\beta]$ and $w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$ be real m-tuple such that satisfying (1.20) and G_p $(p = 1,2,3,4)$ is defined as in (2.47)-(2.50) respectively. Let also G_n be defined as in (2.4) and $f \in C^{2n}[\alpha,\beta]$, then we have the following identities for $p = 1,2,3,4$,*

$$\begin{aligned}
&\sum_{i=1}^{m} w_i f(x_i) - \sum_{i=1}^{m} w_i f(y_i) \\
&= \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k \left(\frac{\beta-s}{\beta-\alpha} \right) ds \\
&+ \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k \left(\frac{s-\alpha}{\beta-\alpha} \right) ds \\
&+ (\beta-\alpha)^{2n-1} \\
&\int_{\alpha}^{\beta} f^{(2n)}(t) \left(\int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) G_n'' \left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha} \right) ds \right) dt,
\end{aligned} \tag{2.57}$$

where G_n'' means second derivative with respect to 's'.

Proof. Fix $p = 1,2,3,4$, substituting the identities (2.46), (2.51), (2.52) and (2.53) into majorization difference, we get

$$\sum_{i=1}^{m} w_i f(x_i) - \sum_{i=1}^{m} w_i f(y_i) = \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) f''(s) ds. \tag{2.58}$$

We use Widder's Lemma for representation of function in the form:

$$f(x) = \sum_{k=0}^{n-1} (\beta-\alpha)^{2k} \left[f^{(2k)}(\alpha)\Lambda_k\left(\frac{\beta-x}{\beta-\alpha}\right) + f^{(2k)}(\beta)\Lambda_k\left(\frac{x-\alpha}{\beta-\alpha}\right) \right]$$
$$+ (\beta-\alpha)^{2n-1} \int_{\alpha}^{\beta} G_n\left(\frac{x-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha}\right) f^{(2n)}(t) dt,$$

where, Λ_k is a Lidstone polynomial.
Therefore, differentiating twice with respect to s, we get

$$f''(s) = \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} \left[f^{(2k+2)}(\alpha)\Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) + f^{(2k+2)}(\beta)\Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) \right]$$
$$+ (\beta-\alpha)^{2n-1} \int_{\alpha}^{\beta} G_n''\left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha}\right) f^{(2n)}(t) dt. \tag{2.59}$$

Using value of $f''(s)$ from (2.59) in (2.58), we have

$$\sum_{i=1}^{m} w_i f(x_i) - \sum_{i=1}^{m} w_i f(y_i)$$
$$= \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) ds$$
$$+ \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) ds$$
$$+ (\beta-\alpha)^{2n-1}$$
$$\int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \left(\int_{\alpha}^{\beta} G_n''\left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha}\right) f^{(2n)}(t) dt \right) ds,$$

after applying Fubini's theorem we get (2.57). □

Integral version of the above theorem can be stated as follows.

Theorem 2.18 *Let $n \in \mathbb{N}$ be such that $n \geq 3$, $x, y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions that satisfy*

$$\int_a^b w(r)y(r)dr = \int_a^b w(r)x(r)dr, \tag{2.60}$$

and G_p $(p = 1,2,3,4)$ is defined as in (2.47)-(2.50) respectively. Let also G_n be defined as in (2.4) and $f \in C^{2n}[\alpha,\beta]$, then we have the following identities for $p = 1,2,3,4,$

$$\begin{aligned}
&\int_a^b w(r) f(x(r))\,dr - \int_a^b w(r) f(y(r))\,dr \\
&= \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_a^b w(r) \left[\int_\alpha^\beta \left(G_p(x(r),s) - G_p(y(r),s)\right) \Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) ds \right] dr \\
&+ \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_a^b w(r) \left[\int_\alpha^\beta \left(G_p(x(r),s) - G_p(y(r),s)\right) \Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) ds \right] dr \\
&+ (\beta-\alpha)^{2n-1} \int_\alpha^\beta f^{(2n)}(t) \left[\int_a^b w(r) \right. \\
&\left. \left(\int_\alpha^\beta \left(G_p(x(r),s) - G_p(y(r),s)\right) G_n''\left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha}\right) ds \right) dr \right] dt.
\end{aligned}$$

The following theorem is the generalization of majorization theorem i.e., Fuchs's theorem.

Theorem 2.19 *Let all the assumptions of Theorem 2.17 be satisfied. If for all $s \in [\alpha,\beta]$*

$$\int_\alpha^\beta \left(\sum_{i=1}^m w_i G_p(x_i,s) - \sum_{i=1}^m w_i G_p(y_i,s) \right) G_n''\left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha}\right) ds \geq 0,$$
$$\text{for } p = 1,2,3,4. \tag{2.61}$$

then for every $(2n)$-convex function $f : [\alpha,\beta] \to \mathbb{R}$, we have

$$\begin{aligned}
&\sum_{i=1}^m w_i f(x_i) - \sum_{i=1}^m w_i f(y_i) \\
&\geq \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_\alpha^\beta \left(\sum_{i=1}^m w_i G_p(x_i,s) - \sum_{i=1}^m w_i G_p(y_i,s) \right) \Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) ds \\
&+ \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_\alpha^\beta \left(\sum_{i=1}^m w_i G_p(x_i,s) - \sum_{i=1}^m w_i G_p(y_i,s) \right) \Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) ds.
\end{aligned} \tag{2.62}$$

If the reverse inequality in (2.61) holds, then also the reverse inequality in (2.62) holds.

Proof. If the function f is $2n$-convex, without loss of generality we can assume that f is $2n$-times differentiable, we have $f^{(2n)}(x) \geq 0$, for all $x \in [\alpha,\beta]$ (see [144], p.19 and p.293). Therefore substituting (2.61) and $f^{(2n)}(x) \geq 0$ in (2.57), we get (2.62). □

Integral version of the above theorem which is in fact the generalization of the weighted integral majorization theorem can be stated as:

Theorem 2.20 *Let all the assumptions of Theorem 2.18 be satidfied.*
If for all $s \in [\alpha,\beta]$

$$\int_a^b w(r)\left(\int_\alpha^\beta \left(G_p\left(x(r),s\right) - G_p(y(r),s)\right) G_n''\left(\frac{s-\alpha}{\beta-\alpha},\frac{t-\alpha}{\beta-\alpha}\right) ds\right) dr \geq 0,$$
$$\text{for } p = 1,2,3,4. \quad (2.63)$$

then for every $(2n)$-convex function $f : [\alpha,\beta] \to \mathbb{R}$, we have

$$\begin{aligned}
&\int_a^b w(r) f\left(x(r)\right) dr - \int_a^b w(r) f\left(y(r)\right) dr \\
&\geq \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_a^b w(r)\left[\int_\alpha^\beta \left(G_p\left(x(r),s\right) - G_p(y(r),s)\right) \Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) ds\right] dr \\
&+ \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_a^b w(r)\left[\int_\alpha^\beta \left(G_p\left(x(r),s\right) - G_p(y(r),s)\right) \Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) ds\right] dr.
\end{aligned} \quad (2.64)$$

If the reverse inequality in (2.63) holds, then also the reverse inequality in (2.64) holds.

The following theorem is majorization theorem for $2n$-convex function:

Theorem 2.21 *Let $n \in \mathbb{N}$ such that $n \geq 3$ and $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ be two decreasing real m-tuples with x_i, $y_i \in [\alpha,\beta]$ $(i = 1,\ldots,m)$ and $\mathbf{w} = (w_1,\ldots,w_m)$ be a real m-tuple such that which satisfies (1.19), (1.20). Let also G_p $(p = 1,2,3,4)$ be defined as in (2.47)-(2.50) respectively.*
Let the inequality (2.62) holds and let $\mathbb{F} : [\alpha,\beta] \to \mathbb{R}$ be a function defined for $p = 1,2,3,4$ as

$$\begin{aligned}
\mathbb{F}(.) :=& \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_\alpha^\beta \Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) G_p(.,s) ds \\
&+ \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_\alpha^\beta \Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) G_p(.,s) ds.
\end{aligned} \quad (2.65)$$

If $\mathbb{F}$ is a convex function, then the right hand side of (2.62) is non-negative that is the following weighted majorization inequality holds

$$\sum_{i=1}^m w_i f\left(y_i\right) \leq \sum_{i=1}^m w_i f\left(x_i\right). \quad (2.66)$$

Proof. We can easily get the equivalent form of the inequality (2.62) as

$$\sum_{i=1}^m w_i f\left(x_i\right) - \sum_{i=1}^m w_i f\left(y_i\right) \geq \sum_{i=1}^m w_i \mathbb{F}\left(x_i\right) - \sum_{i=1}^m w_i \mathbb{F}\left(y_i\right).$$

By using majorization conditions (1.19), (1.20) and the fact that $\mathbb{F}$ is a convex function, we can apply weighted majorization inequality, which implies immediately the non-negativity of the right hand side of (2.62) and we have the inequality (2.66). $\square$

The following theorem is majorization theorem for $2n$-convex function in integral case:

Theorem 2.22 *Let $n \in \mathbb{N}$ such that $n \geq 3$, $x, y : [a,b] \to [\alpha, \beta]$ be decreasing and $w : [a,b] \to \mathbb{R}$ be any continuous functions satisfying*

$$\int_a^{\upsilon} w(r)y(r)dr \leq \int_a^{\upsilon} w(r)x(r)dr, \quad for\ \upsilon \in [a,b] \tag{2.67}$$

and

$$\int_a^b w(r)y(r)dr = \int_a^b w(r)x(r)dr. \tag{2.68}$$

Let also G_p $(p = 1,2,3,4)$ is defined as in (2.47)-(2.50) respectively.
Let the inequality (2.63) holds and let $\mathbb{F} : [\alpha, \beta] \to \mathbb{R}$ be a function defined in (2.65) is a convex function, then the right hand side of (2.63) is non-negative that is the following weighted majorization inequality in integral case holds

$$\int_a^b w(r)f(y(r))dr \leq \int_a^b w(r)f(x(r))dr. \tag{2.69}$$

In the next part of this subsection, we give the upper bounds like the Grüss and Ostrowki type for our generalized results.

Let $\boldsymbol{x}, \boldsymbol{y}$ be two decreasing real $m-$tuples, let $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)$ be a real m-tuple such that satisfying (1.20). Also let G_n and $G_p (p = 1,2,3,4)$ be as defined in (2.4), (2.47), (2.48), (2.49), (2.50) respectively. Then consider

$$\Upsilon_1(s) = \int_\alpha^\beta \left(\sum_{i=1}^m w_i G_p(x_i, s) - \sum_{i=1}^m w_i G_p(y_i, s) \right) G_n'' \left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha} \right) ds, \tag{2.70}$$

where $p = 1,2,3,4$ and $s \in [\alpha, \beta]$. Similarly for $x, y : [a,b] \to [\alpha, \beta]$ and $w : [a,b] \to \mathbb{R}$ be continuous functions that satisfy (2.68) define $\Upsilon_2(s)$ as

$$\Upsilon_2(s) = \int_a^b w(r) \left(\int_\alpha^\beta (G_p(x(r), s) - G_p(y(r), s))\, G_n'' \left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha} \right) ds \right) dr, \tag{2.71}$$

where $p = 1,2,3,4$ and $s \in [\alpha, \beta]$.

Consider the Čebyšev functional defined as

$$T(\Upsilon_u, \Upsilon_u) = \frac{1}{\beta-\alpha} \int_\alpha^\beta \Upsilon_u^2(s)ds - \left(\frac{1}{\beta-\alpha} \int_\alpha^\beta \Upsilon_u(s)ds \right)^2, \quad u = 1,2. \tag{2.72}$$

Theorem 2.23 *Let $n \in \mathbb{N}$ be such that $n \geq 3$ and $f : [\alpha,\beta] \to \mathbb{R}$ be such that $f \in C^{2n}[\alpha,\beta]$ with $(.-\alpha)(\beta-.)\left[f^{(2n+1)}\right]^2 \in L[\alpha,\beta]$, and also $\boldsymbol{x},\boldsymbol{y}$ be two decreasing real m-tuples such that $x_i, y_i \in [\alpha,\beta]$ and $w_i \in \mathbb{R}$ $(i = 1,2,\ldots,m)$ satisfying (1.20). Let also the functions $G_p\,(p = 1,2,3,4)$ be defined as in (2.47)-(2.50) respectively and Υ_1 be defined in(2.70). Then the remainder $\mathbb{REM}(f;\alpha,\beta)$ defined for $p = 1,2,3,4$ as*

$$\begin{aligned}
\mathbb{REM}(f;\alpha,\beta) &= \sum_{i=1}^{m} w_i f(x_i) - \sum_{i=1}^{m} w_i f(y_i) \\
&- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k \left(\frac{\beta-s}{\beta-\alpha} \right) ds \\
&- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k \left(\frac{s-\alpha}{\beta-\alpha} \right) ds \\
&- (\beta-\alpha)^{2n-2} \left(f^{(2n-1)}(\beta) - f^{(2n-1)}(\alpha) \right) \int_{\alpha}^{\beta} \Upsilon_1(t)dt, \qquad (2.73)
\end{aligned}$$

satisfies the estimation

$$|\mathbb{REM}(f;\alpha,\beta)| \leq \frac{(\beta-\alpha)^{2n-\frac{1}{2}}}{\sqrt{2}} \left[T(\Upsilon_1,\Upsilon_1)\right]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta-t) \left[f^{(2n+1)}(t)\right]^2 dt \right|^{\frac{1}{2}}. \qquad (2.74)$$

Proof. Comparing (2.73) and (2.57) we have

$$\mathbb{REM}(f;\alpha,\beta) = (\beta-\alpha)\,T(\Upsilon_1, f^{(2n)}).$$

Applying Theorem 1.10 to the functions Υ and $f^{(2n)}$ we obtain (2.74). □

Integral case of the above theorem can be given as follows.

Theorem 2.24 *Let $n \in \mathbb{N}$ such that $n \geq 3$ and $f : [\alpha,\beta] \to \mathbb{R}$ be such that $f \in C^{2n}[\alpha,\beta]$ with $(.-\alpha)(\beta-.)\left[f^{(2n+1)}\right]^2 \in L[\alpha,\beta]$, and $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions satisfying (2.68). Let also the functions $G_p\,(p = 1,2,3,4)$ be defined as in (2.47)-(2.50) respectively and Υ_2 be defined as in (2.71). Then the remainder $\widetilde{\mathbb{REM}}(f;\alpha,\beta)$ defined for $p = 1,2,3,4$ as*

$$\begin{aligned}
\widetilde{\mathbb{REM}}(f;\alpha,\beta) &= \int_a^b w(r) f(x(r))\,dr - \int_a^b w(r) f(y(r))\,dr \\
&- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_a^b w(r) \left[\int_{\alpha}^{\beta} (G_p(x(r),s) - G_p(y(r),s)) \Lambda_k \left(\frac{\beta-s}{\beta-\alpha} \right) ds \right] dr \\
&- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_a^b w(r) \left[\int_{\alpha}^{\beta} (G_p(x(r),s) - G_p(y(r),s)) \Lambda_k \left(\frac{s-\alpha}{\beta-\alpha} \right) ds \right] dr \\
&- (\beta-\alpha)^{2n-2} \left(f^{(2n-1)}(\beta) - f^{(2n-1)}(\alpha) \right) \int_{\alpha}^{\beta} \Upsilon_2(s)ds \qquad (2.75)
\end{aligned}$$

satisfies the estimation

$$\left|\widetilde{\mathbb{REM}}(f;\alpha,\beta)\right| \leq \frac{(\beta-\alpha)^{2n-\frac{1}{2}}}{\sqrt{2}}\left[T(\Upsilon_2,\Upsilon_2)\right]^{\frac{1}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)\left[f^{(2n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.$$

Using Theorem 1.11 we obtain the following Grüss type inequality.

Theorem 2.25 *Let $n\in\mathbb{N}$ such that $n\geq 3$ and $f:[\alpha,\beta]\to\mathbb{R}$ be such that $f\in C^{2n}[\alpha,\beta]$ and also $f^{(2n+1)}\geq 0$ on $[\alpha,\beta]$. Let the function Υ_1 be defined as in (2.70). Then the remainder $\mathbb{REM}(f;\alpha,\beta)$ defined by (2.73) satisfies the estimation*

$$|\mathbb{REM}(f;\alpha,\beta)|\leq(\beta-\alpha)^{2n-1}\left\|\Upsilon_1'\right\|_\infty\left\{\frac{f^{(2n-1)}(\beta)+f^{(2n-1)}(\alpha)}{2}-\frac{f^{(2n-2)}(\beta)-f^{(2n-2)}(\alpha)}{\beta-\alpha}\right\}. \tag{2.76}$$

Proof. Since $\mathbb{REM}(f;\alpha,\beta)=(\beta-\alpha)^{2n}T(\Upsilon_1,f^{(2n)})$, applying Theorem 1.11 to the functions Υ_1 and $f^{(2n)}$ we get (2.76). □

Integral version of the above theorem can be given as follows.

Theorem 2.26 *Let $n\in\mathbb{N}$ such that $n\geq 3$ and $f:[\alpha,\beta]\to\mathbb{R}$ be such that $f\in C^{2n}[\alpha,\beta]$ and also $f^{(2n+1)}\geq 0$ on $[\alpha,\beta]$. Let also the function Υ_2 be defined as in (2.71). Then the remainder $\widetilde{\mathbb{REM}}(f;\alpha,\beta)$ defined by (2.75) satisfies the estimation*

$$\left|\widetilde{\mathbb{REM}}(f;\alpha,\beta)\right|\leq(\beta-\alpha)^{2n-1}\left\|\Upsilon_2'\right\|_\infty\left\{\frac{f^{(2n-1)}(\beta)+f^{(2n-1)}(\alpha)}{2}-\frac{f^{(2n-2)}(\beta)-f^{(2n-2)}(\alpha)}{\beta-\alpha}\right\}.$$

We give the Ostrowski-type inequality related to our generalized result.

Theorem 2.27 *Suppose that all the assumptions of Theorem 2.17 hold. Assume (u,v) is a pair of conjugate exponents, that is $1\leq u,v\leq\infty$, $\frac{1}{u}+\frac{1}{v}=1$. Let $\left|f^{(2n)}\right|^u:[\alpha,\beta]\to\mathbb{R}$ be an R-integrable function for some $n\in\mathbb{N}$. Then we have*

$$\begin{aligned}
&\Bigg|\sum_{i=1}^m w_i f(x_i)-\sum_{i=1}^m w_i f(y_i)\\
&-\sum_{k=0}^{n-3}(\beta-\alpha)^{2k}f^{(2k+2)}(\alpha)\int_\alpha^\beta\left(\sum_{i=1}^m w_iG_p(x_i,s)-\sum_{i=1}^m w_iG_p(y_i,s)\right)\Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right)ds\\
&-\sum_{k=0}^{n-3}(\beta-\alpha)^{2k}f^{(2k+2)}(\beta)\int_\alpha^\beta\left(\sum_{i=1}^m w_iG_p(x_i,s)-\sum_{i=1}^m w_iG_p(y_i,s)\right)\Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right)ds\Bigg|\\
&\leq(\beta-\alpha)^{2n-1}\left\|f^{(2n)}\right\|_u\\
&\left(\int_\alpha^\beta\left|\int_\alpha^\beta\left(\sum_{i=1}^m w_iG_p(x_i,s)-\sum_{i=1}^m w_iG_p(y_i,s)\right)G_n''\left(\frac{s-\alpha}{\beta-\alpha},\frac{t-\alpha}{\beta-\alpha}\right)ds\right|^v dt\right)^{\frac{1}{v}}
\end{aligned}$$

where $p=1,2,3,4$.

(2.77)

The constant on the right-hand side of (2.77) is sharp for $1 < u \leq \infty$ *and the best possible for* $u = 1$.

Proof. Let us denote

$$\Psi(t) = (\beta-\alpha)^{2n-1}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s)\right) G_n''\left(\frac{s-\alpha}{\beta-\alpha},\frac{t-\alpha}{\beta-\alpha}\right) ds,$$

for $p = 1,2,3,4$.
Using the identity (2.57) and applying Hölder's inequality we obtain

$$\begin{aligned}
&\left|\sum_{i=1}^{m} w_i f(x_i) - \sum_{i=1}^{m} w_i f(y_i)\right.\\
&-\sum_{k=0}^{n-3}(\beta-\alpha)^{2k} f^{(2k+2)}(\alpha)\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s)\right)\Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) ds\\
&\left.-\sum_{k=0}^{n-3}(\beta-\alpha)^{2k} f^{(2k+2)}(\beta)\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s)\right)\Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) ds\right|\\
&=\left|\int_{\alpha}^{\beta}\Psi(t) f^{(2n)}(t)dt\right| \leq \left\|f^{(2n)}\right\|_u\left(\int_{\alpha}^{\beta}|\Psi(t)|^v dt\right)^{\frac{1}{v}}.
\end{aligned}$$

The proof of the sharpness of the constant $\left(\int_{\alpha}^{\beta}|\Psi(t)|^v dt\right)^{\frac{1}{v}}$ is analogous to the proof of Theorem 2.11. □

Integral version of the above theorem can be stated as follows.

Theorem 2.28 *Suppose that all the assumptions of Theorem 2.18 hold. Assume* (u,v) *is a pair of conjugate exponents, that is* $1 \leq u,v \leq \infty$, $\frac{1}{u}+\frac{1}{v} = 1$. *Let* $\left|f^{(2n)}\right|^u : [\alpha,\beta] \to \mathbb{R}$ *be an R-integrable function for some* $n \in \mathbb{N}$. *Then we have the following identities for* $p = 1,2,3,4$

$$\begin{aligned}
&\left|\int_a^b w(r) f(x(r))\,dr - \int_a^b w(r) f(y(r))\,dr\right.\\
&-\sum_{k=0}^{n-3}(\beta-\alpha)^{2k} f^{(2k+2)}(\alpha)\int_a^b w(r)\left[\int_{\alpha}^{\beta}\left(G_p(x(r),s) - G_p(y(r),s)\right)\Lambda_k\left(\frac{\beta-s}{\beta-\alpha}\right) ds\right] dr\\
&\left.-\sum_{k=0}^{n-3}(\beta-\alpha)^{2k} f^{(2k+2)}(\beta)\int_a^b w(r)\left[\int_{\alpha}^{\beta}\left(G_p(x(r),s) - G_p(y(r),s)\right)\Lambda_k\left(\frac{s-\alpha}{\beta-\alpha}\right) ds\right] dr\right|\\
&\leq (\beta-\alpha)^{2n-1}\left\|f^{(2n)}\right\|_u\left(\int_{\alpha}^{\beta}\left|\int_a^b w(r)\right.\right.\\
&\left.\left.\left(\int_{\alpha}^{\beta}\left(G_p(x(r),s) - G_p(y(r),s)\right) G_n''\left(\frac{s-\alpha}{\beta-\alpha},\frac{t-\alpha}{\beta-\alpha}\right) ds\right) dr\right|^v dt\right)^{\frac{1}{v}}.
\end{aligned}$$

The constant on the right-hand side of (2.78) is sharp for $1 < u \leq \infty$ and the best possible for $u = 1$.

Motivated by the inequalities (2.62) and (2.64), we define functionals $\Theta_1(f)$ and $\Theta_2(f)$ by

$$\begin{aligned}\Theta_1(f) &= \sum_{i=1}^{m} w_i f(x_i) - \sum_{i=1}^{m} w_i f(y_i) \\ &- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_\alpha^\beta \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k \left(\frac{\beta - s}{\beta - \alpha} \right) ds \\ &- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_\alpha^\beta \left(\sum_{i=1}^{m} w_i G_p(x_i,s) - \sum_{i=1}^{m} w_i G_p(y_i,s) \right) \Lambda_k \left(\frac{s-\alpha}{\beta - \alpha} \right) ds, \end{aligned} \tag{2.78}$$

$$\begin{aligned}\Theta_2(f) &= \int_a^b w(r) f(x(r))\, dr - \int_a^b w(r) f(y(r))\, dr \\ &- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\alpha) \int_a^b w(r) \left[\int_\alpha^\beta (G_p(x(r),s) - G_p(y(r),s)) \Lambda_k \left(\frac{\beta - s}{\beta-\alpha} \right) ds \right] dr \\ &- \sum_{k=0}^{n-3} (\beta-\alpha)^{2k} f^{(2k+2)}(\beta) \int_a^b w(r) \left[\int_\alpha^\beta (G_p(x(r),s) - G_p(y(r),s)) \Lambda_k \left(\frac{s-\alpha}{\beta-\alpha} \right) ds \right] dr. \end{aligned} \tag{2.79}$$

The Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems:

Theorem 2.29 *Let $f : [\alpha,\beta] \to \mathbb{R}$ be such that $f \in C^{2n}[\alpha,\beta]$. If the inequalities in (2.61) and (2.63) hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\Theta_i(f) = f^{(2n)}(\xi_i)\Theta_i(\eta), \quad i = 1,2, \tag{2.80}$$

where $\eta(x) = \frac{x^{2n}}{(2n)!}$.

Proof. Similar to the proof of Theorem 7 in [30]. □

Theorem 2.30 *Let $f,g : [\alpha,\beta] \to \mathbb{R}$ be such that $f,g \in C^{2n}[\alpha,\beta]$. If the inequalities in (2.61) and (2.63) hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\frac{\Theta_i(f)}{\Theta_i(g)} = \frac{f^{(2n)}(\xi_i)}{g^{(2n)}(\xi_i)}, \quad i = 1,2. \tag{2.81}$$

provided that the denominators are not zero.

Proof. Similar to the proof of Corollary 12 in [30]. □

For example, in the papers [97] and [96] the results about majorization in the form of n-exponentially, exponentially and logarithmically convex functions as well as generalized Cauchy mean value theorems for class of convex functions f are presented, but here we present these results for the class of convex functions $f(x)/x$ and also an important thing is to construct examples for such type of results. So first we give the classical results for convex function $f(x)/x$ and then make functionals for obtaining n-exponentially, exponentially and logarithmically convex functions.

Theorem 2.31 *Let $I_+ \subset \mathbb{R}^+$ be an interval and $\boldsymbol{x} = (x_1, \ldots, x_m)$, $\boldsymbol{y} = (y_1, \ldots, y_m) \in I_+^m$. Let $f : I_+ \to \mathbb{R}$ be a continuous function. Then a function $\mathbb{F} : I_+^m \to \mathbb{R}$, defined by*

$$\mathbb{F}(\mathbf{x}) = \sum_{i=1}^{m} \frac{f(x_i)}{x_i}, \tag{2.82}$$

is Schur-convex on I_+^m iff $\frac{f(x)}{x}$ is convex on I_+.

Proof. In this proof we use Abel's transformation.
Without loss of generality, assume that $x_i \neq y_i$, define

$$\Delta_i = \frac{\frac{f(y_i)}{y_i} - \frac{f(x_i)}{x_i}}{y_i - x_i}, \quad i = (1, \ldots, m). \tag{2.83}$$

Since the function $\frac{f(x)}{x}$ is convex, therefore we have $\Delta_{i+1} \leq \Delta_i$, for $(i = 1, 2, \ldots, m)$, which means that Δ_i is decreasing.
The proof follows from [144, p.323-324]. □

The weighted version of the above theorem is stated as follows.

Theorem 2.32 *Let $\mathbf{x}$, $\mathbf{y}$ be two decreasing m-tuples with entries from $I_+ \subset \mathbb{R}^+$, let $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ be a real m-tuple such that*

$$\sum_{i=1}^{k} w_i y_i \leq \sum_{i=1}^{k} w_i x_i \text{ for } k = 1, \ldots, m-1, \tag{2.84}$$

and

$$\sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i x_i. \tag{2.85}$$

Then for every convex function $\frac{f(x)}{x} : I_+ \to \mathbb{R}$, we have

$$\sum_{i=1}^{m} w_i \frac{f(y_i)}{y_i} \leq \sum_{i=1}^{m} w_i \frac{f(x_i)}{x_i}. \tag{2.86}$$

Proof. The proof is similar to the Theorem 2.31. □

Motivated by the inequalities (2.82) and (2.86) that are linear in f, we define the linear functionals under the assumptions of Theorem 2.31 and Theorem 2.32:

$$\Lambda_1(\boldsymbol{x}, \boldsymbol{y}, f) = \sum_{i=1}^{m} \frac{f(x_i)}{x_i} - \sum_{i=1}^{m} \frac{f(y_i)}{y_i}, \tag{2.87}$$

and

$$\Lambda_2(\boldsymbol{x},\boldsymbol{y},f) = \sum_{i=1}^{m} w_i \frac{f(x_i)}{x_i} - \sum_{i=1}^{m} w_i \frac{f(y_i)}{y_i}. \tag{2.88}$$

Under the assumptions of Theorem 2.31 and Theorem 2.32, it holds $\Lambda_l(f) \geq 0$, $l = 1,2$, for all convex functions $\frac{f(x)}{x}$.

The following Lemma is given in [99]:

Lemma 2.3 *Let $f \in C^2(I)$ for an interval $I \subset \mathbb{R}\setminus\{0\}$ and consider $m, M \in \mathbb{R}$ such that*

$$m \leq \frac{x^2 f''(x) - 2xf'(x) + 2f(x)}{x^3} \leq M.$$

Also, let f_1, f_2 be real valued functions defined on I as follows

$$f_1(x) = M\frac{x^3}{2} - f(x),$$

$$f_2(x) = f(x) - m\frac{x^3}{2}.$$

Then $\frac{f_1(x)}{x}$ and $\frac{f_2(x)}{x}$ are convex.

The Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems:

Theorem 2.33 *Let $\boldsymbol{x},\boldsymbol{y}$ be two real m-tuples.*

- *$\boldsymbol{x} \succ \boldsymbol{y}$ for $l = 1$,*
- *$\boldsymbol{x},\boldsymbol{y}$ be decreasing and let $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ be a real m-tuple such that satisfying (1.19) and (1.20) for $l = 2$.*

Let $[\alpha,\beta] \subset \mathbb{R}^+$ and $f \in C^2([\alpha,\beta])$ then there exists $\xi_l \in [\alpha,\beta]$ such that

$$\Lambda_l(\boldsymbol{x},\boldsymbol{y},f) = \frac{\xi_l^2 f''(\xi_l) - 2\xi_l f'(\xi_l) + 2f(\xi_l)}{2\xi_l^3}\Lambda_l(\boldsymbol{x},\boldsymbol{y},x^3), \;\; l = 1,2. \tag{2.89}$$

Proof. Fix $l = 1,2$ (see Theorem 2.8 in [99]), by convexity of $f_1(x)$ and $f_2(x)$ from Lemma 2.3 therefore (2.82) changes to

$$\Lambda_l(\boldsymbol{x},\boldsymbol{y},f) \leq \frac{M}{2}\Lambda_l(\boldsymbol{x},\boldsymbol{y},x^3), \tag{2.90}$$

$$\frac{m}{2}\Lambda_l(\boldsymbol{x},\boldsymbol{y},x^3) \leq \Lambda_l(\boldsymbol{x},\boldsymbol{y},f). \tag{2.91}$$

Since $\Lambda_l(\boldsymbol{x},\boldsymbol{y},x^3) \neq 0$, so from (2.90) and (2.91) we have

$$m \leq \frac{2\Lambda_l(\boldsymbol{x},\boldsymbol{y},f)}{\Lambda_l(\boldsymbol{x},\boldsymbol{y},x^3)} \leq M. \tag{2.92}$$

Therefore we get the required result by using Lemma 2.3. □

Theorem 2.34 *Let $\boldsymbol{x},\boldsymbol{y}$ be two real m-tuples.*

- $\boldsymbol{x} \succ \boldsymbol{y}$ *for* $l = 1$,

- $\mathbf{x}$, $\mathbf{y}$ *be decreasing and let* $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ *be a real m-tuple such that satisfying (1.19) and (1.20) for* $l = 2$.

Let $[\alpha,\beta] \subset \mathbb{R}^+$ *and* $f,g \in C^2([\alpha,\beta])$, *then there exists* $\xi_l \in [\alpha,\beta]$ *such that*

$$\frac{\Lambda_l(\boldsymbol{x},\boldsymbol{y},f)}{\Lambda_l(\boldsymbol{x},\boldsymbol{y},g)} = \frac{\xi_l^2 f''(\xi_l) - 2\xi_l f'(\xi_l) + 2f(\xi_l)}{\xi_l^2 g''(\xi_l) - 2\xi_l g'(\xi_l) + 2g(\xi_l)}, \quad l = 1,2, \tag{2.93}$$

provided that denominators are non-zero.

Proof. Fix $l = 1,2$ (Theorem 2.9 of [99]), define $h \in C^2([\alpha,\beta])$ in the way that

$$h = c_1 f - c_2 g, \quad \textit{where} \quad c_1 = \Lambda_l(\boldsymbol{x},\boldsymbol{y},g) \ \textit{and} \ c_2 = \Lambda_l(\boldsymbol{x},\boldsymbol{y},f).$$

Now using (2.89) with $f = h$, we have

$$\left(c_1\left(\frac{\xi^2 f''(\xi) - 2\xi f'(\xi) + 2f(\xi)}{2\xi^3}\right) - c_2\left(\frac{\xi^2 g''(\xi) - 2\xi g'(\xi) + 2g(\xi)}{2\xi^3}\right)\right)\Lambda_l(\boldsymbol{x},\boldsymbol{y},x^3) = 0. \tag{2.94}$$

Since $\Lambda_l(\boldsymbol{x},\boldsymbol{y},x^3) \neq 0$, we get the required result. □

In order to obtain results regarding the exponential convexity, we define the families of functions as follows.
For every choice of $t+1$ mutually different points $z_0, \ldots, z_t \in [\alpha,\beta]$ we define

- $\mathbb{F}_1 = \{f_v : [\alpha,\beta] \to \mathbb{R} : v \in J \ \textit{and} \ v \mapsto [z_0, \ldots, z_t, \frac{f_v(x)}{x}]$ is n-exponentially convex in the Jensen sense on $J\ \}$

- $\mathbb{F}_2 = \{f_v : [\alpha,\beta] \to \mathbb{R} : v \in J \ \textit{and} \ v \mapsto [z_0, \ldots, z_t, \frac{f_v(x)}{x}]$ is exponentially convex in the Jensen sense on $J\ \}$

- $\mathbb{F}_3 = \{f_v : [\alpha,\beta] \to \mathbb{R} : v \in J \ \textit{and} \ v \mapsto [z_0, \ldots, z_t, \frac{f_v(x)}{x}]$ is 2-exponentially convex in the Jensen sense on $J\ \}$

Theorem 2.35 *Let Λ_l $(l = 1,2)$ be the linear functionals defined by (2.87) and (2.88) associated with family $\mathbb{F}_1$. Then the following statements hold:*

(i) The function $v \mapsto \Lambda_l(f_v)$ is an n-exponentially convex function in the Jensen sense on J and the matrix $\left[\Lambda_l\left(f_{\frac{v_i+v_j}{2}}\right)\right]_{i,j=1}^{p}$ is a positive semi-definite. Particularly

$$\det\left[\Lambda_l\left(f_{\frac{v_i+v_j}{2}}\right)\right]_{i,j=1}^{p} \geq 0$$

holds for all $p \in \mathbb{N}$, $p \leq n$, $v_1,\ldots,v_p \in J$.

(ii) If the function $v \mapsto \Lambda_l(f_v)$ is continuous on J, then it is n-exponentially convex function on J.

Proof. (i) For fixed $l = 1,2$, $\vartheta_i \in \mathbb{R}$ and $v_i \in J$, $i = 1,\ldots,n$ we define the function

$$\delta(x) = \sum_{i,j=1}^{n} \vartheta_i \vartheta_j \frac{f_{\frac{v_i+v_j}{2}}(x)}{x}.$$

Using the assumption we have

$$[z_0,\ldots,z_t,\delta] = \sum_{i,j=1}^{n} \vartheta_i \vartheta_j \left[z_0,\ldots,z_t,\frac{f_{\frac{v_i+v_j}{2}}(x)}{x}\right] \geq 0,$$

which in turn implies the required results (see Theorem 10 in [99]). □

The following corollaries are immediate consequences of the above theorem.

Corollary 2.3 *Let Λ_l $(l = 1,2)$ be the linear functionals defined by (2.87) and (2.88) associated with family $\mathbb{F}_2$. Then the following statements hold:*

(i) The function $v \mapsto \Lambda_l(f_v)$ is an exponentially convex function in the Jensen sense on J.

(ii) If the function $v \mapsto \Lambda_l(f_v)$ is continuous on J, then it is exponentially convex function on J.

Corollary 2.4 *Let Λ_l $(l = 1,2)$ be the linear functionals defined by (2.87) and (2.88) associated with family $\mathbb{F}_3$. Then the following statements hold:*

(i) If the function $v \mapsto \Lambda_l(f_v)$ is continuous on J, then it is 2-exponentially convex function on J. If $v \mapsto \Lambda_l(f_v)$ is additionally strictly positive, then it is log-convex *on J. Furthermore, for every choice $q,u,w \in J$, such that $q < u < w$, Lypunov's inequality holds:*

$$[\Lambda_l(f_u)]^{w-q} \leq [\Lambda_l(f_q)]^{w-u}[\Lambda_l(f_w)]^{u-q}.$$

(ii) If the function $v \mapsto \Lambda_l(f_v)$ is strictly positive and differentiable on J, then for every $p,q,u,w \in J$, such that $p \le u$ and $q \le w$, we have

$$\mu_{p,q}(\Lambda_l, \Phi) \le \mu_{u,w}(\Lambda_l, \Phi), \tag{2.95}$$

where

$$\mu_{p,q}(\Lambda_l, \Phi) = \begin{cases} \left(\frac{\Lambda_l(f_p)}{\Lambda_l(f_q)}\right)^{\frac{1}{p-q}}, & p \ne q \\ \exp\left(\frac{\frac{d}{dp}\Lambda_l(f_p)}{\Lambda_l(f_q)}\right), & p = q \end{cases}. \tag{2.96}$$

Proof. (i) This is an immediate consequence of Theorem 2.35 and Remark 1.16.
(ii) Fix $l = 1,2$, since $v \mapsto \Lambda_l(f_v)$ is positive and continuous, by (i) we have that the function $v \mapsto \Lambda_l(f_v)$ is log-convex on J. So, for $p,q,u,w \in J$, such that $p \ne q$ and $u \ne w$ and $p \le u$ and $q \le w$, we have

$$\frac{\log \Lambda_l(f_p) - \log \Lambda_l(f_q)}{p-q} \le \frac{\log \Lambda_l(f_u) - \log \Lambda_l(f_w)}{u-w}, \tag{2.97}$$

i.e. we conclude that

$$\mu_{p,q}(\Lambda_l, \Phi) \le \mu_{u,w}(\Lambda_l, \Phi).$$

Cases $p = q$ and $u = w$ follows from (2.97) as limiting cases. □

Remark 2.4 *Note that the results from Theorem 2.35, Corollary 2.3 and Corollary 2.4 still hold when two of the points $z_0, \dots, z_t \in [\alpha, \beta]$ coincide, say $z_1 = z_0$, for a family of differentiable functions f_v such that the function $v \mapsto \left[z_0, \dots, z_t, \frac{f_v(x)}{x}\right]$ is an n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense,* log-convex *in the Jensen sense), and furthermore, they still hold when all $(t+1)$ points coincide for a family of t differentiable functions with the same property. The proofs are obtained by (1.80) and suitable characterization of convexity.*

Remark 2.5 *We can give the similar results as Theorem 2.35, Corollary 2.3, Corollary 2.4 and Remark 2.4 for $(2t+1)$-points as to prove $(2n)$-exponentially convex functions.*

At the end of this subsection, we give some applications of our generalized results about the upper bounds as well as exponential convex functions.
Firstly, we consider some related inequalities by using our generalized results of upper bounds.

Example 2.1 *By using Ostrowski-type inequality (2.77) for $n = 3$ as an upper bound of our generalized results:*

- *let $f(x) = e^x, x \in \mathbb{R}$, then*

$$0 \le \left| \sum_{i=1}^{m} w_i e^{x_i} - \sum_{i=1}^{m} w_i e^{y_i} \right| \le \frac{(\beta - \alpha)^5}{u^{\frac{1}{u}}} (e^{u\beta} - e^{u\alpha})^{\frac{1}{u}} \parallel \mathbb{G}_p \parallel_v,$$

- *let $f(x) = x^r$, $[0,\infty)$ for $r > 1$, then*

$$0 \leq \left| \sum_{i=1}^{m} w_i x_i^r - \sum_{i=1}^{m} w_i y_i^r \right| \leq (\beta-\alpha)^5 \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{(u(r-6)+1)^{\frac{1}{u}}} \left(\beta^{u(r-6)+1} - \alpha^{u(r-6)+1}\right)^{\frac{1}{u}} \parallel \mathbb{G}_p \parallel_v,$$

- *let $f(x) = x \log x$, $x \in (0,\infty)$, then*

$$0 \leq \left| \sum_{i=1}^{m} w_i x_i \log x_i - \sum_{i=1}^{m} w_i y_i \log y_i \right| \leq \frac{24(\beta-\alpha)^5}{(1-5u)^{\frac{1}{u}}} \left(\beta^{1-5u} - \alpha^{1-5u}\right)^{\frac{1}{u}} \parallel \mathbb{G}_p \parallel_v,$$

- *let $f(x) = -\log x$, $x \in (0,\infty)$, then*

$$0 \leq \left| \sum_{i=1}^{m} w_i \log y_i - \sum_{i=1}^{m} w_i \log x_i \right| \leq \frac{120(\beta-\alpha)^5}{(1-6u)^{\frac{1}{u}}} \left(\beta^{1-6u} - \alpha^{1-6u}\right)^{\frac{1}{u}} \parallel \mathbb{G}_p \parallel_v,$$

where $\mathbb{G}_p = \int_\alpha^\beta \left(\sum_{i=1}^m w_i G_p(x_i,s) - \sum_{i=1}^m w_i G_p(y_i,s)\right) G_3''\left(\frac{s-\alpha}{\beta-\alpha}, \frac{t-\alpha}{\beta-\alpha}\right) ds$, $(p = 1,2,3,4)$.
We can also give the particular cases of above results for $u = 1$ and $v = \infty$.

Now, we construct exponentially convex function by using family of convex functions defined on $(0,\infty)$.

Example 2.2 *Let*

$$\mathbb{E}_1 = \{\theta_v : (0,\infty) \to (0,\infty) : v \in \mathbb{R}\}$$

be a family of continuous convex functions defined by

$$\theta_v(x) = \begin{cases} \frac{xe^{vx}}{v^2}, & v \neq 0; \\ \frac{x^3}{2}, & v = 0. \end{cases}$$

We have $v \mapsto \left(\frac{\theta_v(x)}{x}\right)''$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in \mathbb{R}$. Using analogous arguing as in the proof of Theorem 2.35 we also have that $v \mapsto \theta_v[z_0,\ldots,z_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 2.3 we conclude that $v \mapsto \Lambda_l(\theta_v)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $v \mapsto \psi_v$ is not continuous for $v = 0$), so it is exponentially convex.
For this family of functions, $\mu_{v,q}(\Theta,\Lambda_2)$ from (2.96), becomes

$$\mu_{t,s}(\mathbb{E}_1,\Lambda_2) = \left(\frac{\mathbb{E}_1(\theta_t)}{\mathbb{E}_1(\theta_s)}\right)^{\frac{1}{t-s}}, \quad t \neq s, t, s \neq 0;$$

$$\mu_{t,t}(\mathbb{E}_1,\Lambda_2) = \exp\left(\frac{\sum_{i=1}^n p_i x_i^2 e^{tx_i} - \sum_{i=1}^n p_i y_i^2 e^{ty_i}}{\sum_{i=1}^n p_i x_i e^{tx_i} - \sum_{i=1}^n p_i y_i e^{ty_i}} - \frac{2}{t}\right), \ t = s \neq 0;$$

$$\mu_{0,0}(\mathbb{E}_1, \Lambda_2) = \exp\left(\frac{1}{3}\frac{\sum_{i=1}^n p_i x_i^4 - \sum_{i=1}^n p_i y_i^4}{\sum_{i=1}^n p_i x_i^3 - \sum_{i=1}^n p_i x_i^3}\right).$$

Now using (2.95), $\mu_{t,s}$ is monotone function in parameters t and s.

We observe here that $\left(\frac{\frac{d^2\theta_t}{dx^2}}{\frac{d^2\theta_s}{dx^2}}\right)^{\frac{1}{t-s}}(\ln x) = x$ *so using Theorem 2.34 it follows that*

$$M_{t,s}(\mathbb{E}_1, \Lambda_2) = \ln \mu_{t,s}(\mathbb{E}_1, \Lambda_2),$$

satisfies

$$\alpha \leq M_{t,s}(\mathbb{E}_1, \Lambda_2) \leq \beta.$$

This shows that $M_{t,s}(\mathbb{E}_1, \Lambda_2)$ is a mean. Because of the above inequality (2.95), this mean is also monotonic.

Remark 2.6 *We can construct other examples for exponentially convex functions as Example 2 for the families of continuous convex functions:*

- $$\mathbb{E}_2 = \{\mu_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

  where

  $$\mu_t(x) = \begin{cases} \frac{x^{t+1}}{t(t-1)}, & t \neq 0, 1; \\ -x\log x, & t = 0; \\ x^2 \log x, & t = 1. \end{cases}$$

- $$\mathbb{E}_3 = \{\chi_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

  where,

  $$\chi_t(x) = \begin{cases} \frac{xt^{-x}}{\log^2 t}, & t \neq 1; \\ \frac{x^3}{2}, & t = 1. \end{cases}$$

- $$\mathbb{E}_4 = \{\delta_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

  where,

  $$\delta_t(x) := \frac{xe^{-x\sqrt{t}}}{t}.$$

2.1.3 Results Obtained for the Jensen and the Jensen-Steffensen Inequalities and their Converses via Lidstone Polynomial

In this subsection (see [31]), using majorization theorems and Lidstone's interpolating polynomials we obtain results concerning Jensen's and the Jensen-Steffensen inequalities and their converses in both an integral and discrete case. We give bounds for identities related to these inequalities by using Čebyšev functionals. We also give the Grüss and Ostrowsky type inequalities for these functionals. Also we use these generalizations to construct a linear functionals and we present mean value theorems and n-exponential convexity which leads to exponential convexity and then log-convexity for these functionals. We give some families of functions which enable us to construct a large families of functions that are exponentially convex and give Stolarsky type means.

We will use the following notations for composition of functions:

$$\Lambda_k\left(\tfrac{x-a}{b-a}\right) = \tilde{\Lambda}_k(x),\, x \in [a,b],\, k = 0,1,\ldots,n-1, \tag{2.98}$$

$$\Lambda_k\left(\tfrac{b-x}{b-a}\right) = \hat{\Lambda}_k(x),\, x \in [a,b],\, k = 0,1,\ldots,n-1. \tag{2.99}$$

Theorem 2.36 *Let $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1,\ldots,x_m)$, and $\boldsymbol{w} = (w_1,\ldots,w_m)$ be m-tuples such that $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and $\phi \in C^{2n}[a,b]$. Then*

$$\begin{aligned}
&\frac{1}{W_m}\sum_{i=1}^m w_i\phi(x_i) - \phi(\overline{x}) \qquad (2.100)\\
&= \sum_{k=0}^{n-1} \phi^{(2k)}(a)(b-a)^{2k}\left[\frac{1}{W_m}\sum_{i=1}^m w_i\hat{\Lambda}_k(x_i) - \hat{\Lambda}_k(\overline{x})\right]\\
&+ \sum_{k=0}^{n-1} \phi^{(2k)}(b)(b-a)^{2k}\left[\frac{1}{W_m}\sum_{i=1}^m w_i\tilde{\Lambda}_k(x_i) - \tilde{\Lambda}_k(\overline{x})\right]\\
&+\ (b-a)^{2n-1}\int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G_n\left(\tfrac{x_i-a}{b-a},\tfrac{t-a}{b-a}\right) - G_n\left(\tfrac{\overline{x}-a}{b-a},\tfrac{t-a}{b-a}\right)\right]\phi^{(2n)}(t)\,dt.
\end{aligned}$$

Proof. Consider

$$\frac{1}{W_m}\sum_{i=1}^m w_i\phi(x_i) - \phi(\overline{x}). \tag{2.101}$$

By Widder's lemma we can represent every function $\phi \in C^{2n}[a,b]$ in the form:

$$\begin{aligned}
\phi(x) \ &= \sum_{k=0}^{n-1}(b-a)^{2k}\left[\phi^{(2k)}(a)\hat{\Lambda}_k(x) + \phi^{(2k)}(b)\tilde{\Lambda}_k(x)\right]\\
&+\ (b-a)^{2n-1}\int_a^b G_n\left(\tfrac{x-a}{b-a},\tfrac{t-a}{b-a}\right)\phi^{(2n)}(t)dt,
\end{aligned} \tag{2.102}$$

where Λ_k is a Lidstone polynomial. Using (2.102) we calculate $\phi(x_i)$ and $\phi(\overline{x})$ and from (2.101) we obtain (2.100) □

Using Theorem 2.5 (a) we give generalization of Jensen's inequality for $(2n)$-convex function.

Theorem 2.37 *Let $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \ldots, x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1, \ldots, m$, let $\mathbf{w} = (w_1, \ldots, w_m)$ be positive m-tuple, $W_m = \sum_{i=1}^m w_i$ and $\overline{x} = \frac{1}{W_m} \sum_{i=1}^m w_i x_i$.*

(i) If n is odd, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned} &\frac{1}{W_m} \sum_{i=1}^m w_i \phi(x_i) - \phi(\overline{x}) \\ &\geq \sum_{k=1}^{n-1} \phi^{(2k)}(a)(b-a)^{2k} \left[\frac{1}{W_m} \sum_{i=1}^m w_i \hat{\Lambda}_k(x_i) - \hat{\Lambda}_k(\overline{x}) \right] \\ &+ \sum_{k=1}^{n-1} \phi^{(2k)}(b)(b-a)^{2k} \left[\frac{1}{W_m} \sum_{i=1}^m w_i \tilde{\Lambda}_k(x_i) - \tilde{\Lambda}_k(\overline{x}) \right]. \end{aligned} \tag{2.103}$$

Moreover, we define function $F : [a,b] \to \mathbb{R}$, such that

$$F(x) = \sum_{k=1}^{n-1} (b-a)^{2k} \left[\phi^{(2k)}(a)\hat{\Lambda}_k(x) + \phi^{(2k)}(b)\tilde{\Lambda}_k(x) \right]. \tag{2.104}$$

If F is convex function, then the right hand side of (2.103) is non-negative and

$$\frac{1}{W_m} \sum_{i=1}^m w_i \phi(x_i) - \phi(\overline{x}) \geq 0. \tag{2.105}$$

(ii) If n is even, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, the reverse inequality in (2.103) holds.
Moreover, if F is concave function, then the reverse inequality in (2.105) is valid.

Proof. For $l = 1, \ldots, k$, such that $x_k \geq \overline{x}$ we get

$$\sum_{i=1}^{l} w_i \overline{x} \leq \sum_{i=1}^{l} w_i x_i.$$

If $l = k+1, \ldots, m-1$, such that $x_{k+1} < \overline{x}$ we have

$$\sum_{i=1}^{l} w_i x_i = \sum_{i=1}^{m} w_i x_i - \sum_{i=l+1}^{m} w_i x_i > \sum_{i=1}^{m} w_i \overline{x} - \sum_{i=l+1}^{m} w_i \overline{x} = \sum_{i=1}^{l} w_i \overline{x}.$$

So,

$$\sum_{i=1}^{l} w_i \overline{x} \leq \sum_{i=1}^{l} w_i x_i \text{ for all } l = 1, \ldots, m-1 \tag{2.106}$$

and obviously

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} w_i x_i. \tag{2.107}$$

Now, we put $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (\overline{x}, \ldots, \overline{x})$ in Theorem 2.5 (a) to get inequality (2.103).

For inequality (2.105) we use fact that for convex function F we have

$$\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \geq 0.$$

□

Remark 2.7 *For $x : [\alpha,\beta] \to \mathbb{R}$ continuous decreasing function, such that $x([\alpha,\beta]) \subseteq [a,b]$ and $\lambda : [\alpha,\beta] \to \mathbb{R}$ increasing, bounded function with $\lambda(\alpha) \neq \lambda(\beta)$ and $\overline{x} = \frac{\int_\alpha^\beta x(t)\,d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$, for $x(\gamma) \geq \overline{x}$, we have:*

$$\int_\alpha^\gamma x(t)\,d\lambda(t) \geq \int_\alpha^\gamma x(\gamma)\,d\lambda(t) \geq \int_\alpha^\gamma \overline{x}\,d\lambda(t),\ \gamma \in [\alpha,\beta]. \tag{2.108}$$

If $x(\gamma) < \overline{x}$ we have

$$\begin{aligned}\int_\alpha^\gamma x(t)\,d\lambda(t) &= \int_\alpha^\beta x(t)\,d\lambda(t) - \int_\gamma^\beta x(t)\,d\lambda(t) \\ &> \int_\alpha^\beta \overline{x}\,d\lambda(t) - \int_\gamma^\beta \overline{x}\,d\lambda(t) = \int_\alpha^\gamma \overline{x}\,d\lambda(t),\ \gamma \in [\alpha,\beta].\end{aligned} \tag{2.109}$$

Equality

$$\int_\alpha^\beta x(t)\,d\lambda(t) = \int_\alpha^\beta \overline{x}\,d\lambda(t) \tag{2.110}$$

obviously holds.

So, If $n \in \mathbb{N}$ is odd, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, we obtain integral version of the inequality (2.103) from the above theorem

$$\begin{aligned}&\frac{\int_\alpha^\beta \phi(x(t))\,d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \phi(\overline{x}) \\ &\geq \sum_{k=1}^{n-1} \phi^{(2k)}(a)(b-a)^{2k}\left[\frac{\int_\alpha^\beta \hat{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \hat{\Lambda}_k(\overline{x})\right] \\ &+ \sum_{k=1}^{n-1} \phi^{(2k)}(b)(b-a)^{2k}\left[\frac{\int_\alpha^\beta \tilde{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \tilde{\Lambda}_k(\overline{x})\right],\end{aligned} \tag{2.111}$$

which is result proved in [28].

Moreover, for the convex function F defined in (2.104) the right hand side of (2.111) is non-negative and

$$\frac{\int_\alpha^\beta \phi(x(t))\,d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \phi(\overline{x}) \geq 0. \tag{2.112}$$

If n is even, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$ the reverse inequality in (2.111) holds. Moreover, if F is concave function, then the reverse inequality in (2.112) is also valid.

Remark 2.8 *Motivated by the inequalities (2.103) and (2.111), we define functionals $\Theta_1(\phi)$ and $\Theta_2(\phi)$ by*

$$\begin{aligned}\Theta_1(\phi) \;&=\; \tfrac{1}{W_m}\sum_{i=1}^{m} w_i\phi(x_i) - \phi(\overline{x}) \\ &- \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\left[\tfrac{1}{W_m}\sum_{i=1}^{m} w_i\hat{\Lambda}_k(x_i) - \hat{\Lambda}_k(\overline{x})\right] \\ &- \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\left[\tfrac{1}{W_m}\sum_{i=1}^{m} w_i\tilde{\Lambda}_k(x_i) - \tilde{\Lambda}_k(\overline{x})\right]\end{aligned} \tag{2.113}$$

and

$$\begin{aligned}\Theta_2(\phi) \;&=\; \frac{\int_\alpha^\beta \phi\left(x(t)\right) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \phi(\overline{x}) \\ &- \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\left[\frac{\int_\alpha^\beta \hat{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \hat{\Lambda}_k(\overline{x})\right] \\ &- \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\left[\frac{\int_\alpha^\beta \tilde{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \tilde{\Lambda}_k(\overline{x})\right],\end{aligned} \tag{2.114}$$

Similarly as in [28] we can construct new families of exponentially convex function and Cauchy type means by looking at these linear functionals. The monotonicity property of the generalized Cauchy means obtained via these functionals can be prove by using the properties of the linear functionals associated with this error representation, such as n-exponential and logarithmic convexity.

Using majorization theorem for $(2n)$-convex function we give generalization of the Jensen-Steffensen inequality.

Theorem 2.38 *Let $n \in \mathbb{N}$, $\mathbf{x} = (x_1,\ldots,x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1,\ldots,m$, let $\mathbf{w} = (w_1,\ldots,w_m)$ be real m-tuple such that $0 \le W_k \le W_m$, $k = 1,\ldots,m$, $W_m > 0$, where $W_k = \sum_{i=1}^k w_i$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$.*

(i) If n is odd, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, the inequality (2.103) holds.
Moreover, for the convex function F defined in (2.104) the inequality (2.105) is also valid.

(ii) If n is even, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, the reverse inequality in (2.103) holds.
Moreover, for the concave function F defined in (2.104) the reverse inequality in (2.105) is also valid.

Proof. For $l = 1,\ldots,k$, such that $x_k \ge \overline{x}$ we have

$$\sum_{i=1}^{l} w_i x_i - W_l x_l = \sum_{i=1}^{l-1}(x_i - x_{i+1})W_i \ge 0 \tag{2.115}$$

and so we get

$$\sum_{i=1}^{l} w_i\bar{x} = W_l\bar{x} \le W_l x_l \le \sum_{i=1}^{l} w_i x_i .$$

For $l = k+1,\ldots,m-1$, such that $x_{k+1} < \bar{x}$ we have

$$x_l\left(W_m - W_l\right) - \sum_{i=l+1}^{m} w_i x_i = \sum_{i=l+1}^{m} (x_{i-1} - x_i)(W_m - W_{i-1}) \ge 0 \tag{2.116}$$

and now

$$\sum_{i=l+1}^{m} w_i\bar{x} = (W_m - W_l)\bar{x} > (W_m - W_l)\,x_l \ge \sum_{i=l+1}^{m} w_i x_i . \tag{2.117}$$

So, similarly as in Theorem 2.37, we get that conditions (1.19) and (1.20) for majorization are satisfied, so inequalities (2.103) and (2.105) are valid. □

Remark 2.9 *For $x : [\alpha,\beta] \to \mathbb{R}$ continuous, decreasing function, such that $x([\alpha,\beta]) \subseteq [a,b]$ and $\lambda : [\alpha,\beta] \to \mathbb{R}$ is either continuous or of bounded variation satisfying $\lambda(\alpha) \le \lambda(t) \le \lambda(\beta)$ for all $x \in [\alpha,\beta]$ and $\bar{x} = \frac{\int_\alpha^\beta x(t)\,d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$, for $x(\gamma) \ge \bar{x}$, we have:*

$$\int_\alpha^\gamma x(t)d\lambda(t) - x(\gamma)\int_\alpha^\gamma d\lambda(t) = -\int_\alpha^\gamma x'(t)\left(\int_\alpha^t d\lambda(x)\right)dt \ge 0$$

and so

$$\bar{x}\int_\alpha^\gamma d\lambda(t) \le x(\gamma)\int_\alpha^\gamma d\lambda(t) \le \int_\alpha^\gamma x(t)d\lambda(t).$$

If $x(\gamma) < \bar{x}$ we have

$$x(\gamma)\int_\gamma^\beta d\lambda(t) - \int_\gamma^\beta x(t)d\lambda(t) = -\int_\gamma^\beta x'(t)\left(\int_t^\beta d\lambda(x)\right)dt \ge 0$$

and now

$$\bar{x}\int_\gamma^\beta d\lambda(t) > x(\gamma)\int_\gamma^\beta d\lambda(t) \ge \int_\gamma^\beta x(t)d\lambda(t).$$

Similarly as in the Remark 2.7 we get that conditions for majorization are satisfied, so inequalities (2.111) and (2.112) are valid.

Theorem 2.39 *Let $n \in \mathbb{N}$, $\mathbf{x} = (x_1,\ldots,x_r)$ be real r-tuple with $x_i \in [m,M] \subseteq [a,b]$, $i = 1,\ldots,r$, let $\mathbf{w} = (w_1,\ldots,w_r)$ be positive r-tuple, $W_r = \sum_{i=1}^r w_i$ and $\bar{x} = \frac{1}{W_r}\sum_{i=1}^r w_i x_i$.*

(i) If n is odd, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\tfrac{1}{W_r}\sum_{i=1}^{r} w_i\phi(x_i) \le \tfrac{\bar{x}-m}{M-m}\phi(M) + \tfrac{M-\bar{x}}{M-m}\phi(m) \\
&- \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\cdot\left[\tfrac{\bar{x}-m}{M-m}\hat{\Lambda}_k(M) + \tfrac{M-\bar{x}}{M-m}\hat{\Lambda}_k(m) - \tfrac{1}{W_r}\sum_{i=1}^{r} w_i\hat{\Lambda}_k(x_i)\right] \\
&- \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\cdot\left[\tfrac{\bar{x}-m}{M-m}\tilde{\Lambda}_k(M) + \tfrac{M-\bar{x}}{M-m}\tilde{\Lambda}_k(m) - \tfrac{1}{W_r}\sum_{i=1}^{r} w_i\tilde{\Lambda}_k(x_i)\right].
\end{aligned} \tag{2.118}$$

Moreover, for the convex function F defined in (2.104), we have

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i\phi(x_i) \leq \frac{\overline{x}-m}{M-m}\phi(M) + \frac{M-\overline{x}}{M-m}\phi(m). \tag{2.119}$$

(ii) If n is even, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, the reverse inequality in (2.118) holds.
Moreover, for the concave function F defined in (2.104) the reverse inequality in (2.119) is also valid.

Proof. Using inequality (2.103) we have

$$\begin{aligned}
\frac{1}{W_r}\sum_{i=1}^{r} w_i\phi(x_i) &= \frac{1}{W_r}\sum_{i=1}^{r} w_i\phi\left(\frac{x_i-m}{M-m}M + \frac{M-x_i}{M-m}m\right) \\
&\leq \frac{\overline{x}-m}{M-m}\phi(M) + \frac{M-\overline{x}}{M-m}\phi(m) \\
&- \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\left[\frac{\overline{x}-m}{M-m}\hat{\Lambda}_k(M) + \frac{M-\overline{x}}{M-m}\hat{\Lambda}_k(m) - \frac{1}{W_r}\sum_{i=1}^{r} w_i\hat{\Lambda}_k(x_i)\right] \\
&- \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\left[\frac{\overline{x}-m}{M-m}\tilde{\Lambda}_k(M) + \frac{M-\overline{x}}{M-m}\tilde{\Lambda}_k(m) - \frac{1}{W_r}\sum_{i=1}^{r} w_i\tilde{\Lambda}_k(x_i)\right].
\end{aligned}$$

Hence, for any odd n and $(2n)$-convex function ϕ we get (2.118).

For inequality (2.119) we use the fact that for convex function F we have

$$\frac{1}{W_r}\sum_{i=1}^{r} w_iF(x_i) \leq \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m).$$

(ii) Similar to the part (i) □

Corollary 2.5 *Let $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1,\ldots,x_r)$ be real r-tuple with $x_i \in [m,M]$, let $\boldsymbol{w} = (w_1,\ldots,w_r)$ be positive r-tuple, $W_r = \sum_{i=1}^{r} w_i$ and $\overline{x} = \frac{1}{W_r}\sum_{i=1}^{r} w_ix_i$.*
If n is odd then for every $(2n)$-convex function $\phi : [m,M] \to \mathbb{R}$ it holds

$$\sum_{i=1}^{r} w_i\phi(x_i) \leq \sum_{k=0}^{n-1}(M-m)^{2k}\sum_{i=1}^{r} w_i\left[\phi^{(2k)}(m)\hat{\Lambda}_k(x_i) + \phi^{(2k)}(M)\tilde{\Lambda}_k(x_i)\right]. \tag{2.120}$$

If n is even, then the reverse inequality in (2.120) is valid.

Proof. We use inequality (2.118) for $m = a$ and $M = b$ and (2.2). □

Remark 2.10 *For $x : [\alpha,\beta] \to \mathbb{R}$ continuous function, such that $x([\alpha,\beta]) \subseteq [m,M] \subseteq [a,b]$ and $\lambda : [\alpha,\beta] \to \mathbb{R}$ increasing, bounded function with $\lambda(\alpha) \neq \lambda(\beta)$ and $\overline{x} = \frac{\int_\alpha^\beta x(t)\,d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$, similarly as in Theorem 2.39 we get integral version of converse of Jensen's inequality.*

For odd $n \in \mathbb{N}$ and for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$ we have:

$$\frac{\int_\alpha^\beta \phi(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \leq \frac{\bar{x}-m}{M-m}\phi(M) + \frac{M-\bar{x}}{M-m}\phi(m) - \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}$$
$$\cdot\left[\frac{\bar{x}-m}{M-m}\hat{\Lambda}_k(M) + \frac{M-\bar{x}}{M-m}\hat{\Lambda}_k(m) - \frac{\int_\alpha^\beta \hat{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right]$$
$$-\sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\left[\frac{\bar{x}-m}{M-m}\tilde{\Lambda}_k(M) + \frac{M-\bar{x}}{M-m}\tilde{\Lambda}_k(m) - \frac{\int_\alpha^\beta \tilde{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right],$$

which is result proved in [28].
Moreover, for the convex function F defined in (2.104) we have

$$\frac{\int_\alpha^\beta \phi(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \leq \frac{\bar{x}-m}{M-m}\phi(M) + \frac{M-\bar{x}}{M-m}\phi(m). \tag{2.121}$$

If n is even, then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, the reverse inequality in (2.121) holds.
Moreover, for the concave function F defined in (2.104) the reverse inequality in (2.121) is also valid.

Remark 2.11 *Motivated by the inequalities (2.118) and (2.121), we define functionals $\Theta_3(\phi)$ and $\Theta_4(\phi)$ by*

$$\begin{aligned}\Theta_3(\phi) &= \frac{1}{W_r}\sum_{i=1}^{r} w_i\phi(x_i) - \frac{\bar{x}-m}{M-m}\phi(M) - \frac{M-\bar{x}}{M-m}\phi(m)\\ &+ \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\cdot\left[\frac{\bar{x}-m}{M-m}\hat{\Lambda}_k(M) + \frac{M-\bar{x}}{M-m}\hat{\Lambda}_k(m) - \frac{1}{W_r}\sum_{i=1}^{r} w_i\hat{\Lambda}_k(x_i)\right]\\ &+ \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\cdot\left[\frac{\bar{x}-m}{M-m}\tilde{\Lambda}_k(M) + \frac{M-\bar{x}}{M-m}\tilde{\Lambda}_k(m) - \frac{1}{W_r}\sum_{i=1}^{r} w_i\tilde{\Lambda}_k(x_i)\right],\end{aligned} \tag{2.122}$$

and

$$\begin{aligned}\Theta_4(\phi) &= \frac{\int_\alpha^\beta \phi(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \frac{\bar{x}-m}{M-m}\phi(M) - \frac{M-\bar{x}}{M-m}\phi(m)\\ &+ \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\cdot\left[\frac{\bar{x}-m}{M-m}\hat{\Lambda}_k(M) + \frac{M-\bar{x}}{M-m}\hat{\Lambda}_k(m) - \frac{\int_\alpha^\beta \hat{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right]\\ &+ \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\cdot\left[\frac{\bar{x}-m}{M-m}\tilde{\Lambda}_k(M) + \frac{M-\bar{x}}{M-m}\tilde{\Lambda}_k(m) - \frac{\int_\alpha^\beta \tilde{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right].\end{aligned}$$

Now, we can observe the same results which are mentioned in Remark 2.23.

In the sequel we use the above theorems to obtain generalizations of the previous results.

For m-tuples $\boldsymbol{w} = (w_1, \dots, w_m)$, $\boldsymbol{x} = (x_1, \dots, x_m)$ with $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1, \dots, m$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and function G_n as defined in (2.4), we denote

$$\Upsilon(t) = \frac{1}{W_m}\sum_{i=1}^m w_i G_n\left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a}\right) - G_n\left(\frac{\overline{x}-a}{b-a}, \frac{t-a}{b-a}\right). \tag{2.123}$$

Similarly for $x : [\alpha,\beta] \to [a,b]$ continuous function, $\lambda : [\alpha,\beta] \to \mathbb{R}$ as defined in Remark 2.7 or in Remark 2.9 and for all $s \in [a,b]$ denote

$$\tilde{\Upsilon}(s) = \frac{\int_\alpha^\beta G_n\left(\frac{x(t)-a}{b-a}, \frac{s-a}{b-a}\right) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - G_n\left(\frac{\overline{x}-a}{b-a}, \frac{s-a}{b-a}\right). \tag{2.124}$$

We have the Čebyšev functionals defined as:

$$T(\Upsilon,\Upsilon) = \frac{1}{b-a}\int_a^b \Upsilon^2(t)dt - \left(\frac{1}{b-a}\int_a^b \Upsilon(t)dt\right)^2, \tag{2.125}$$

$$T(\tilde{\Upsilon},\tilde{\Upsilon}) = \frac{1}{b-a}\int_a^b \tilde{\Upsilon}^2(s)ds - \left(\frac{1}{b-a}\int_a^b \tilde{\Upsilon}(s)ds\right)^2. \tag{2.126}$$

Theorem 2.40 *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)\left[\phi^{(2n+1)}\right]^2 \in L[a,b]$ and $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,2,\dots,m$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and let the functions G_n, Υ and T be defined in (2.4), (2.123) and (2.125). Then*

$$\begin{aligned}\frac{1}{W_m}\sum_{i=1}^m w_i\phi(x_i) - \phi(\overline{x}) &= \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\left[\frac{1}{W_m}\sum_{i=1}^m w_i\hat{\Lambda}_k(x_i) - \hat{\Lambda}_k(\overline{x})\right] \\ &+ \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\left[\frac{1}{W_m}\sum_{i=1}^m w_i\tilde{\Lambda}_k(x_i) - \tilde{\Lambda}_k(\overline{x})\right] \\ &+ (b-a)^{2n-1}\left(\phi^{(2n-1)}(b) - \phi^{(2n-1)}(a)\right)\cdot \\ &\left\{\frac{1}{W_m}\sum_{i=1}^m w_i\left[\tilde{\Lambda}_n(x_i) + \hat{\Lambda}_n(x_i)\right] - \left[\tilde{\Lambda}_n(\overline{x}) + \hat{\Lambda}_n(\overline{x})\right]\right\} \\ &+ H_n^1(\phi;a,b),\end{aligned} \tag{2.127}$$

where the remainder $H_n^1(\phi;a,b)$ satisfies the estimation

$$\begin{aligned}\mid H_n^1(\phi;a,b)\mid &\leq \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}}\left[T(\Upsilon,\Upsilon)\right]^{\frac{1}{2}} \\ &\cdot\left|\int_a^b (t-a)(b-t)\left[\phi^{(2n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.\end{aligned} \tag{2.128}$$

Proof. If we apply Theorem 1.10 for $f \to \Upsilon$ and $h \to \phi^{(2n)}$ we obtain

$$\left|\tfrac{1}{b-a}\int_a^b \Upsilon(t)\phi^{(2n)}(t)dt - \tfrac{1}{b-a}\int_a^b \Upsilon(t)dt \cdot \tfrac{1}{b-a}\int_a^b \phi^{(2n)}(t)dt\right|$$
$$\leq \tfrac{1}{\sqrt{2}}\left[T(\Upsilon,\Upsilon)\right]^{\frac{1}{2}} \tfrac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(2n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.$$

Therefore we have

$$\begin{aligned}(b-a)^{2n-1}&\int_a^b \Upsilon(t)\phi^{(2n)}(t)dt\\ &= (b-a)^{2n-2}\left(\phi^{(2n-1)}(b)-\phi^{(2n-1)}(a)\right)\int_a^b \Upsilon(t)dt + H_n^1(\phi;a,b),\end{aligned}$$

where the remainder $H_n^1(\phi;a,b)$ satisfies the estimation (2.433). Now from identity (2.100) and fact that $\Lambda_n(1-t)=\int_0^1 G_n(t,s)(1-s)ds$ (see [16]) we obtain (2.127). □

Integral case of the above theorem can be given as follows.

Theorem 2.41 *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)\left[\phi^{(2n+1)}\right]^2 \in L[a,b]$, let $x : [\alpha,\beta] \to \mathbb{R}$ be continuous functions such that $x([\alpha,\beta]) \subseteq [a,b]$ and $\lambda : [\alpha,\beta] \to \mathbb{R}$ be as defined in Remark 2.7 or in Remark 2.9 and $\overline{x} = \frac{\int_\alpha^\beta x(t)\,d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$. Let the functions G_n, $\tilde{\Upsilon}$ and T be defined in (2.4), (2.124) and (2.126). Then*

$$\begin{aligned}\frac{\int_\alpha^\beta \phi(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \phi(\overline{x}) &= \sum_{k=1}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\left[\frac{\int_\alpha^\beta \hat{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \hat{\Lambda}_k(\overline{x})\right]\\ &+ \sum_{k=1}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\left[\frac{\int_\alpha^\beta \tilde{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \tilde{\Lambda}_k(\overline{x})\right]\\ &+ (b-a)^{2n-1}\left(\phi^{(2n-1)}(b)-\phi^{(2n-1)}(a)\right)\cdot\\ &\left\{\frac{\int_\alpha^\beta\left[\tilde{\Lambda}_n(x(t))+\hat{\Lambda}_n(x(t))\right]d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \left[\tilde{\Lambda}_n(\overline{x})+\hat{\Lambda}_n(\overline{x})\right]\right\}\\ &+ \tilde{H}_n^1(\phi;a,b), \qquad (2.129)\end{aligned}$$

where the remainder $\tilde{H}_n^1(\phi;a,b)$ satisfies the estimation

$$|\tilde{H}_n^1(\phi;a,b)| \leq \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}}\left[T(\tilde{\Upsilon},\tilde{\Upsilon})\right]^{\frac{1}{2}}\left|\int_a^b (s-a)(b-s)\left[\phi^{(2n+1)}(s)\right]^2 ds\right|^{\frac{1}{2}}. \qquad (2.130)$$

Using Theorem 1.11 we also get the following Grüss type inequality.

Theorem 2.42 *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ and $\phi^{(2n+1)} \geq 0$ on $[a,b]$ and let the function Υ be defined in (2.123). Then we have the representation (2.127) and the remainder $H_n^1(\phi;a,b)$ satisfies the bound*

$$| H_n^1(\phi;a,b) | \leq (b-a)^{2n-1} \|\Upsilon'\|_\infty \left\{ \frac{\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)}{2} - \frac{\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)}{b-a} \right\}. \tag{2.131}$$

Proof. Applying Theorem 1.11 for $f \to \Upsilon$ and $h \to \phi^{(2n)}$ we obtain

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b \Upsilon(t)\phi^{(2n)}(t)dt - \frac{1}{b-a} \int_a^b \Upsilon(t)dt \cdot \frac{1}{b-a} \int_a^b \phi^{(2n)}(t)dt \right| \\ &\leq \frac{1}{2(b-a)} \|\Upsilon'\|_\infty \int_a^b (t-a)(b-t)\phi^{(2n+1)}(t)dt. \end{aligned} \tag{2.132}$$

Since

$$\begin{aligned} &\int_a^b (t-a)(b-t)\phi^{(2n+1)}(t)dt = \int_a^b [2t-(a+b)]\,\phi^{(2n)}(t)dt \\ &= (b-a)\left[\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)\right] - 2\left(\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)\right), \end{aligned}$$

using the identity (2.100) and (2.132) we deduce (2.131). □

Integral version of the above theorem can be given as follows.

Theorem 2.43 *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ and $\phi^{(2n+1)} \geq 0$ on $[a,b]$ and let the function $\tilde{\Upsilon}$ be defined in (2.124). Then we have the representation (2.129) and the remainder $\tilde{H}_n^1(\phi;a,b)$ satisfies the bound*

$$| \tilde{H}_n^1(\phi;a,b) | \leq (b-a)^{2n-1} \|\tilde{\Upsilon}'\|_\infty \left\{ \frac{\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)}{2} - \frac{\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)}{b-a} \right\}.$$

We also give the Ostrowsky type inequality related to the generalization of majorization inequality.

Theorem 2.44 *Let $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,2,\ldots,m$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and let (p,q) be a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $\phi \in C^{2n}[a,b]$ be such that $\left|\phi^{(2n)}\right|^p : [a,b] \to \mathbb{R}$ is an R-integrable function for some $\mathbb{N}$. Then we have*

$$\begin{aligned} &\left| \frac{1}{W_m} \sum_{i=1}^m w_i\phi(x_i) - \phi(\overline{x}) - \sum_{k=1}^{n-1} \phi^{(2k)}(a)(b-a)^{2k} \left[\frac{1}{W_m} \sum_{i=1}^m w_i\hat{\Lambda}_k(x_i) - \hat{\Lambda}_k(\overline{x}) \right] \right. \\ &\qquad \left. - \sum_{k=1}^{n-1} \phi^{(2k)}(b)(b-a)^{2k} \left[\frac{1}{W_m} \sum_{i=1}^m w_i\tilde{\Lambda}_k(x_i) - \tilde{\Lambda}_k(\overline{x}) \right] \right| \\ &\qquad \leq (b-a)^{2n-1} \parallel \phi^{(2n)} \parallel_p \cdot \left(\int_a^b \left| \frac{1}{W_m} \sum_{i=1}^m w_i G_n\left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a}\right) - G_n\left(\frac{\overline{x}-a}{b-a}, \frac{t-a}{b-a}\right) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned} \tag{2.133}$$

The constant on the right hand side of (2.133) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Let's denote

$$\Psi(t) = (b-a)^{2n-1}\left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G_n\left(\tfrac{x_i-a}{b-a},\tfrac{t-a}{b-a}\right) - G_n\left(\tfrac{\bar{x}-a}{b-a},\tfrac{t-a}{b-a}\right)\right].$$

Using the identity (2.100) and applying Hölder's inequality we obtain

$$\begin{aligned}&\left|\frac{1}{W_m}\sum_{i=1}^{m} w_i\,\phi(x_i) - \phi(\bar{x}) - \sum_{k=0}^{n-1}\phi^{(2k)}(a)(b-a)^{2k}\left[\frac{1}{W_m}\sum_{i=1}^{m} w_i\hat{\Lambda}_k(x_i) - \hat{\Lambda}_k(\bar{x})\right]\right.\\ &\qquad\left. - \sum_{k=0}^{n-1}\phi^{(2k)}(b)(b-a)^{2k}\left[\frac{1}{W_m}\sum_{i=1}^{m} w_i\tilde{\Lambda}_k(x_i) - \tilde{\Lambda}_k(\bar{x})\right]\right|\\ &\qquad = \left|\int_a^b \Psi(t)\phi^{(2n)}(t)dt\right| \le ||\phi^{(2n)}||_p\left(\int_a^b |\psi(t)|^q ds\right)^{1/q}.\end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_a^b |\Psi(t)|^q\,dt\right)^{1/q}$ let us find a function ϕ for which the equality in (2.133) is obtained.
For $1 < p < \infty$ take ϕ to be such that

$$\phi^{(2n)}(t) = sgn\,\Psi(t)\,|\Psi(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\phi^{(2n)}(t) = sgn\,\Psi(t)$.
For $p = 1$ we prove that

$$\left|\int_a^b \Psi(t)\phi^{(2n)}(t)dt\right| \le \max_{t\in[a,b]}|\Psi(t)|\left(\int_a^b \left|\phi^{(2n)}(t)\right|dt\right) \tag{2.134}$$

is the best possible inequality. Suppose that $|\Psi(t)|$ attains its maximum at $t_0 \in [a,b]$. First we assume that $\Psi(t_0) > 0$. For ε small enough we define $\phi_\varepsilon(t)$ by

$$\phi_\varepsilon(t) = \begin{cases} 0, & a \le t \le t_0,\\ \frac{1}{\varepsilon\, n!}(t-t_0)^n, & t_0 \le t \le t_0+\varepsilon,\\ \frac{1}{(n-1)!}(t-t_0)^{n-1}, & t_0+\varepsilon \le t \le b.\end{cases}$$

Then for ε small enough

$$\left|\int_a^b \Psi(t)\phi^{(2n)}(t)dt\right| = \left|\int_{t_0}^{t_0+\varepsilon}\Psi(t)\frac{1}{\varepsilon}dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Psi(t)dt.$$

Now from the inequality (2.134) we have

$$\tfrac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Psi(t)dt \le \Psi(t_0)\int_{t_0}^{t_0+\varepsilon}\tfrac{1}{\varepsilon}dt = \Psi(t_0).$$

Since

$$\lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\Psi(t)dt = \Psi(t_0)$$

the statement follows. In the case $\Psi(t_0) < 0$, we define $\phi_\varepsilon(t)$ by

$$\phi_\varepsilon(t) = \begin{cases} \frac{1}{(n-1)!}(t-t_0-\varepsilon)^{n-1}, & a \le t \le t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \le t \le t_0+\varepsilon, \\ 0, & t_0+\varepsilon \le t \le b, \end{cases}$$

and the rest of the proof is the same as above. □

Integral version of the above theorem can be stated as follows.

Theorem 2.45 *Let $x : [\alpha,\beta] \to \mathbb{R}$ be continuous functions such that $x([\alpha,\beta]) \subseteq [a,b], \lambda : [\alpha,\beta] \to \mathbb{R}$ be as defined in Remark 2.7 or in Remark 2.9, $\overline{x} = \frac{\int_\alpha^\beta x(t)\,d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$ and let (p,q) be a pair of conjugate exponents, that is $1 \le p,q \le \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\phi \in C^{2n}[a,b]$ be such that $\left|\phi^{(2n)}\right|^p : [a,b] \to \mathbb{R}$ is an R-integrable function for some $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\left| \frac{\int_\alpha^\beta \phi(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \phi(\overline{x}) - \sum_{k=1}^{n-1} \phi^{(2k)}(a)(b-a)^{2k} \left[\frac{\int_\alpha^\beta \hat{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \hat{\Lambda}_k(\overline{x}) \right] \right. \qquad (2.135) \\ &\qquad \left. - \sum_{k=1}^{n-1} \phi^{(2k)}(b)(b-a)^{2k} \left[\frac{\int_\alpha^\beta \tilde{\Lambda}_k(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \tilde{\Lambda}_k(\overline{x}) \right] \right| \\ &\le (b-a)^{2n-1} ||\phi^{(2n)}||_p \left(\int_a^b \left| \frac{\int_\alpha^\beta G_n\left(\frac{x(t)-a}{b-a}, \frac{s-a}{b-a}\right) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - G_n\left(\frac{\overline{x}-a}{b-a}, \frac{s-a}{b-a}\right) \right|^q ds \right)^{\frac{1}{q}}. \end{aligned}$$

The constant on the right hand side of (2.135) is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

2.2 Majorization and Hermite Interpolation Polynomial

In Section 2.3 we discuss about results obtained by approximation with Taylor's polynomials. Taylor's polynomials are useful over small intervals for functions whose derivatives exist and are easily evaluated. Unlike that, Lagrange polynomials can be determined simply by specifying certain points on the plane through which they must pass. They agree with a function ϕ at specified points. The values of ϕ are often determined from observation, and in some situations it is possible to determine the derivative of ϕ as well. For example, if the independent variable is time and the function describes the position of an object, the derivative of the function in this case is the velocity, which might be available. In this section we consider Hermite's interpolation which determines a polynomial that agrees with the function and its first derivative at specified points. It includes Lagrange's interpolating polynomial as particular case. When we consider Hermite's interpolation, we may discusse about different an error function $e(t) = \phi(t) - \rho_H(t)$, where $\phi \in C^n[\alpha,\beta]$ and $\rho_H(t)$ is Hermite's interpolating polynomial of the function ϕ. On the basis of various applications, several representations with different kind of error function $e(t)$ can be obtained like Peano's representation, Cauchy's representation, Newton's representation etc. Here we discusse about Hermite's interpolation with Peano's representation and its particular cases namely, Lagrange interpolating polynomial, $(m,n-m)$ interpolating polynomial, two-point Taylor interpolating polynomial. Using interpolation by Hermite's polynomials we give new generalizations of majorization inequalities. We also give bounds for the identities related to the generalizations of majorization inequalities by using Čebyšev functionals and derive the Grüss and Ostrowski type inequalities for these functionals. We present mean value theorems which lead to exponential convexity and log-convexity for these functionals, i.e. enable us to construct families of exponentially convex functions and as consequences Stolarsky type of means.

2.2.1 Results Obtained by Hermite Interpolation Polynomial

Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points. For $\phi \in C^n[\alpha,\beta]$ a unique polynomial $\rho_H(s)$ of degree $(n-1)$ exists satisfying any of the following conditions:

Hermite conditions:

$$\rho_H^{(i)}(a_j) = \phi^i(a_j);\ 0 \leq i \leq k_j,\ 1 \leq j \leq r,\ \sum_{j=1}^{r} k_j + r = n. \tag{H}$$

It is of great interest to note that Hermite conditions include the following particular cases:

Lagrange conditions:$(r = n,\ k_j = 0$ for all $j)$

$$\rho_L(a_j) = \phi(a_j), 1 \leq j \leq n,$$

Type $(m,n-m)$ **conditions:** $(r = 2, 1 \leq m \leq n-1,\ k_1 = m-1,\ k_2 = n-m-1)$

$$\rho_{(m,n)}^{(i)}(\alpha) = \phi^{(i)}(\alpha), 0 \leq i \leq m-1,$$
$$\rho_{(m,n)}^{(i)}(\beta) = \phi^{(i)}(\beta), 0 \leq i \leq n-m-1,$$

Two-point Taylor conditions: $(n = 2m,\ r = 2,\ k_1 = k_2 = m-1)$

$$\rho_{2T}^{(i)}(\alpha) = \phi^{(i)}(\alpha), \rho_{2T}^{(i)}(\beta) = \phi^{(i)}(\beta), 0 \leq i \leq m-1.$$

We have the following result from [16].

Theorem 2.46 *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$. Then we have*

$$\phi(t) = \rho_H(t) + R_{H,n}(\phi,t) \tag{2.136}$$

*where $\rho_H(t)$ is **Hermite's interpolating polynomial**, i.e.*

$$\rho_H(t) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i)}(a_j);$$

the H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!} \frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dt^k} \left(\frac{(t-a_j)^{k_j+1}}{\omega(t)} \right) \bigg|_{t=a_j} (t-a_j)^k, \tag{2.137}$$

with

$$\omega(t) = \prod_{j=1}^{r} (t-a_j)^{k_j+1}, \tag{2.138}$$

and the remainder is given by

$$R_{H,n}(\phi,t) = \int_{\alpha}^{\beta} G_{H,n}(t,s)\phi^{(n)}(s)ds$$

where $G_{H,n}(t,s)$ (Peano's kernel) is defined by

$$G_{H,n}(t,s) = \begin{cases} \sum\limits_{j=1}^{l} \sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t); \; s \le t, \\ -\sum\limits_{j=l+1}^{r} \sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t); \; s \ge t, \end{cases} \tag{2.139}$$

for all $a_l \le s \le a_{l+1}$; $l = 0, \ldots, r$ with $a_0 = \alpha$ and $a_{r+1} = \beta$.

Remark 2.12 *In particular cases,*
a) for Lagrange conditions, from Theorem 2.46 we have

$$\phi(t) = \rho_L(t) + R_L(\phi,t)$$

where $\rho_L(t)$ is the Lagrange interpolating polynomial i.e,

$$\rho_L(t) = \sum_{j=1}^{n} \prod_{\substack{k=1 \\ k \ne j}}^{n} \left(\frac{t-a_k}{a_j-a_k} \right) \phi(a_j)$$

and the remainder $R_L(\phi,t)$ is given by

$$R_L(\phi,t) = \int_{\alpha}^{\beta} G_L(t,s)\phi^{(n)}(s)ds$$

with

$$G_L(t,s) = \frac{1}{(n-1)!} \begin{cases} \sum\limits_{j=1}^{l} (a_j-s)^{n-1} \prod\limits_{\substack{k=1 \\ k \ne j}}^{n} \left(\frac{t-a_k}{a_j-a_k} \right), & s \le t \\ -\sum\limits_{j=l+1}^{n} (a_j-s)^{n-1} \prod\limits_{\substack{k=1 \\ k \ne j}}^{n} \left(\frac{t-a_k}{a_j-a_k} \right), & s \ge t \end{cases} \tag{2.140}$$

$a_l \le s \le a_{l+1}$, $l = 1, 2, \ldots, n-1$ with $a_1 = \alpha$ and $a_n = \beta$;

b) for type $(m,n-m)$ conditions, from Theorem 2.46 we have

$$\phi(t)=\rho_{(m,n)}(t)+R_{(m,n)}(\phi,t)$$

where $\rho_{(m,n)}(t)$ is $(m,n-m)$ interpolating polynomial, i.e

$$\rho_{(m,n)}(t)=\sum_{i=0}^{m-1}\tau_i(t)\phi^i(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^i(\beta),$$

with

$$\tau_i(t)=\frac{1}{i!}(t-\alpha)^i\left(\frac{t-\beta}{\alpha-\beta}\right)^{n-m}\sum_{k=0}^{m-1-i}\binom{n-m+k-1}{k}\left(\frac{t-\alpha}{\beta-\alpha}\right)^k \tag{2.141}$$

and

$$\eta_i(t)=\frac{1}{i!}(t-\beta)^i\left(\frac{t-\alpha}{\beta-\alpha}\right)^m\sum_{k=0}^{n-m-1-i}\binom{m+k-1}{k}\left(\frac{t-\beta}{\alpha-\beta}\right)^k. \tag{2.142}$$

and the remainder $R_{(m,n)}(\phi,t)$ is given by

$$R_{(m,n)}(\phi,t)=\int_\alpha^\beta G_{(m,n)}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{(m,n)}(t,s)=\begin{cases}\sum_{j=0}^{m-1}\left[\sum_{p=0}^{m-1-j}\binom{n-m+p-1}{p}\left(\frac{t-\alpha}{\beta-\alpha}\right)^p\right]\cdot\\ \frac{(t-\alpha)^j(\alpha-s)^{n-j-1}}{j!(n-j-1)!}\left(\frac{\beta-t}{\beta-\alpha}\right)^{n-m}, & \alpha\le s\le t\le\beta\\ -\sum_{i=0}^{n-m-1}\left[\sum_{q=0}^{n-m-i-1}\binom{m+q-1}{q}\left(\frac{\beta-t}{\beta-\alpha}\right)^q\right]\cdot\\ \frac{(t-\beta)^i(\beta-s)^{n-i-1}}{i!(n-i-1)!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m, & \alpha\le t\le s\le\beta;\end{cases} \tag{2.143}$$

c) for Type Two-point Taylor conditions, from Theorem 2.46 we have

$$\phi(t)=\rho_{2T}(t)+R_{2T}(\phi,t)$$

where $\rho_{2T}(t)$is the two-point Taylor interpolating polynomial i.e,

$$\rho_{2T}(t)=\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\left[\frac{(t-\alpha)^i}{i!}\left(\frac{t-\beta}{\alpha-\beta}\right)^m\left(\frac{t-\alpha}{\beta-\alpha}\right)^k\phi^i(\alpha)\right.$$
$$\left.+\frac{(t-\beta)^i}{i!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m\left(\frac{t-\beta}{\alpha-\beta}\right)^k\phi^i(\beta)\right] \tag{2.144}$$

and the remainder $R_{2T}(\phi,t)$ *is given by*

$$R_{2T}(\phi,t) = \int_{\alpha}^{\beta} G_{2T}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{2T}(t,s) = \begin{cases} \frac{(-1)^m}{(2m-1)!} p^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (t-s)^{m-1-j} q^j(t,s), & s \leq t; \\ \frac{(-1)^m}{(2m-1)!} q^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (s-t)^{m-1-j} p^j(t,s), & s \geq t; \end{cases} \tag{2.145}$$

where $p(t,s) = \frac{(s-\alpha)(\beta-t)}{\beta-\alpha}$, $q(t,s) = p(t,s), \forall t,s \in [\alpha,\beta]$.

Now we give identities related to generalizations of majorization inequality obtained by interpolation by Hermite's polynomials.

Theorem 2.47 ([8]) *Let* $-\infty < \alpha < \beta < \infty$ *and* $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ *be the given points, and* $\phi \in C^n[\alpha,\beta]$ *and* $\mathbf{w} = (w_1,\ldots,w_m)$, $\mathbf{x} = (x_1,\ldots,x_m)$ *and* $\mathbf{y} = (y_1,\ldots,y_m)$ *be m-tuples such that* x_l, $y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l = 1,\ldots,m)$*. Also let* H_{ij} *be the fundamental polynomials of the Hermite basis and* $G_{H,n}$ *be the Green function as defined by (2.137) and (2.139) respectively. Then*

$$\begin{aligned} \sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j) \left[\sum_{l=1}^{m} w_l\,(H_{ij}(x_l) - H_{ij}(y_l))\right] \\ &+ \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G_{H,n}(x_l,s) - G_{H,n}(y_l,s))\right] \phi^{(n)}(s)ds. \end{aligned} \tag{2.146}$$

Proof. Using (2.136) in $\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l)$ we obtain (2.146). □

Integral version of the above theorem can be stated as follows.

Theorem 2.48 ([8]) *Let* $-\infty < \alpha < \beta < \infty$ *and* $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ *be the given points,* $\phi \in C^n[\alpha,\beta]$ *and* $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ *be continuous functions. Also let* H_{ij} *be the fundamental polynomials of the Hermite basis and* $G_{H,n}$ *be the Green function as defined by (2.137) and (2.139) respectively. Then*

$$\begin{aligned} &\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\ &\quad = \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j) \left[\int_a^b w(\tau)\,(H_{ij}(x(\tau)) - H_{ij}(y(\tau)))\,d\tau\right] \\ &\quad + \int_{\alpha}^{\beta} \left[\int_a^b w(\tau)\,(G_{H,n}(x(\tau),s) - G_{H,n}(y(\tau),s))\,d\tau\right] \phi^{(n)}(s)ds. \end{aligned} \tag{2.147}$$

In the following theorem we give generalized majorization inequality.

Theorem 2.49 ([8]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, and $\mathbf{w} = (w_1, \ldots, w_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be m-tuples such that x_l, $y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, \ldots, m)$. Also let H_{ij} be the fundamental polynomials of the Hermite basis and $G_{H,n}$ be the Green function as defined by (2.137) and (2.139) respectively. If $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex and*

$$\sum_{l=1}^{m} w_l \left(G_{H,n}(x_l, s) - G_{H,n}(y_l, s)\right) \ge 0, \quad s \in [\alpha, \beta]. \tag{2.148}$$

Then

$$\begin{aligned}&\sum_{l=1}^{m} w_l\, \phi\,(x_l) - \sum_{l=1}^{m} w_l\, \phi\,(y_l)\\ &\quad \ge \sum_{l=1}^{m} w_l \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) H_{ij}(x_l) - \sum_{l=1}^{m} w_l \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) H_{ij}(y_l).\end{aligned} \tag{2.149}$$

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)} \ge 0$ see [144, p. 16 and p. 293]. Hence, we can apply Theorem 2.47 to obtain (2.149). □

Integral version of the above theorem can be stated as follows.

Theorem 2.50 ([8]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, and $x, y : [a, b] \to [\alpha, \beta]$, $w : [a, b] \to \mathbb{R}$ be continuous functions. Also let H_{ij} be the fundamental polynomials of the Hermite basis and $G_{H,n}$ be the Green function as defined by (2.137) and (2.139) respectively. If $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex and*

$$\int_a^b w(\tau) \left(G_{H,n}(x(\tau), s) - G_{H,n}(y(\tau), s)\right) d\tau \ge 0, \quad s \in [\alpha, \beta]. \tag{2.150}$$

Then

$$\begin{aligned}&\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau\\ &\quad \ge \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) \left[\int_a^b w(\tau) \left(H_{ij}(x(\tau)) - H_{ij}(y(\tau))\right) d\tau\right].\end{aligned} \tag{2.151}$$

In the following theorem we discuss the case for majorized tuples.

Theorem 2.51 *Let all the assumptions of Theorem 2.49 be satisfied. Additionally, let $\mathbf{y} \prec \mathbf{x}$. If the inequality (2.149) holds for $w_l = 1$, $l = 1, .., m$ and the function*

$$\bar{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_i) H_{ij}(\cdot) \tag{2.152}$$

is convex on $[\alpha, \beta]$, then the following inequality holds.

$$\sum_{i=1}^{m} \phi\,(y_i) \le \sum_{i=1}^{m} \phi\,(x_i)\,. \tag{2.153}$$

Proof. If (2.149) holds, the right hand side of (2.149) can be written in the form

$$\sum_{l=1}^{m} \overline{F}(x_l) - \sum_{l=1}^{m} \overline{F}(y_l),$$

where $\overline{F}$ is defined by (2.152). If $\overline{F}$ is convex, then by majorization theorem we have

$$\sum_{l=1}^{m} \overline{F}(x_p) - \sum_{l=1}^{m} \overline{F}(y_l) \geq 0,$$

i.e. the right hand side of (2.149) is nonnegative, so (2.153) immediately follows. □

The weighted version of the above theorem can be presented as follows.

Theorem 2.52 *Let all the assumptions of Theorem 2.49 be satisfied. Additionally, let* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_m)$ *be decreasing m-tuples such that*

$$\sum_{i=1}^{l} w_i y_i \leq \sum_{i=1}^{l} w_i x_i \text{ for } l = 1, \ldots, m-1, \tag{2.154}$$

and

$$\sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i x_i. \tag{2.155}$$

hold. If the inequality (2.149) holds and the function

$$\overline{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_i) H_{ij}(\cdot) \tag{2.156}$$

is convex on $[\alpha, \beta]$*, then the following inequality holds.*

$$\sum_{i=1}^{m} w_i \phi(y_i) \leq \sum_{i=1}^{m} w_i \phi(x_i). \tag{2.157}$$

Proof. The proof is similar to the proof of Theorem 2.51. □

The integral version of the above theorem can be stated as follows.

Theorem 2.53 *Let all the assumptions of Theorem 2.50 be satisfied. Additionally, let x and y be decreasing functions such that (2.67) and (2.68) hold. If the inequality (2.149) holds and the function*

$$\overline{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_i) H_{ij}(\cdot) \tag{2.158}$$

is convex on $[\alpha, \beta]$*, then the following inequality holds.*

$$\int_a^b w(\tau)\phi(y(\tau))d\tau \leq \int_a^b w(\tau)\phi(x(\tau))d\tau. \tag{2.159}$$

By using Lagrange conditions we can give the following result.

Corollary 2.6 *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_n \leq \beta$, $(n \geq 2)$ be the given points, and $\mathbf{w} = (w_1, \ldots, w_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be m-tuples such that x_l, $y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, \ldots, m)$. Let G_L be the Green function as defined in (2.140).*

(i) If $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex and

$$\sum_{l=1}^{m} w_l \left(G_L(x_l, s) - G_L(y_l, s)\right) \geq 0, \quad s \in [\alpha, \beta].$$

Then

$$\begin{aligned} &\sum_{l=1}^{m} w_l \, \phi(x_l) - \sum_{l=1}^{m} w_l \, \phi(y_l) \\ &\qquad \geq \sum_{l=1}^{m} w_l \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{x_l - a_k}{a_j - a_k} \right) - \sum_{l=1}^{m} w_l \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{y_l - a_k}{a_j - a_k} \right). \end{aligned} \tag{2.160}$$

(i) If the inequality (2.160) holds and $\mathbf{x}$, $\mathbf{y}$ are decreasing m-tuples such that (2.154), (2.155) hold and the function

$$\tilde{F}(\cdot) = \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{u=1 \\ u \neq j}}^{n} \left(\frac{\cdot - a_u}{a_j - a_u} \right)$$

is convex on $[\alpha, \beta]$, then the inequality (2.157) holds.

By using type $(m, n-m)$ conditions we can give the following result.

Corollary 2.7 *Let $[\alpha, \beta]$ be an interval and $\mathbf{w} = (w_1, \ldots, w_p)$, $\mathbf{x} = (x_1, \ldots, x_p)$ and $\mathbf{y} = (y_1, \ldots, y_p)$ be p-tuples such that x_l, $y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, \ldots, p)$. Let $G_{(m,n)}$ be the Green function as defined by (2.143) and τ_i, η_i be as defined in (2.141) and (2.142) respectively.*

(i) If $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex and

$$\sum_{l=1}^{p} w_l \left(G_{(m,n)}(x_l, s) - G_{(m,n)}(y_l, s)\right) \geq 0, \quad s \in [\alpha, \beta].$$

Then

$$\begin{aligned} \sum_{l=1}^{p} w_l \, \phi(x_l) - \sum_{l=1}^{p} w_l \, \phi(y_l) \geq &\sum_{i=0}^{m-1} \sum_{l=1}^{p} w_l (\tau_i(x_l) - \tau_i(y_l)) \phi^i(\alpha) \\ &+ \sum_{i=0}^{n-m-1} \sum_{l=1}^{p} w_l (\eta_i(x_l) - \eta_i(y_l)) \phi^i(\beta). \end{aligned} \tag{2.161}$$

(ii) If (2.161) holds and $\mathbf{x}$, $\mathbf{y}$ *are decreasing p-tuples such that (2.154), (2.155) hold and the function*

$$\hat{F}(\cdot) = \sum_{i=0}^{m-1} \tau_i(\cdot)\phi^i(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(\cdot)\phi^i(\beta)$$

is convex on $[\alpha,\beta]$, *then*

$$\sum_{l=1}^{p} w_l\phi(y_l) \leq \sum_{l=1}^{p} w_l\phi(x_p).$$

By using Two-point Taylor conditions we can give the following result.

Corollary 2.8 *Let* $[\alpha,\beta]$ *be an interval and* $\mathbf{w} = (w_1,\ldots,w_p)$, $\mathbf{x} = (x_1,\ldots,x_p)$ *and* $\mathbf{y} = (y_1,\ldots,y_p)$ *be p-tuples such that* $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}\ (l = 1,\ldots,p)$. *Let* ρ_{2T} *and* G_{2T} *be as defined by (2.144) and (2.145), respectively.*

(i) If $\phi : [\alpha,\beta] \to \mathbb{R}$ *is n-convex and*

$$\sum_{l=1}^{p} w_l\left(G_{2T}(x_l,s) - G_{2T}(y_l,s)\right) \geq 0, \quad s \in [\alpha,\beta].$$

Then

$$\sum_{l=1}^{p} w_l\,\phi\,(x_l) - \sum_{l=1}^{p} w_l\,\phi\,(y_l) \geq \sum_{l=1}^{p} w_l\rho_{2T}(x_l) - \sum_{l=1}^{p} w_l\rho_{2T}(y_l). \tag{2.162}$$

(ii) If (2.162) holds and $\mathbf{x}$, $\mathbf{y}$ *are decreasing p-tuples such that (2.154), (2.155) hold and the function* ρ_{2T} *is convex on* $[\alpha,\beta]$, *then*

$$\sum_{l=1}^{p} w_l\,\phi\,(y_l) \leq \sum_{l=1}^{p} w_l\,\phi\,(x_l).$$

Remark 2.13 *Similarly we can give related results to Corollary 2.6-2.8 for majorized tuples. Also we can give related integral version.*

In the sequel (see [8]) we use the above theorems to obtain generalizations of the previous results.

For m-tuples $\mathbf{w} = (w_1,\ldots,w_m)$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ with $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}\ (l = 1,\ldots,m)$ and the Green function $G_{H,n}$ as defined in (2.139), denote

$$\mathfrak{R}_H(s) = \sum_{l=1}^{m} w_l\left(G_{H,n}(x_l,s) - G_{H,n}(y_l,s)\right), \quad s \in [\alpha,\beta], \tag{2.163}$$

similarly for continuous functions $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ and the Green function $G_{H,n}$ as defined in (2.139), denote

$$\mathfrak{B}_H(s) = \int_a^b w(\tau)\left(G_{H,n}(x(\tau),s) - G_{H,n}(y(\tau),s)\right)d\tau, \quad s \in [\alpha,\beta]. \tag{2.164}$$

Consider the Čebyšev functionals $\mathbb{T}(\mathfrak{R}_H,\mathfrak{R}_H)$, $\mathbb{T}(\mathfrak{B}_H,\mathfrak{B}_H)$ are given by:

$$T(\mathfrak{R}_H,\mathfrak{R}_H) = \frac{1}{\beta-\alpha}{}_{\alpha}^{\beta}\mathfrak{R}_H^2(s)ds - \left(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}_H(s)ds\right)^2$$

$$T(\mathfrak{B}_H,\mathfrak{B}_H) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{B}_H^2(s)ds - \left(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{B}_H(s)ds\right)^2$$

Theorem 2.54 ([8]) *Let $-\infty<\alpha<\beta<\infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and $\boldsymbol{w}=(w_1,\ldots,w_m)$, $\boldsymbol{x}=(x_1,\ldots,x_m)$ and $\boldsymbol{y}=(y_1,\ldots,y_m)$ be m-tuples such that x_l, $y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l=1,\ldots,m)$. Also let H_{ij} be the fundamental polynomials of the Hermite basis and the functions $G_{H,n}$ and $\mathfrak{R}_H$ be defined by (2.139) and (2.163) respectively. Then*

$$\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) = \sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\sum_{l=1}^{m} w_l\left(H_{ij}(x_l)-H_{ij}(y_l)\right)\right] + \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{R}_H(s)ds + \kappa_H(\phi;\alpha,\beta). \tag{2.165}$$

where the remainder $\kappa_H(\phi;\alpha,\beta)$ satisfies the estimation

$$|\kappa_H(\phi;\alpha,\beta)| \le \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left[T(\mathfrak{R}_H,\mathfrak{R}_H)\right]^{\frac{1}{2}}\left|\int_{\alpha}^{\beta}(s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right|^{\frac{1}{2}}.$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

The integral version of the above theorem can be stated as follows.

Theorem 2.55 ([8]) *Let $-\infty<\alpha<\beta<\infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and $x,y:[a,b]\to[\alpha,\beta]$, $w:[a,b]\to\mathbb{R}$ be continuous functions. Also let H_{ij} be the fundamental polynomials of the Hermite basis and the functions $G_{H,n}$ and $\mathfrak{B}_H$ be defined by (2.139) and (2.164) respectively. Then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \tag{2.166}$$

$$= \sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\int_a^b w(\tau)\left(H_{ij}(x(\tau))-H_{ij}(y(\tau))\right)d\tau\right] + \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{B}_H(s)ds + \tilde{\kappa}_H(\phi;\alpha,\beta). \tag{2.167}$$

where the remainder $\tilde{\kappa}_H(\phi;\alpha,\beta)$ satisfies the estimation

$$|\tilde{\kappa}_H(\phi;\alpha,\beta)| \le \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left[T(\mathfrak{B}_H,\mathfrak{B}_H)\right]^{\frac{1}{2}}\left|\int_{\alpha}^{\beta}(s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right|^{\frac{1}{2}}.$$

Using Theorem 1.11 we obtain the following Grüss type inequalities.

Theorem 2.56 ([8]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $\phi^{(n)}$ is monotonic non decreasing on $[\alpha,\beta]$ and let $\mathfrak{R}_H$ be defined by (2.163). Then the representation (2.165) holds and the remainder $\kappa_H(\phi;\alpha,\beta)$ satisfies the bound*

$$|\kappa_H(\phi;\alpha,\beta)| \leq \|\mathfrak{R}'_H\|_\infty \left\{ \frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha} \right\}.$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

Integral case of the above theorem can be given as follows.

Theorem 2.57 ([8]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $\phi^{(n)}$ is monotonic non decreasing on $[\alpha,\beta]$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and the functions $G_{H,n}$ and $\mathfrak{B}_H$ be defined by (2.139) and (2.164) respectively. Then we have the representation (2.166) and the remainder $\tilde{\kappa}_{H,n}(\phi;\alpha,\beta)$ satisfies the bound*

$$|\tilde{\kappa}_H(\phi;\alpha,\beta)| \leq \|\mathfrak{B}'_H\|_\infty \left\{ \frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha} \right\}.$$

We present the Ostrowski type inequalities related to generalizations of majorization inequality.

Theorem 2.58 ([8]) *Suppose that all assumptions of Theorem 2.47 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $\left|\phi^{(n)}\right|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function. Then we have:*

$$\begin{aligned}\left| \sum_{l=1}^{m} w_l\, \phi\,(x_l) - \sum_{l=1}^{m} w_l\, \phi\,(y_l) - \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j) \left[\sum_{l=1}^{m} w_l\,(H_{ij}(x_l) - H_{ij}(y_l)) \right] \right| \\ \leq \left\|\phi^{(n)}\right\|_p \|\mathfrak{R}_H\|_q, \end{aligned} \tag{2.168}$$

where $\mathfrak{R}_H$ is defined in (2.163).

The constant on the right-hand side of (2.168) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Integral version of the above theorem can be given as follows.

Theorem 2.59 ([8]) *Suppose that all assumptions of Theorem 2.48 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $\left|\phi^{(n)}\right|^p$: $[\alpha,\beta] \to \mathbb{R}$ be an R-integrable function. Then we have:*

$$\left| \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \right.$$
$$\left. - \sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\int_a^b w(\tau)\left(H_{ij}(x(\tau)) - H_{ij}(y(\tau))\right)d\tau\right] \right| \leq \left\|\phi^{(n)}\right\|_p \|\mathfrak{B}_H\|_q, \quad (2.169)$$

where $\mathfrak{B}_H$ is defined in (2.164).

The constant on the right-hand side of (2.169) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Remark 2.14 ([8]) *We can give all these results of bounds for the Lagrange conditions, Type $(m, n-m)$ conditions, Two-point Taylor conditions.*

Motivated (see [8]) by inequalities (2.149) and (2.151), under the assumptions of Theorems 2.49 and 2.50 we define the following linear functionals:

$$\digamma_1^H(\phi) = \sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l)$$
$$- \sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\sum_{l=1}^{m} w_l\left(H_{ij}(x_l) - H_{ij}(y_l)\right)\right]. \quad (2.170)$$

$$\digamma_2^H(\phi) = \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$- \sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\int_a^b w(\tau)\left(H_{ij}(x(\tau)) - H_{ij}(y(\tau))\right)d\tau\right]. \quad (2.171)$$

Remark 2.15 ([8]) Under the assumptions of Theorems 2.49 and 2.50, it holds $\digamma_i^H(\phi) \geq 0$, $i = 1,2$, for all n-convex functions ϕ.

The Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

Theorem 2.60 ([8]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha,\beta]$. If the inequalities in (2.148) and (2.150) hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\digamma_i^H(\phi) = \phi^{(n)}(\xi_i)\digamma_i^H(\varphi), \quad i = 1,2$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\digamma_i^H$, $i = 1,2$ are defined by (2.170) and(2.171).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (see also the proof of Theorem 4.1 in [86]). □

Theorem 2.61 ([8]) *Let $\phi,\psi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi,\psi\in C^n[\alpha,\beta]$. If the inequalities in (2.148) and (2.150) hold, then there exist $\xi_i\in[\alpha,\beta]$ such that*

$$\frac{F_i^H(\phi)}{F_i^H(\psi)}=\frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)},\quad i=1,2$$

provided that the denominators are non-zero and F_i^H, $i=1,2$, are defined by (2.170) and(2.171).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.14 (see also the proof of Corollary 4.2 in [86]). □

We use an idea from [142] and produce n-exponentially and exponentially convex functions.

Theorem 2.62 ([8]) *Let $\Omega=\{\phi_t : t\in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha,\beta]$ such that the function $t\mapsto[x_0,\ldots,x_n;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n\in[\alpha,\beta]$. Then for the linear functionals $F_i^H(\phi_t)$ $(i=1,2)$ as defined by (2.170) and (2.171), the following statements hold:*

(i) The function $t\to F_i^H(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[F_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m\in\mathbb{N}, m\le n$, $t_1,..,t_m\in J$. Particularly,

$$\det[F_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m\ge 0 \text{ for all } m\in\mathbb{N},\ m=1,2,\ldots,n.$$

(ii) If the function $t\to F_i^H(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39. □

The following corollary is an immediate consequence of the above theorem

Corollary 2.9 ([8]) *Let $\Omega=\{\phi_t : t\in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha,\beta]$ such that the function $t\mapsto[x_0,\ldots,x_n;\phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n\in[\alpha,\beta]$. Then for the linear functionals $F_i^H(\phi_t)$ $(i=1,2)$ as defined by (2.170) and (2.171), the following statements hold:*

(i) The function $t\to F_i^H(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[F_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m\in\mathbb{N}, m\le n$, $t_1,..,t_m\in J$. Particularly,

$$\det[F_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m\ge 0 \text{ for all } m\in\mathbb{N},\ m=1,2,\ldots,n.$$

(ii) If the function $t\to F_i^H(\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 2.10 ([8]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha,\beta]$ such that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in [\alpha,\beta]$. Let F_i^H, $i = 1,2$ be linear functionals defined by (2.170) and (2.171). Then the following statements hold:*

(i) If the function $t \mapsto F_i^H(\phi_t)$ is continuous on J, then it is 2-exponentially convex function on J. If $t \mapsto F_i^H(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[F_i^H(\phi_s)]^{t-r} \le \left[F_i^H(\phi_r)\right]^{t-s} \left[F_i^H(\phi_t)\right]^{s-r}, \quad i = 1,2$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto F_i^H(\phi_t)$ is strictly positive and differentiable on J, then for every $p,q,u,v \in J$, such that $p \le u$ and $q \le v$, we have

$$\mu_{p,q}(F_i^H,\Omega) \le \mu_{u,v}(F_i^H,\Omega), \tag{2.172}$$

where

$$\mu_{p,q}(F_i^H,\Omega) = \begin{cases} \left(\dfrac{F_i^H(\phi_p)}{F_i^H(\phi_q)}\right)^{\frac{1}{p-q}}, & p \ne q, \\ \exp\left(\dfrac{\frac{d}{dp}F_i^H(\phi_p)}{F_i^H(\phi_p)}\right), & p = q, \end{cases} \tag{2.173}$$

for $\phi_p, \phi_q \in \Omega$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10. □

Remark 1.19 is also valid for these functionals.

Remark 2.16 ([8]) *Similar examples can be discussed as given in Section 1.4.*

2.2.2 Results Obtained by Green's Function and Hermite Interpolation Polynomial

In this subsection, using interpolation by Hermite interpolating polynomials in combination with Green's function (1.180) we establish new identities for majorization inequalities which enable us to obtain new generalization of majorization inequalities. Using new identities we present analogous results as in the previous subsection.

The following lemma describes the positivity of function (2.139) (see [43], [111]).

Lemma 2.4 *The Green function $G_{H,n}(t,s)$ has the following properties:*

(i) $\frac{G_{H,n}(t,s)}{w(t)} > 0, a_1 \le t \le a_r, a_1 < s < a_r;$

(ii) $G_{H,n}(t,s) \le \frac{1}{(n-1)!(\beta-\alpha)}|w(t)|;$

(iii) $\int_\alpha^\beta G_{H,n}(t,s)ds = \frac{w(t)}{n!}.$

In the following theorem we some identities related to generalizations of majorization inequality.

Theorem 2.63 ([10]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ and $\mathbf{w} = (w_1,\ldots,w_m)$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that x_l, $y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l = 1,\ldots,m)$. Also let $H_{ij}, G_{H,n}$ and G be as defined in (2.137), (2.139) and (1.180) respectively. Then*

$$\sum_{l=1}^{m} w_l\, \phi(x_l) - \sum_{l=1}^{m} w_l\, \phi(y_l) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^{m} w_l\,(x_l - y_l)$$
$$+ \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G(x_l,t) - G(y_l,t)) \right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i+2)}(a_j) H_{ij}(t)dt \tag{2.174}$$
$$+ \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G(x_l,t) - G(y_l,t)) \right] G_{H,n-2}(t,s)\phi^{(n)}(s)dsdt.$$

Proof. Using (1.181) in $\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l)$ we have

$$\sum_{l=1}^{m} w_l\, \phi(x_l) - \sum_{l=1}^{m} w_l\, \phi(y_l)$$
$$= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^{m} w_l\,(x_l - y_l) + \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l G(x_l,t) - \sum_{l=1}^{m} w_l G(y_l,t) \right] \phi''(t)dt. \tag{2.175}$$

By Theorem 2.46, $\phi''(t)$ can be expressed as

$$\phi''(t) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j) + \int_{\alpha}^{\beta} G_{H,n-2}(t,s)\phi^{(n)}(s)ds. \tag{2.176}$$

Using (2.176) in (2.175) we get (2.174). □

Integral version of the above theorem can be stated as follows.

Theorem 2.64 ([10]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, $\phi \in C^n[\alpha,\beta]$ and $x, y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions. Also let $H_{ij}, G_{H,n}$ and G be as defined in (2.137), (2.139) and (1.180) respectively. Then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+ \int_{\alpha}^{\beta} \left[\int_a^b w(\tau)\,(G(x(\tau),t) - G(y(\tau),t))\,d\tau \right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i+2)}(a_j) H_{ij}(t)dt$$
$$+ \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} \left[\int_a^b w(\tau)\,(G(x(\tau),t) - G(y(\tau),t))\,d\tau \right] G_{H,n-2}(t,s)\phi^{(n)}(s)dsdt. \tag{2.177}$$

Theorem 2.65 ([10]) *Let $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $(r \geq 2)$ be the given points, $\mathbf{w} = (w_1, \ldots, w_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be m-tuples such that x_l, $y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, \ldots, m)$ and H_{ij}, G be as defined in (2.137) and (1.180) respectively. Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be n-convex and*

$$\sum_{l=1}^{m} w_l \left(G(x_l, t) - G(y_l, t)\right) \geq 0, \quad t \in [\alpha, \beta]. \tag{2.178}$$

Consider the inequality

$$\sum_{l=1}^{m} w_l\, \phi\,(x_l) - \sum_{l=1}^{m} w_l\, \phi\,(y_l) \geq \frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha} \sum_{l=1}^{m} w_l\,(x_l - y_l)$$
$$+ \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l \left(G(x_l, t) - G(y_l, t)\right)\right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i+2)}(a_j) H_{ij}(t) dt. \tag{2.179}$$

(i) If k_j is odd for each $j = 2, .., r$, then the inequality (2.179) holds.

(ii) If k_j is odd for each $j = 2, .., r-1$ and k_r is even, then the reverse inequality in (2.179) holds.

Proof.

(i) Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is $n-$times differentiable and $\phi^{(n)} \geq 0$ see [144, p. 16 and p. 293]. Also as it is given that k_j is odd for each $j = 1, 2, .., r$, therefore we have $\omega(t) \geq 0$ and by using Lemma 2.4(i) we have $G_{H,n-2}(t,s) \geq 0$. Hence, we can apply Theorem 2.63 to obtain (2.179).

(ii) If k_r is even then $(t - a_r)^{k_r+1} \leq 0$ for any $t \in [\alpha, \beta]$. Also clearly $(t - a_1)^{k_1+1} \geq 0$ for any $t \in [\alpha, \beta]$ and $\prod_{j=2}^{r-1}(t - a_j)^{k_j+1} \geq 0$ for $t \in [\alpha, \beta]$ if k_j is odd for each $j = 2, .., r-1$, therefore combining all these we have $\omega(t) = \prod_{j=1}^{r}(t - a_j)^{k_j+1} \leq 0$ for any $t \in [\alpha, \beta]$ and by using Lemma 2.4(i) we have $G_{H,n-2}(t,s) \leq 0$. Hence, we can apply Theorem 2.63 to obtain reverse inequality in (2.179).

□

Integral version of the above theorem can be stated as follows.

Theorem 2.66 ([10]) *Let $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $(r \geq 2)$ be given points and $x, y : [a,b] \to [\alpha, \beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and H_{ij} and G be as defined in (2.137) and (1.180) respectively. Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be n-convex and*

$$\int_{a}^{b} w(\tau) \left(G(x(\tau), t) - G(y(\tau), t)\right) d\tau \geq 0, \quad t \in [\alpha, \beta]. \tag{2.180}$$

Consider the inequality

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \geq \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\int_\alpha^\beta \left[\int_a^b w(\tau)\left(G(x(\tau),t)-G(y(\tau),t)\right)d\tau\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i+2)}(a_j)H_{ij}(t)dt. \tag{2.181}$$

(i) If k_j is odd for each $j=2,..,r$, then the inequality (2.181) holds.

(ii) If k_j is odd for each $j=2,..,r-1$ and k_r is even, then the reverse inequality in (2.181) holds.

By using type $(m,n-m)$ conditions we can give the following result.

Corollary 2.11 ([10]) *Let $[\alpha,\beta]$ be an interval and $\boldsymbol{w}=(w_1,\ldots,w_p)$, $\boldsymbol{x}=(x_1,\ldots,x_p)$ and $\boldsymbol{y}=(y_1,\ldots,y_p)$ be p-tuples such that $x_l,\ y_l\in[\alpha,\beta], w_l\in\mathbb{R}$ $(l=1,\ldots,p)$. Let G be the Green function as defined in (1.180) and τ_i,η_i be as defined in (2.141) and (2.142) respectively. Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be n-convex and the inequality (2.178) holds for p-tuples. Consider the inequality*

$$\sum_{l=1}^{p}w_l\,\phi\left(x_l\right)-\sum_{l=1}^{p}w_l\,\phi\left(y_l\right)\geq\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{p}w_l\left(x_l-y_l\right)$$
$$+\int_\alpha^\beta\left[\sum_{l=1}^{p}w_l\left(G(x_l,t)-G(y_l,t)\right)\right]\left(\sum_{i=0}^{m-1}\tau_i(t)\phi^{(i+2)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^{(i+2)}(\beta)\right)dt. \tag{2.182}$$

(i) If $n-m$ is even, then the inequality (2.182) holds.

(ii) If $n-m$ is odd, then the reverse inequality in (2.182) holds.

By using Two-point Taylor conditions we can give the following result.

Corollary 2.12 ([10]) *Let $[\alpha,\beta]$ be an interval, $\boldsymbol{w}=(w_1,\ldots,w_p)$, $\boldsymbol{x}=(x_1,\ldots,x_p)$ and $\boldsymbol{y}=(y_1,\ldots,y_p)$ be p-tuples such that $x_l,\ y_l\in[\alpha,\beta], w_l\in\mathbb{R}$ $(l=1,\ldots,p)$ and G be the Green function as defined in (1.180). Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be n-convex and the inequality (2.178) holds for p-tuples. Consider the inequality*

$$\sum_{l=1}^{p}w_l\,\phi\left(x_l\right)-\sum_{l=1}^{p}w_l\,\phi\left(y_l\right)\geq\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{p}w_l\left(x_l-y_l\right)$$
$$+\int_\alpha^\beta\left[\sum_{l=1}^{p}w_l\left(G(x_l,t)-G(y_l,t)\right)\right]$$
$$\left[\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\left[\frac{(t-\alpha)^i}{i!}\left(\frac{t-\beta}{\alpha-\beta}\right)^m\left(\frac{t-\alpha}{\beta-\alpha}\right)^k\phi^{(i+2)}(\alpha)\right.\right.$$
$$\left.\left.+\frac{(t-\beta)^i}{i!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m\left(\frac{t-\beta}{\alpha-\beta}\right)^k\phi^{(i+2)}(\beta)\right]\right]dt. \tag{2.183}$$

(i) If m is even, then the inequality (2.183) holds.

(ii) If m is odd, then the reverse inequality in (2.183) holds.

Remark 2.17 *Similarly, one can also easily obtain the integral variants of Corollaries 2.11,2.12.*

The following generalization of majorization theorem is valid.

Theorem 2.67 ([10]) *Let* $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $(r \geq 2)$ *be the given points,* $\mathbf{x} = (x_1, \ldots, x_m)$ *and* $\mathbf{y} = (y_1, \ldots, y_m)$ *be m-tuples such that* $\mathbf{y} \prec \mathbf{x}$ *with* x_l, $y_l \in [\alpha, \beta]$ $(l = 1, \ldots, m)$. *Let* H_{ij} *be as defined in (2.137) and* $\phi : [\alpha, \beta] \to \mathbb{R}$ *be n-convex. Consider*

$$\sum_{l=1}^{m} \phi(x_l) - \sum_{l=1}^{m} \phi(y_l) \geq \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} (G(x_l, t) - G(y_l, t)) \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i+2)}(a_j) H_{ij}(t) dt. \tag{2.184}$$

(i) If k_j *is odd for each* $j = 2, .., r$, *then the inequality (2.184) holds.*

(ii) If k_j *is odd for each* $j = 2, .., r-1$ *and* k_r *is even, then the reverse inequality in (2.184) holds.*

(iii) If the inequality (reverse inequality) in (2.184) holds and the function

$$\bar{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \int_{\alpha}^{\beta} G(\cdot, t) H_{ij}(t) \phi^{(i+2)}(a_j) dt \tag{2.185}$$

is convex (concave) on $[\alpha, \beta]$, *then the right hand side of (2.184) will be non negative (non positive) that is the inequality (reverse inequality) in (2.153) will holds.*

Proof. (i) Since the function G is convex and $\boldsymbol{y} \prec \boldsymbol{x}$ therefore by Theorem 1.12, the inequality (2.178) holds for $w_l = 1$. Hence by Theorem 2.65(i) the inequality (2.184) holds.
(ii) Similar to part (ii).
(iii) The proof is similar to the proof of Theorem 2.51. □

In the following theorem we give generalization of Fuch's majorization theorem.

Theorem 2.68 ([10]) *Let* $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $(r \geq 2)$ *be the given points,* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_m)$ *be decreasing m-tuples and* $\boldsymbol{w} = (w_1, \ldots, w_m)$ *be any m-tuple with* x_l, $y_l \in [\alpha, \beta], w_l \in \mathbb{R}$ $(l = 1, \ldots, m)$ *which satisfy (1.19) and (1.20). Let* H_{ij} *be as defined in (2.137)and* $\phi : [\alpha, \beta] \to \mathbb{R}$ *be n-convex, then*

$$\sum_{l=1}^{m} w_l \phi(x_l) - \sum_{l=1}^{m} w_l \phi(y_l)$$
$$\geq \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l (G(x_l, t) - G(y_l, t)) \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i+2)}(a_j) H_{ij}(t) dt. \tag{2.186}$$

(i) *If k_j is odd for each $j = 2,..,r$, then the inequality (2.186) holds.*

(ii) *If k_j is odd for each $j = 2,..,r-1$ and k_r is even, then the reverse inequality in (2.186) holds.*

(iii) *If the inequality (reverse inequality) in (2.186) holds and the function*

$$\bar{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \int_{\alpha}^{\beta} G(\cdot,t) H_{ij}(t) \phi^{(i+2)}(a_j) dt \tag{2.187}$$

is convex (concave) on $[\alpha,\beta]$, then the right hand side of (2.186) will be non negative (non positive) that is the inequality (reverse inequality) in (2.157) will hold.

Proof. Similar to the proof of Theorem 2.67. □

In the following theorem we give generalized majorization integral inequality.

Theorem 2.69 ([10]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha = a_1 < a_2 \cdots < a_r = \beta$, $(r \geq 2)$ be the given points, and $x, y : [a,b] \to [\alpha,\beta]$ be decreasing and $w : [a,b] \to \mathbb{R}$ be continuous functions such that (1.27) and (1.28) hold. Also let H_{ij} be as defined in (2.137) and $\phi : [\alpha,\beta] \to \mathbb{R}$ be n-convex and consider the inequality*

$$\begin{aligned} &\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\ &\geq \int_{\alpha}^{\beta} \left[\int_a^b w(\tau) \left(G(x(\tau),t) - G(y(\tau),t) \right) d\tau \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i+2)}(a_j) H_{ij}(t) dt. \end{aligned} \tag{2.188}$$

(i) *If k_j is odd for each $j = 2,..,r$, then the inequality (2.188) holds.*

(ii) *If k_j is odd for each $j = 2,..,r-1$ and k_r is even, then the reverse inequality in (2.188) holds.*

(iii) *If the inequality (reverse inequality) in (2.188) holds and the function*

$$\bar{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \int_{\alpha}^{\beta} G(\cdot,t) H_{ij}(t) \phi^{(i+2)}(a_j) dt \tag{2.189}$$

is convex (concave), then the right hand side of (2.188) will be non negative (non positive) that is the inequality (reverse inequality) in (2.159) will hold.

By using type $(m, n-m)$ conditions we can give generalization of majorization inequality for majorized tuples:

Corollary 2.13 ([10]) *Let $[\alpha,\beta]$ be an interval, $\mathbf{x} = (x_1,\ldots,x_p)$ and $\mathbf{y} = (y_1,\ldots,y_p)$ be any p-tuple such that $\mathbf{y} \prec \mathbf{x}$ with $x_l, y_l \in [\alpha,\beta]$ $(l = 1,\ldots,p)$. Let τ_i and η_i be as defined in (2.141) and (2.142) respectively and $\phi : [\alpha,\beta] \to \mathbb{R}$ be n-convex. Consider*

$$\sum_{l=1}^{p}\phi(x_l)-\sum_{l=1}^{p}\phi(y_l)$$
$$\geq \int_{\alpha}^{\beta}\left[\sum_{l=1}^{p}(G(x_l,t)-G(y_l,t))\right]\left(\sum_{i=0}^{m-1}\tau_i(t)\phi^{(i+2)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^{(i+2)}(\beta)\right)dt. \tag{2.190}$$

(i) If $n-m$ is even, then the inequality (2.190) holds.

(ii) If $n-m$ is odd, then the reverse inequality in (2.190) holds.

(iii) If the inequality (reverse inequality) in (2.190) holds and the function

$$\tilde{F}(\cdot)=\int_{\alpha}^{\beta}G(\cdot,t)\left(\sum_{i=0}^{m-1}\tau_i(t)\phi^{i}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^{i}(\beta)\right)dt \tag{2.191}$$

is convex (concave) on $[\alpha,\beta]$, then the right hand side of (2.190) will be non negative (non positive) that is the inequality (reverse inequality) in (2.153) will hold.

By using Two-point Taylor conditions we can give generalization of majorization inequality for majorized tuples:

Corollary 2.14 ([10]) *Let $[\alpha,\beta]$ be an interval and $\mathbf{x}=(x_1,\ldots,x_p)$, $\mathbf{y}=(y_1,\ldots,y_p)$ be decreasing p-tuples such that $\mathbf{y}\prec\mathbf{x}$ with $x_l,y_l\in[\alpha,\beta]$ $(l=1,\ldots,p)$. Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be n-convex. Consider*

$$\sum_{l=1}^{p}\phi(x_l)-\sum_{l=1}^{p}\phi(y_l)\geq\int_{\alpha}^{\beta}\left[\sum_{l=1}^{p}(G(x_l,t)-G(y_l,t))\right]F(t)dt, \tag{2.192}$$

$$\text{where } F(t)=\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\left[\frac{(t-\alpha)^i}{i!}\left(\frac{t-\beta}{\alpha-\beta}\right)^m\left(\frac{t-\alpha}{\beta-\alpha}\right)^k\phi^{(i+2)}(\alpha)\right.$$
$$\left.+\frac{(t-\beta)^i}{i!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m\left(\frac{t-\beta}{\alpha-\beta}\right)^k\phi^{(i+2)}(\beta)\right].$$

(i) If m is even, then the inequality (2.192) holds.

(ii) If m is odd, then the reverse inequality in (2.192) holds.

(iii) If the inequality (reverse inequality) in (2.192) holds and the function

$$\hat{F}(\cdot)=\int_{\alpha}^{\beta}G(\cdot,t)F(t)dt$$

is convex (concave) on $[\alpha,\beta]$, then the right hand side of (2.192) will be non negative (non positive) that is the inequality (reverse inequality) in (2.153) will hold.

Remark 2.18 *Similarly we can give the weighted and integral version of Corollaries 2.13,2.14.*

In the sequel (see [10]) we use the above theorems to obtain generalizations of the previous results.

For m-tuples $\boldsymbol{w} = (w_1,\dots,w_m)$, $\boldsymbol{x} = (x_1,\dots,x_m)$ and $\boldsymbol{y} = (y_1,\dots,y_m)$ with $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}\ (l = 1,\dots,m)$ and the Green functions G and $G_{H,n}$ be as defined in (1.180) and (2.139) respectively, denote

$$\mathfrak{L}(s) = \int_\alpha^\beta \left[\sum_{l=1}^m w_l\,(G(x_l,t) - G(y_l,t))\right] G_{H,n-2}(t,s)dt, \quad s \in [\alpha,\beta], \tag{2.193}$$

similarly for continuous functions $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ and the Green function G and $G_{H,n}$ be as defined in (1.180) and (2.139) respectively, denote

$$\mathfrak{J}(s) = \int_\alpha^\beta \left[\int_a^b w(\tau)\,(G(x(\tau),t) - G(y(\tau),t))\right] d\tau G_{H,n-2}(t,s)dt, \quad s \in [\alpha,\beta]. \tag{2.194}$$

Consider the Čebyšev functionals $T(\mathfrak{L},\mathfrak{L})$, $T(\mathfrak{J},\mathfrak{J})$ are given by:

$$T(\mathfrak{L},\mathfrak{L}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{L}^2(s)ds - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{L}(s)ds\right)^2, \tag{2.195}$$

$$T(\mathfrak{J},\mathfrak{J}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{J}^2(s)ds - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{J}(s)ds\right)^2. \tag{2.196}$$

Theorem 2.70 ([10]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and $\boldsymbol{w} = (w_1,\dots,w_m)$, $\boldsymbol{x} = (x_1,\dots,x_m)$ and $\boldsymbol{y} = (y_1,\dots,y_m)$ be m-tuples such that $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}\ (l = 1,\dots,m)$. Also let H_{ij} be the fundamental polynomials of the Hermite basis and the functions G and $\mathfrak{L}$ be defined by (1.180) and (2.193) respectively. Then*

$$\begin{aligned}\sum_{l=1}^m w_l\,\phi\,(x_l) - \sum_{l=1}^m w_l\,\phi\,(y_l) &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^m w_l\,(x_l - y_l)\\ &+ \int_\alpha^\beta \left[\sum_{l=1}^m w_l\,(G(x_l,t) - G(y_l,t))\right] \sum_{j=1}^r \sum_{i=0}^{k_j} \phi^{(i+2)}(a_j)H_{ij}(t)dt\\ &+ \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta \mathfrak{L}(s)ds + \kappa(\phi;\alpha,\beta).\end{aligned} \tag{2.197}$$

where the remainder $\kappa(\phi;\alpha,\beta)$ satisfies the estimation

$$|\kappa(\phi;\alpha,\beta)| \le \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\,[T(\mathfrak{L},\mathfrak{L})]^{\frac{1}{2}} \left|\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right|^{\frac{1}{2}}. \tag{2.198}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

The integral version of the above theorem can be stated as follows.

Theorem 2.71 ([10]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $(\cdot - \alpha)(\beta - \cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions. Also let H_{ij} be the fundamental polynomials of the Hermite basis and the functions G and $\mathfrak{J}$ be defined by (1.180) and (2.194) respectively. Then*

$$\begin{aligned}\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \\ &+ \int_\alpha^\beta \left[\int_a^b w(\tau)\left(G(x(\tau),t)-G(y(\tau),t)\right)d\tau\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i+2)}(a_j)H_{ij}(t)dt \\ &+ \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta \mathfrak{J}(s)ds + \tilde{\kappa}(\phi;\alpha,\beta),\end{aligned} \tag{2.199}$$

where the remainder $\tilde{\kappa}(\phi;\alpha,\beta)$ satisfies the estimation

$$|\tilde{\kappa}(\phi;\alpha,\beta)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left[T(\mathfrak{J},\mathfrak{J})\right]^{\frac{1}{2}}\left|\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right|^{\frac{1}{2}}. \tag{2.200}$$

Using Theorem 1.11 we obtain the following Grüss type inequalities.

Theorem 2.72 ([10]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $\phi^{(n)}$ is monotonic non decreasing on $[\alpha,\beta]$ and let $\mathfrak{L}$ be defined by (2.193). Then the representation* (2.197) *holds and the remainder $\kappa(\phi;\alpha,\beta)$ satisfies the bound*

$$|\kappa(\phi;\alpha,\beta)| \leq \|\mathfrak{L}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \tag{2.201}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

Integral case of the above theorem can be given follows.

Theorem 2.73 ([10]) *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $\phi \in C^n[\alpha,\beta]$ such that $\phi^{(n)}$ is monotonic non decreasing on $[\alpha,\beta]$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and the functions G and $\mathfrak{J}$ be defined by (1.180) and (2.194) respectively. Then we have the representation* (2.199) *and the remainder $\tilde{\kappa}(\phi;\alpha,\beta)$ satisfies the bound*

$$|\tilde{\kappa}(\phi;\alpha,\beta)| \leq \|\mathfrak{J}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \tag{2.202}$$

We present the Ostrowski type inequalities related to generalizations of majorization inequality.

Theorem 2.74 ([10]) *Suppose that all assumptions of Theorem 2.63 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $1/p+1/q=1$. Let $\left|\phi^{(n)}\right|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$. Then we have:*

$$\left|\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l\,(x_l-y_l)\right.$$
$$\left. - \int_\alpha^\beta \left[\sum_{l=1}^{m} w_l\,(G(x_l,t)-G(y_l,t))\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i+2)}(a_j)H_{ij}(t)dt\right|$$
$$\le \left\|\phi^{(n)}\right\|_p \|\mathfrak{L}\|_q, \tag{2.203}$$

where $\mathfrak{L}$ is defined in (2.193).

The constant on the right-hand side of (2.203) is sharp for $1 < p \le \infty$ and the best possible for $p=1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Integral version of the above theorem can be given as follows.

Theorem 2.75 ([10]) *Suppose that all assumptions of Theorem 2.64 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $1/p+1/q=1$. Let $\left|\phi^{(n)}\right|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$. Then we have:*

$$\left|\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau\right.$$
$$\left. - \int_\alpha^\beta \left[\int_a^b w(\tau)\,(G(x(\tau),t)-G(y(\tau),t))\,d\tau\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i+2)}(a_j)H_{ij}(t)dt\right|$$
$$\le \left\|\phi^{(n)}\right\|_p \|\mathfrak{I}\|_q, \tag{2.204}$$

where $\mathfrak{I}$ is defined in (2.194).

The constant on the right-hand side of (2.204) is sharp for $1 < p \le \infty$ and the best possible for $p=1$.

Motivated (see [10]) by inequalities (2.179) and (2.181), under the assumptions of Theorems 2.65 and 2.66 we define the following linear functionals:

$$F_1^H(\phi) = \sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l\,(x_l-y_l)$$
$$- \int_\alpha^\beta \left[\sum_{l=1}^{m} w_l\,(G(x_l,t)-G(y_l,t))\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i+2)}(a_j)H_{ij}(t)dt. \tag{2.205}$$

$$
\begin{aligned}
\digamma_2^H(\phi) = & \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \\
& - \int_\alpha^\beta \left[\int_a^b w(\tau)\left(G(x(\tau),t)-G(y(\tau),t)\right)d\tau\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i+2)}(a_j)H_{ij}(t)dt.
\end{aligned}
\tag{2.206}
$$

Remark 2.19 ([10]) Under the assumptions of Theorems 2.65 and 2.66, it holds $\digamma_i^H(\phi) \geq 0$, $i = 1,2$, for all n-convex functions ϕ.

Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

Theorem 2.76 ([10]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha,\beta]$. If the inequalities (2.178) and (2.180) hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$
\digamma_i^H(\phi) = \phi^{(n)}(\xi_i)\digamma_i^H(\varphi), \quad i = 1,2 \tag{2.207}
$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\digamma_i^H$, $i = 1,2$ are defined by (2.205) and(2.206).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (see also the proof of Theorem 4.1 in [86]). □

Theorem 2.77 ([10]) *Let $\phi,\psi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi,\psi \in C^n[\alpha,\beta]$. If the inequalities (2.178) and (2.180) hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$
\frac{\digamma_i^H(\phi)}{\digamma_i^H(\psi)} = \frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)}, \quad i = 1,2 \tag{2.208}
$$

provided that the denominators are non-zero and $\digamma_i^H$, $i = 1,2$, are defined by (2.205) and(2.206).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.14 (see also the proof of Corollary 4.2 in [86]). □

Now we present results for n-exponentially and exponentially convex functions.

Theorem 2.78 ([10]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha,\beta]$ such that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in [\alpha,\beta]$. Then for the linear functionals $\digamma_i^H(\phi_t)$ $(i = 1,2)$ as defined by (2.205) and (2.206), the following statements hold:*

(i) The function $t \to \digamma_i^H(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\digamma_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1,..,t_m \in J$. Particularly,

$$
\det[\digamma_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \textit{ for all } m \in \mathbb{N},\ m = 1,2,\ldots,n.
$$

(ii) If the function $t \to \digamma_i^H(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39. □

The following corollary is an immediate consequence of the above theorem

Corollary 2.15 ([10]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha, \beta]$ such that the function $t \mapsto [x_0, \dots, x_n; \phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in [\alpha, \beta]$. Then for the linear functionals $F_i^H(\phi_t)$ $(i = 1, 2)$ as defined by (2.205) and (2.206), the following statements hold:*

(i) The function $t \to F_i^H(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[F_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det[F_i^H(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, \dots, n.$$

(ii) If the function $t \to F_i^H(\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 2.16 ([10]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha, \beta]$ such that the function $t \mapsto [x_0, \dots, x_n; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in [\alpha, \beta]$. Let F_i^H, $i = 1, 2$ be linear functionals defined by (2.205) and (2.206). Then the following statements hold:*

(i) If the function $t \mapsto F_i^H(\phi_t)$ is continuous on J, then it is 2-exponentially convex function on J. If $t \mapsto F_i^H(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[F_i^H(\phi_s)]^{t-r} \leq \left[F_i^H(\phi_r)\right]^{t-s} \left[F_i^H(\phi_t)\right]^{s-r}, \quad i = 1, 2$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto F_i^H(\phi_t)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(F_i^H, \Omega) \leq \mu_{u,v}(F_i^H, \Omega), \tag{2.209}$$

where

$$\mu_{p,q}(F_i^H, \Omega) = \begin{cases} \left(\frac{F_i^H(\phi_p)}{F_i^H(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp} F_i^H(\phi_p)}{F_i^H(\phi_p)}\right), & p = q, \end{cases} \tag{2.210}$$

for $\phi_p, \phi_q \in \Omega$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10. □

Remark 1.19 is also valid for these functionals.

Remark 2.20 ([10]) *Similar examples can be discussed as given in Section 1.4.*

2.2.3 Results Obtained by New Green's Functions and Hermite Interpolation Polynomial

In this subsection (see [10]), using interpolation by Hermite interpolating polynomials in combination with newly defined Green's functions $G_c(c = 1,2,3,4)$, defined as in (2.47)-(2.50), we present analogous results as in the previous subsection.

We begin with identities related to the generalizations of majorization inequality via Peano's representation of Hermite's polynomial and new Green's functions.

Theorem 2.79 *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$, $(r \geq 2)$ be the given points, and $f \in C^n[\alpha,\beta]$ and $\mathbf{w} = (w_1,\ldots,w_m)$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l = 1,\ldots,m)$. Also let $H_{ij}, G_{H,n}$ and $G_c(c = 1,2,3,4)$ be as defined in (2.137), (2.139) and (2.47)-(2.50) respectively. Then we have the following identities for $c = 1,2,3,4$,*

$$\sum_{l=1}^{m} w_l f(x_l) - \sum_{l=1}^{m} w_l f(y_l) = \left(\sum_{l=1}^{m} w_l x_l - \sum_{l=1}^{m} w_l y_l\right) f'(\alpha)$$
$$+ \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l \left(G_c(x_l,t) - G_c(y_l,t)\right)\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j) H_{ij}(t)dt \tag{2.211}$$
$$+ \int_{\alpha}^{\beta} f^{(n)}(s) \left[\int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l \left(G_c(x_l,t) - G_c(y_l,t)\right)\right] G_{H,n-2}(t,s)dt\right] ds,$$

where Peano's kernel is defined as

$$G_{H,n-2}(t,s) = \begin{cases} \sum\limits_{j=1}^{l}\sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-3}}{(n-i-3)!} H_{ij}(t);\ s \leq t, \\ -\sum\limits_{j=l+1}^{r}\sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-3}}{(n-i-3)!} H_{ij}(t);\ s \geq t, \end{cases} \tag{2.212}$$

for all $a_l \leq s \leq a_{l+1}$; $l = 0,\ldots,r$ with $a_0 = \alpha$ and $a_{r+1} = \beta$.

Proof. Fix $c = 1,2,3,4$, evaluating the identities one by one (2.46), (2.51), (2.52) and (2.53) into majorization difference, we get

$$\sum_{l=1}^{m} w_l f(x_l) - \sum_{l=1}^{m} w_l f(y_l)$$
$$= \left(\sum_{l=1}^{m} w_l x_l - \sum_{l=1}^{m} w_l y_l\right) f'(\alpha) + \int_{\alpha}^{\beta} \left(\sum_{l=1}^{m} w_l G_c(x_l,t) - \sum_{l=1}^{m} w_l G_c(y_l,t)\right) f''(t)dt. \tag{2.213}$$

By the Peano's representation of Hermite's interpolatinhg polynomial Theorem 2.46, $f''(t)$ can be expressed as

$$f''(t) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t) f^{(i+2)}(a_j) + \int_{\alpha}^{\beta} G_{H,n-2}(t,s) f^{(n)}(s)ds. \tag{2.214}$$

Using (2.214) in (2.213) we get

$$\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) = \left(\sum_{l=1}^{m} w_l x_l - \sum_{l=1}^{m} w_l y_l\right) f'(\alpha)$$
$$+ \int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\left(G_c(x_l,t) - G_c(y_l,t)\right)\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)H_{ij}(t)dt$$
$$+ \int_{\alpha}^{\beta}\left(\sum_{l=1}^{m} w_l\left(G_c(x_l,t) - G_c(y_l,t)\right)\right)\left(\int_{\alpha}^{\beta} G_{H,n-2}(t,s)f^{(n)}(s)ds\right)dt.$$

after applying Fubini's theorem we get (2.211). □

Integral version of the above theorem can be stated as follows.

Theorem 2.80 *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$, $(r \ge 2)$ be the given points, $f \in C^n[\alpha,\beta]$ and $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions. Also let $H_{ij}, G_{H,n-2}$ and $G_c(c = 1,2,3,4)$ be as defined in (2.137), (2.212) and (2.47)-(2.50) respectively. Then we have the following identities for $c = 1,2,3,4$,*

$$\int_a^b w(\tau)f(x(\tau))d\tau - \int_a^b w(\tau)f(y(\tau))d\tau = \left(\int_a^b w(\tau)x(\tau)d\tau - \int_a^b w(\tau)y(\tau)d\tau\right) f'(\alpha)$$
$$+ \int_{\alpha}^{\beta}\left[\int_a^b w(\tau)\left(G_c(x(\tau),t) - G_c(y(\tau),t)\right)d\tau\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)H_{ij}(t)dt$$
$$+ \int_{\alpha}^{\beta} f^{(n)}(s)\left(\int_{\alpha}^{\beta}\left[\int_a^b w(\tau)\left(G_c(x(\tau),t) - G_c(y(\tau),t)\right)d\tau\right] G_{H,n-2}(t,s)dt\right)ds. \tag{2.215}$$

Theorem 2.81 *Let $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $(r \ge 2)$ be the given points, $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ be m-tuples such that $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l = 1,\ldots,m)$ and H_{ij}, $G_c(c = 1,2,3,4)$ be as defined in (2.137) and (2.47)-(2.50) respectively. Let $f : [\alpha,\beta] \to \mathbb{R}$ be n-convex and*

$$\sum_{l=1}^{m} w_l\left(G_c(x_l,t) - G_c(y_l,t)\right) \ge 0, \quad t \in [\alpha,\beta]. \tag{2.216}$$

Consider the inequalities for $c = 1,2,3,4$,

$$\sum_{l=1}^{m} w_l\,f(x_l) - \sum_{l=1}^{m} w_l\,f(y_l) \ge \left(\sum_{l=1}^{m} w_l x_l - \sum_{l=1}^{m} w_l y_l\right) f'(\alpha)$$
$$+ \int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\left(G_c(x_l,t) - G_c(y_l,t)\right)\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)H_{ij}(t)dt. \tag{2.217}$$

(i) If k_j is odd for each $j = 2,..,r$, then the inequalities for $c = 1,2,3,4$, in (2.217) hold.

(ii) If k_j is odd for each $j = 2,..,r-1$ and k_r is even, then the reverse inequalities for $c = 1,2,3,4$, in (2.217) hold.

Proof.

(i) Since the function f is n-convex, therefore without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$ see [144, p. 16 and p. 293]. Also the given condition is that k_j is odd for each $j = 1,2,..,r$ implies that

$$\omega(t) = \prod_{j=1}^{r} (t - a_j)^{k_j+1} \geq 0.$$

By using the first part of Lemma 2.4 we have that the Peano's kernel $G_{H,n-2}(t,s) \geq 0$. Hence, we can apply Theorem 2.79 to obtain (2.217).

(ii) If k_r is even then $(t - a_r)^{k_r+1} \leq 0$ for any $t \in [\alpha, \beta]$. Also clearly $(t - a_1)^{k_1+1} \geq 0$ for any $t \in [\alpha, \beta]$ and $\prod_{j=2}^{r-1}(t - a_j)^{k_j+1} \geq 0$ for $t \in [\alpha, \beta]$ if k_j is odd for each $j = 2,..,r-1$, therefore combining all these we have $\omega(t) = \prod_{j=1}^{r}(t - a_j)^{k_j+1} \leq 0$ for any $t \in [\alpha, \beta]$ and by using the first part of Lemma 2.4 we have $G_{H,n-2}(t,s) \leq 0$. Hence, we can apply Theorem 2.79 to obtain reverse inequality in (2.217).

□

Integral version of the above theorem can be stated as follows.

Theorem 2.82 *Let $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $(r \geq 2)$ be given points and $x, y : [a,b] \to [\alpha, \beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and H_{ij} and $G_c (c = 1,2,3,4)$ be as defined in (2.137) and (2.47)-(2.50) respectively. Let $f : [\alpha, \beta] \to \mathbb{R}$ be n-convex and*

$$\int_a^b w(\tau) \left(G_c(x(\tau),t) - G_c(y(\tau),t) \right) d\tau \geq 0, \quad t \in [\alpha, \beta]. \tag{2.218}$$

Consider the inequalities for $c = 1,2,3,4$,

$$\begin{aligned} &\int_a^b w(\tau) f(x(\tau)) d\tau - \int_a^b w(\tau) f(y(\tau)) d\tau \geq \left(\int_a^b w(\tau) x(\tau) d\tau - \int_a^b w(\tau) y(\tau) d\tau \right) f^{'}(\alpha) \\ &+ \int_\alpha^\beta \left[\int_a^b w(\tau) \left(G_c(x(\tau),t) - G_c(y(\tau),t) \right) d\tau \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+2)}(a_j) H_{ij}(t) dt. \end{aligned} \tag{2.219}$$

(i) If k_j is odd for each $j = 2,..,r$, then the inequalities for $c = 1,2,3,4$, in (2.219) hold.

(ii) If k_j is odd for each $j = 2,..,r-1$ and k_r is even, then the reverse inequalities for $c = 1,2,3,4$, in (2.219) hold.

By using type $(m, n-m)$ conditions we can give the following result.

Corollary 2.17 *Let $[\alpha, \beta]$ be an interval and $\mathbf{w} = (w_1, \ldots, w_p)$, $\mathbf{x} = (x_1, \ldots, x_p)$ and $\mathbf{y} = (y_1, \ldots, y_p)$ be p-tuples such that x_l, $y_l \in [\alpha, \beta]$, $w_l \in \mathbb{R}$ $(l = 1, \ldots, p)$. Let $G_c (c = 1,2,3,4)$ be the Green functions as defined in (2.47)-(2.50) respectively and also τ_i, η_i be as defined in (2.141) and (2.142) respectively. Let $f : [\alpha, \beta] \to \mathbb{R}$ be n-convex and the inequality (2.216) holds for p-tuples. Consider the inequalities for $c = 1,2,3,4$,*

$$\sum_{l=1}^{p} w_l f(x_l) - \sum_{l=1}^{p} w_l f(y_l) \geq \left(\sum_{l=1}^{p} w_l x_l - \sum_{l=1}^{p} w_l y_l\right) f'(\alpha)$$
$$+ \int_{\alpha}^{\beta} \left[\sum_{l=1}^{p} w_l \left(G_c(x_l,t) - G_c(y_l,t)\right)\right] \left(\sum_{i=0}^{m-1} \tau_i(t) f^{(i+2)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(t) f^{(i+2)}(\beta)\right) dt. \tag{2.220}$$

(i) If $n-m$ is even, then the inequalities for $c=1,2,3,4$, in (2.220) hold.

(ii) If $n-m$ is odd, then the reverse inequalities for $c=1,2,3,4$, in (2.220) hold.

By using Two-point Taylor conditions we can give the following result.

Corollary 2.18 *Let $[\alpha,\beta]$ be an interval, $\mathbf{w}=(w_1,\ldots,w_p)$, $\boldsymbol{x}=(x_1,\ldots,x_p)$ and $\mathbf{y}=(y_1,\ldots,y_p)$ be p-tuples such that x_l, $y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l=1,\ldots,p)$ and $G_c(c=1,2,3,4)$ be the Green function as defined in (2.47)-(2.50) respectively. Let $f:[\alpha,\beta]\to\mathbb{R}$ be n-convex and the inequality (2.178) holds for p-tuples. Consider the inequalities for $c=1,2,3,4$,*

$$\sum_{l=1}^{p} w_l f(x_l) - \sum_{l=1}^{p} w_l f(y_l) \geq \left(\sum_{l=1}^{p} w_l x_l - \sum_{l=1}^{p} w_l y_l\right) f'(\alpha)$$
$$+ \int_{\alpha}^{\beta} \left[\sum_{l=1}^{p} w_l \left(G_c(x_l,t) - G_c(y_l,t)\right)\right]$$
$$\left[\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \left[\frac{(t-\alpha)^i}{i!}\left(\frac{t-\beta}{\alpha-\beta}\right)^m \left(\frac{t-\alpha}{\beta-\alpha}\right)^k f^{(i+2)}(\alpha)\right.\right.$$
$$\left.\left. + \frac{(t-\beta)^i}{i!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m \left(\frac{t-\beta}{\alpha-\beta}\right)^k f^{(i+2)}(\beta)\right]\right] dt. \tag{2.221}$$

(i) If m is even, then the inequalities for $c=1,2,3,4$, in (2.221) hold.

(ii) If m is odd, then the reverse inequalities for $c=1,2,3,4$, in (2.221) hold.

Remark 2.21 *Similarly, one can also easily obtain the integral variants of corollaries 2.17,2.18.*

The following generalization of classical majorization theorem is valid.

Theorem 2.83 *Let $-\infty<\alpha=a_1<a_2\cdots<a_r=\beta<\infty$, $(r\geq 2)$ be the given points, $\boldsymbol{x}=(x_1,\ldots,x_m)$ and $\mathbf{y}=(y_1,\ldots,y_m)$ be m-tuples such that $\mathbf{y}\prec\mathbf{x}$ with x_l, $y_l\in[\alpha,\beta]$ $(l=1,\ldots,m)$. Let H_{ij} and $G_c(c=1,2,3,4)$ be as defined in (2.137) and (2.47)-(2.50) respectively and also $f:[\alpha,\beta]\to\mathbb{R}$ be n-convex. Consider the inequalities for $c=1,2,3,4$,*

$$\sum_{l=1}^{m} f(x_l) - \sum_{l=1}^{m} f(y_l) \geq \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} \left(G_c(x_l,t) - G_c(y_l,t)\right)\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j) H_{ij}(t)\, dt. \tag{2.222}$$

(i) If k_j is odd for each $j = 2,..,r$, then the inequalities for $c = 1,2,3,4$, in (2.222) hold.

(ii) If k_j is odd for each $j = 2,..,r-1$ and k_r is even, then the reverse inequalities for $c = 1,2,3,4$, in (2.222) hold.

If the inequalities (reverse inequalities) for $c = 1,2,3,4$, in (2.222) hold and the function $\mathbb{F}(.) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)H_{ij}(.)$ is non negative (non positive), then the right hand side of (2.222) will be non negative (non positive) for each $c = 1,2,3,4$, that is the inequality (reverse inequality) in (1.18) will hold.

Proof. (i) Since the function G_c is convex and $\mathbf{y} \prec \mathbf{x}$ therefore by Theorem 1.12, the inequalities for $c = 1,2,3,4$, in (2.178) hold for $w_l = 1$. Hence by Theorem 2.65(i) the inequalities for $c = 1,2,3,4$, in (2.184) hold. Also if the function $\mathbb{F}$ is convex then by using $\mathbb{F}$ in (1.18) instead of f we get that the right hand side of (2.184) is non negative for each $c = 1,2,3,4$.

Similarly we can prove part (ii). □

2.2.4 Results Obtained for the Jensen and Jensen-Steffensen Inequalities and their Converses via Hermite Interpolation Polynomial

In this section, we present generalizations of the Jensen, the Jensen-Steffensen and converse of the Jensen inequalities by using Hermite's interpolating polynomials. We give bounds for the identities related to the generalization of Jensen's inequality by using Čebyšev functionals. We also give the Grüss and Ostrowski types inequalities related to generalized Jensen type inequalities. The results presented in this section are given in [32].

Theorem 2.84 *Let $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$ be the given points, let $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\mathbf{w} = (w_1,\ldots,w_m)$ be m-tuples such that $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and $F \in C^n[a,b]$. Also let H_{lj}, $G_{H,n}$ and G be as defined in (2.137), (2.139) and (1.180) respectively. Then*

$$\frac{1}{W_m}\sum_{i=1}^m w_i F(x_i) - F(\overline{x}) \tag{2.223}$$
$$= \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\overline{x},s)\right]\sum_{j=1}^r\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds$$
$$+ \int_a^b\int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\overline{x},s)\right] G_{H,n-2}(s,t)F^{(n)}(t)\,dtds.$$

Proof. Consider $\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x})$. Using (1.181), we have

$$\begin{aligned}&\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\ &= \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] F''(s)ds.\end{aligned} \tag{2.224}$$

By Theorem 2.46, $F''(s)$ can be expressed as

$$F''(s) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} H_{lj}(s)F^{(l+2)}(a_j) + \int_a^b G_{H,n-2}(s,t)F^{(n)}(t)dt. \tag{2.225}$$

Using (2.224) and (2.225) we get (2.223). □

Using previous result and Theorem 2.67, here we give generalization of Jensen's inequality for n-convex function.

Theorem 2.85 *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, let $\boldsymbol{x} = (x_1,\ldots,x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1,\ldots,m$, let $\boldsymbol{w} = (w_1,\ldots,w_m)$ be positive m-tuple such that $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and H_{lj} be as defined in (2.137). Let $F : [a,b] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\begin{aligned}&\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\ &\geq \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds.\end{aligned} \tag{2.226}$$

(i) If k_j is odd for every $j = 2,\ldots,r$, then the inequality (2.226) holds.

(ii) If k_j is odd for every $j = 2,\ldots,r-1$ and k_r is even, then the reverse inequality in (2.226) holds.

If the inequality (reverse inequality) in (2.226) holds and the function $\phi(.) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(.)$ is non negative (non positive), then the right hand side of (2.226) will be non negative (non positive), that is the inequality (reverse inequality)

$$\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \geq 0 \tag{2.227}$$

holds.

Proof. For $l = 1,\ldots,k$, such that $x_k \geq \overline{x}$ we get

$$\sum_{i=1}^{l} w_i\overline{x} \leq \sum_{i=1}^{l} w_i x_i.$$

If $l = k+1,\ldots,m-1$, such that $x_{k+1} < \overline{x}$ we have

$$\sum_{i=1}^{l} w_i x_i = \sum_{i=1}^{m} w_i x_i - \sum_{i=l+1}^{m} w_i x_i > \sum_{i=1}^{m} w_i \overline{x} - \sum_{i=l+1}^{m} w_i \overline{x} = \sum_{i=1}^{l} w_i \overline{x}.$$

So,

$$\sum_{i=1}^{l} w_i \overline{x} \le \sum_{i=1}^{l} w_i x_i \text{ for all } l = 1,\ldots,m-1$$

and obviously

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} w_i x_i.$$

Now, we put $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (\overline{x},\ldots,\overline{x})$ in Theorem 2.67 to get inequalities (2.226) and (2.227). □

Using $(p, n-p)$ type conditions, we get the following corollary:

Corollary 2.19 *Let $[a,b]$ be the given interval, $\boldsymbol{x} = (x_1,\ldots,x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1,\ldots,m$, let $\mathbf{w} = (w_1,\ldots,w_m)$ be positive m-tuple such that $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$. Let $F : [a,b] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\begin{aligned}\frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\ \ge \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] \\ \left[\sum_{l=0}^{p-1} F^{(l+2)}(a) H_{l1}(s) + \sum_{l=0}^{n-p-1} F^{(l+2)}(b) H_{l2}(s)\right] ds,\end{aligned} \tag{2.228}$$

where

$$H_{l1}(s) = \frac{1}{l!}(s-a)^l \left(\frac{s-b}{a-b}\right)^{n-p} \sum_{k=0}^{p-1-l} \binom{n-p+k-1}{k} \left(\frac{s-a}{b-a}\right)^k$$

and

$$H_{l2}(s) = \frac{1}{l!}(s-b)^l \left(\frac{s-a}{b-a}\right)^{p} \sum_{k=0}^{n-p-1-l} \binom{p+k-1}{k} \left(\frac{s-b}{a-b}\right)^k.$$

(i) If $n-p$ is even, then the inequality (2.228) holds.

(ii) If $n-p$ is odd, then the reverse inequality in (2.228) holds.

If the inequality (reverse inequality) in (2.228) holds and the function $\phi(.) = \sum_{l=0}^{p-1} F^{(l+2)}(a) H_{l1}(.) + \sum_{l=0}^{n-p-1} F^{(l+2)}(b) H_{l2}(.)$ is non negative (non positive), then the right hand side of (2.228) will be non negative (non positive), that is the inequality (reverse inequality) (2.227) holds.

Using Two-point Taylor conditions, we get the following corollary:

Corollary 2.20 *Let $[a,b]$ be the given interval, $\mathbf{x} = (x_1,\dots,x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1,\dots,m$, let $\mathbf{w} = (w_1,\dots,w_m)$ be positive m-tuple such that $w_i \in \mathbb{R}$, $i = 1,\dots,m$, $W_m = \sum_{i=1}^m w_i$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$. Let $F : [a,b] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\frac{1}{W_m}\sum_{i=1}^m w_i F(x_i) - F(\overline{x}) \tag{2.229}$$
$$\geq \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\overline{x},s)\right] \sum_{l=0}^{p-1}\sum_{k=0}^{p-1-l}\binom{p+k-1}{k}\cdot$$
$$\left[\frac{(s-a)^l}{l!}\left(\frac{s-b}{a-b}\right)^p\left(\frac{s-a}{b-a}\right)^k F^{(l+2)}(a) + \frac{(s-b)^l}{l!}\left(\frac{s-a}{b-a}\right)^p\left(\frac{s-b}{a-b}\right)^k F^{(l+2)}(b)\right] ds$$

(i) If p is even then the inequality (2.229) holds.

(ii) If p is odd then the reverse inequality in (2.229) holds.

If the inequality (reverse inequality) in (2.229) holds and the function $\phi(s) = \sum_{l=0}^{p-1}\sum_{k=0}^{p-1-l}\binom{p+k-1}{k}\left[\frac{(s-a)^l}{l!}\left(\frac{s-b}{a-b}\right)^p\left(\frac{s-a}{b-a}\right)^k F^{(l+2)}(a) + \frac{(s-b)^l}{l!}\left(\frac{s-a}{b-a}\right)^p\left(\frac{s-b}{a-b}\right)^k F^{(l+2)}(b)\right]$ is non negative (non positive), then the right hand side of (2.229) will be non negative (non positive), that is the inequality (reverse inequality) (2.227) holds.

Using Simple Hermite or Osculatory conditions, we get the following corollary:

Corollary 2.21 *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, let $\mathbf{x} = (x_1,\dots,x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1,\dots,m$, let $\mathbf{w} = (w_1,\dots,w_m)$ be positive m-tuple such that $w_i \in \mathbb{R}$, $i = 1,\dots,m$, $W_m = \sum_{i=1}^m w_i$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$. Let $F : [a,b] \to \mathbb{R}$ be $(2r)$-convex function. Then we have*

$$\frac{1}{W_m}\sum_{i=1}^m w_i F(x_i) - F(\overline{x})$$
$$\geq \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\overline{x},s)\right]\sum_{j=1}^r\left[F''(a_j)H_{0j}(s) + F'''(a_j)H_{1j}(s)\right] ds,$$

where

$$H_{0j}(s) = \frac{P_r^2(s)}{(s-a_j)^2\,[P_r'(a_j)]^2}\left(1 - \frac{P_r''(a_j)}{P_r'(a_j)}(s-a_j)\right)$$
$$H_{1j}(s) = \frac{P_r^2(s)}{(s-a_j)\,[P_r'(a_j)]^2},$$

and

$$P_r(s) = \prod_{j=1}^r (s-a_j).$$

Proof. We put $k_j = 1$ for $j = 1, \ldots, r$ in Theorem 2.85. □

In the following remark we give the integral version of Theorem 2.85.

Remark 2.22 *For the given points* $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $r \geq 2$, $x : [a,b] \to \mathbb{R}$ *continuous decreasing function, such that* $x([a,b]) \subseteq [\alpha, \beta]$, $\lambda : [a,b] \to \mathbb{R}$ *increasing, bounded function with* $\lambda(a) \neq \lambda(b)$ *and* $\overline{x} = \frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$, *for* $x(c) \geq \overline{x}$, *we have:*

$$\int_a^c x(t)\,d\lambda(t) \geq \int_a^c x(c)\,d\lambda(t) \geq \int_a^c \overline{x}\,d\lambda(t),\ c \in [a,b].$$

If $x(c) < \overline{x}$ *we have*

$$\begin{aligned}\int_a^c x(t)\,d\lambda(t) &= \int_a^b x(t)\,d\lambda(t) - \int_c^b x(t)\,d\lambda(t)\\ &> \int_a^b \overline{x}\,d\lambda(t) - \int_c^b \overline{x}\,d\lambda(t) = \int_a^c \overline{x}\,d\lambda(t),\ c \in [a,b].\end{aligned}$$

Equality

$$\int_a^b x(t)\,d\lambda(t) = \int_a^b \overline{x}\,d\lambda(t)$$

obviously holds, so majorization conditions (1.27) and (1.28) are satisfied.
Consider the inequality:

$$\begin{aligned}&\frac{\int_a^b F(x(t))\,d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) \\ &\geq \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\,d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x},s)\right] \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s)ds,\end{aligned} \tag{2.230}$$

where H_{lj} *is as defined in (2.137) and* $F : [\alpha, \beta] \to \mathbb{R}$ *is n-convex function.*

(i) If k_j *is odd for every* $j = 2, \ldots, r$, *then the inequality (2.230) holds.*

(ii) If k_j *is odd for every* $j = 2, \ldots, r-1$ *and* k_r *is even, then the reverse inequality in (2.230) holds.*

If the inequality (reverse inequality) in (2.230) holds and the function $\phi(.) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(.)$ *is non negative (non positive), then the right hand side of (2.230) will be non negative (non positive), that is the inequality (reverse inequality)*

$$\frac{\int_a^b F(x(t))\,d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) \geq 0 \tag{2.231}$$

holds.

Remark 2.23 *Motivated by the inequalities (2.226) and (2.230), we define functionals $\Theta_1(F)$ and $\Theta_2(F)$, by*

$$\begin{aligned}\Theta_1(F) &= \frac{1}{W_m}\sum_{i=1}^{m} w_i F(x_i) - F(\overline{x}) \\ &- \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^{m} w_i G(x_i,s) - G(\overline{x},s)\right] \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds,\end{aligned}$$

$$\begin{aligned}\Theta_2(F) &= \frac{\int_a^b F(x(t))\, d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) \\ &- \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\, d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x},s)\right] \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s) ds,\end{aligned}$$

Similarly as in [29] we can construct new families of exponentially convex function and Cauchy type means by looking at these linear functionals. The monotonicity property of the generalized Cauchy means obtained via these functionals can be prove by using the properties of the linear functionals associated with this error representation, such as n-exponential and logarithmic convexity.

Theorem 2.86 *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, let $\boldsymbol{x} = (x_1,\ldots,x_m)$ be decreasing real m-tuple with $x_i \in [a,b]$, $i = 1,\ldots,m$, let $\boldsymbol{w} = (w_1,\ldots,w_m)$ be real m-tuple such that $0 \leq W_k \leq W_m$, $k = 1,\ldots,m$, $W_m > 0$, where $W_k = \sum_{i=1}^{k} w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i$ and H_{lj} be as defined in (2.137). Let $F : [a,b] \to \mathbb{R}$ be n-convex function.*

(i) If k_j is odd for every $j = 2,\ldots,r$, then the inequality (2.226) holds.

(ii) If k_j is odd for every $j = 2,\ldots,r-1$ and k_r is even, then the reverse inequality in (2.226) holds.

If the inequality (reverse inequality) in (2.226) holds and the function $\phi(.) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(.)$ is non negative (non positive), then the right hand side of (2.226) will be non negative (non positive), that is the inequality (reverse inequality) (2.227) holds.

Proof. For $l = 1,\ldots,k$, such that $x_k \geq \overline{x}$ we have

$$\sum_{i=1}^{l} w_i x_i - W_l x_l = \sum_{i=1}^{l-1} (x_i - x_{i+1}) W_i \geq 0$$

and so we get

$$\sum_{i=1}^{l} w_i \overline{x} = W_l \overline{x} \leq W_l x_l \leq \sum_{i=1}^{l} w_i x_i.$$

For $l = k+1, \ldots, m-1$, such that $x_{k+1} < \overline{x}$ we have

$$x_l\,(W_m - W_l) - \sum_{i=l+1}^{m} w_i x_i = \sum_{i=l+1}^{m} (x_{i-1} - x_i)(W_m - W_{i-1}) \geq 0$$

and now
$$\sum_{i=l+1}^{m} w_i \overline{x} = (W_m - W_l)\,\overline{x} > (W_m - W_l)\,x_l \geq \sum_{i=l+1}^{m} w_i x_i.$$

So, similarly as in Theorem 2.85, we get that conditions (1.19) and (1.20) for majorization are satisfied, so inequalities (2.226) and (2.227) are valid. □

Remark 2.24 *For the given points $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $r \geq 2$, $x : [a,b] \to \mathbb{R}$ continuous, decreasing function, such that $x([a,b]) \subseteq [\alpha,\beta]$ and $\lambda : [a,b] \to \mathbb{R}$ is either continuous or of bounded variation satisfying $\lambda(a) \leq \lambda(t) \leq \lambda(b)$ for all $t \in [a,b]$, $\overline{x} = \frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$ and $F : [\alpha,\beta] \to \mathbb{R}$ n-convex function, for $x(c) \geq \overline{x}$, we have:*

$$\int_a^c x(t)d\lambda(t) - x(c)\int_a^c d\lambda(t) = -\int_a^c x'(t)\left(\int_a^t d\lambda(x)\right)dt \geq 0$$

and so
$$\overline{x}\int_a^c d\lambda(t) \leq x(c)\int_a^c d\lambda(t) \leq \int_a^c x(t)d\lambda(t).$$

If $x(c) < \overline{x}$ we have

$$x(c)\int_c^b d\lambda(t) - \int_c^b x(t)d\lambda(t) = -\int_c^b x'(t)\left(\int_t^b d\lambda(x)\right)dt \geq 0$$

and now
$$\overline{x}\int_c^b d\lambda(t) > x(c)\int_c^b d\lambda(t) \geq \int_c^b x(t)d\lambda(t).$$

Similarly as in the Remark 2.22 we get that conditions for majorization are satisfied, so inequalities (2.230) and (2.231) are valid.

Theorem 2.87 *Let $-\infty < a = a_1 < a_2 \cdots < a_r = b < \infty$, $r \geq 2$ be the given points, let $\boldsymbol{x} = (x_1, \ldots, x_p)$ be real p-tuple with $x_i \in [m,M] \subseteq [a,b]$, $i = 1, \ldots, p$, let $\boldsymbol{w} = (w_1, \ldots, w_p)$ be positive p-tuple such that $w_i \in \mathbb{R}$, $i = 1, \ldots, p$, $W_p = \sum_{i=1}^{p} w_i$, $\overline{x} = \frac{1}{W_p}\sum_{i=1}^{p} w_i x_i$ and H_{lj} be as defined in (2.137). Let $F : [a,b] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\begin{aligned}\frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) \leq{}& \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m) \\ &- \int_a^b \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W_p}\sum_{i=1}^{p} w_i G(x_i,s)\right] \\ &\cdot \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s)ds.\end{aligned} \tag{2.232}$$

(i) If k_j is odd for every $j = 2, \ldots, r$, then the inequality (2.232) holds.

(ii) If k_j is odd for every $j = 2,\dots,r-1$, and k_r is even, then the reverse inequality in (2.232) holds.

Moreover, if the inequality (reverse inequality) in (2.232) holds and the function $\phi(.) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(.)$ is non negative (non positive), then the right hand side of (2.232) will be non positive (non negative), that is the inequality (reverse inequality)

$$\frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) \le \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m) \tag{2.233}$$

holds.

Proof. Using inequality (2.226) we have

$$\begin{aligned}
&\frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) = \frac{1}{W_p}\sum_{i=1}^{p} w_i F\left(\frac{x_i-m}{M-m}M + \frac{M-x_i}{M-m}m\right)\\
&\le \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m)\\
&- \int_a^b \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W_p}\sum_{i=1}^{p} w_i G(x_i,s)\right]\\
&\cdot \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds.
\end{aligned}$$

For the inequality (2.233) we use the fact that for every convex function φ we have

$$\frac{1}{W_p}\sum_{i=1}^{p} w_i \varphi(x_i) \le \frac{\overline{x}-m}{M-m}\varphi(M) + \frac{M-\overline{x}}{M-m}\varphi(m).$$

□

Corollary 2.22 *Let $-\infty < m < a_2 \cdots < a_{r-1} < M < \infty$, $r \ge 2$ be the given points, let $\mathbf{x} = (x_1,\dots,x_p)$ be real p-tuple with $x_i \in [m,M]$, $i = 1,\dots,p$, let $\mathbf{w} = (w_1,\dots,w_p)$ be positive p-tuple such that $w_i \in \mathbb{R}$, $i = 1,\dots,p$, $W_p = \sum_{i=1}^{p} w_i$, $\overline{x} = \frac{1}{W_p}\sum_{i=1}^{p} w_i x_i$ and H_{lj} be as defined in (2.137). Let $F : [m,M] \to \mathbb{R}$ be n-convex function. Consider the inequality*

$$\begin{aligned}
\frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) &\le \frac{\overline{x}-m}{M-m}F(M) + \frac{M-\overline{x}}{M-m}F(m)\\
&+ \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)\frac{1}{W_p}\sum_{i=1}^{p} w_i \int_m^M G(x_i,s)H_{lj}(s)ds.
\end{aligned} \tag{2.234}$$

(i) If k_j is odd for every $j = 2,\dots,r$, then the inequality (2.234) holds.

(ii) If k_j is odd for every $j = 2,\dots,r-1$, and k_r is even, then the reverse inequality in (2.234) holds.

Proof. We use inequality (2.232) for $m = a = a_1$ and $M = b = a_r$. Therefore we get $G(m,s) = 0$ and $G(M,s) = 0$ and so obtain inequality (2.234). □

Remark 2.25 *For the given points* $-\infty < \alpha = a_1 < a_2 \cdots < a_r = \beta < \infty$, $r \geq 2$, $x : [a,b] \to \mathbb{R}$ *continuous function, such that* $x([a,b]) \subseteq [m,M] \subseteq [\alpha,\beta]$ *and* $\lambda : [a,b] \to \mathbb{R}$ *increasing, bounded function with* $\lambda(a) \neq \lambda(b)$, $\bar{x} = \frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$, H_{lj} *as defined in (2.137) and* $F : [\alpha,\beta] \to \mathbb{R}$ *n-convex function, consider the inequality*

$$\frac{\int_a^b F(x(t))d\lambda(t)}{\int_a^b d\lambda(t)} \leq \frac{\bar{x}-m}{M-m}F(M) + \frac{M-\bar{x}}{M-m}F(m) \tag{2.235}$$
$$-\int_\alpha^\beta \left[\frac{\bar{x}-m}{M-m}G(M,s) + \frac{M-\bar{x}}{M-m}G(m,s) - \frac{\int_a^b G(x(t),s)d\lambda(t)}{\int_a^b d\lambda(t)}\right] \cdot \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds.$$

(i) *If* k_j *is odd for every* $j = 2,\ldots,r$, *then the inequality (2.235) holds.*

(ii) *If* k_j *is odd for every* $j = 2,\ldots,r-1$, *and* k_r *is even, then the reverse inequality in (2.235) holds.*

Moreover, if the inequality (reverse inequality) in (2.235) holds and the function $\phi(.) = \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(.)$ *is non negative (non positive), then the right hand side of (2.235) will be non positive (non negative), that is the inequality (reverse inequality)*

$$\frac{\int_a^b F(x(t))d\lambda(t)}{\int_a^b d\lambda(t)} \leq \frac{\bar{x}-m}{M-m}F(M) + \frac{M-\bar{x}}{M-m}F(m)$$

holds.

Remark 2.26 *Motivated by the inequalities (2.232) and (2.235), we define functionals* $\Theta_3(F)$ *and* $\Theta_4(F)$ *by*

$$\Theta_3(F) = \frac{1}{W_p}\sum_{i=1}^{p} w_i F(x_i) - \frac{\bar{x}-m}{M-m}F(M) - \frac{M-\bar{x}}{M-m}F(m)$$
$$+\int_a^b \left[\frac{\bar{x}-m}{M-m}G(M,s) + \frac{M-\bar{x}}{M-m}G(m,s) - \frac{1}{W_p}\sum_{i=1}^{p} w_i G(x_i,s)\right] \cdot \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds$$

and

$$\Theta_4(F) = \frac{\int_a^b F(x(t))d\lambda(t)}{\int_a^b d\lambda(t)} - \frac{\bar{x}-m}{M-m}F(M) - \frac{M-\bar{x}}{M-m}F(m)$$
$$+\int_\alpha^\beta \left[\frac{\bar{x}-m}{M-m}G(M,s) + \frac{M-\bar{x}}{M-m}G(m,s) - \frac{\int_a^b G(x(t),s)d\lambda(t)}{\int_a^b d\lambda(t)}\right] \cdot \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds.$$

Now, we can observe the same results which are mentioned in Remark 2.23.

In the sequel we use the above theorems to obtain generalizations of the previous results.

For m-tuples $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ with $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and the Green function's G and $G_{H,n-2}$ as defined in (1.180) and (2.139), respectively, we denote

$$\Upsilon(t) = \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\overline{x},s)\right] G_{H,n-2}(s,t)ds,\ t \in [a,b]. \tag{2.236}$$

Similarly for $x : [a,b] \to [\alpha,\beta]$ continuous function, $\lambda : [a,b] \to \mathbb{R}$ as defined in Remark 2.22 or in Remark 2.24, the Green function's G and $G_{H,n-2}$ as defined in (1.180) and (2.139), respectively, and for all $s \in [\alpha,\beta]$ we denote

$$\tilde{\Upsilon}(t) = \int_\alpha^\beta \left[\frac{\int_a^b G(x(p),s)\,d\lambda(p)}{\int_a^b d\lambda(p)} - G(\overline{x},s)\right] G_{H,n-2}(s,t)ds,\ t \in [\alpha,\beta]. \tag{2.237}$$

Theorem 2.88 *Let $-\infty < a \le a_1 < a_2 \cdots < a_r \le b < \infty$, $r \ge 2$ be the given points, let $F : [a,b] \to \mathbb{R}$ be such that $F \in C^{n+1}[a,b]$ for $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1,\ldots,x_m)$, $\boldsymbol{w} = (w_1,\ldots,w_m)$ be m-tuples such that $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and let the functions H_{lj}, $l = 0,\ldots,k_j$, $j = 1,\ldots,r$, ω, G, Υ and functional T be defined in (2.137), (2.138), (1.180), (2.236) and (1.6), respectively. Then we have*

$$\begin{aligned}
&\frac{1}{W_m}\sum_{i=1}^m w_i F(x_i) - F(\overline{x})\\
&= \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\overline{x},s)\right] \sum_{j=1}^r\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds\\
&\quad + \frac{F^{(n-1)}(b) - F^{(n-1)}(a)}{b-a}\int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\overline{x},s)\right]\frac{\omega(s)}{(n-2)!}ds\\
&\quad + H_n^1(F;a,b)
\end{aligned} \tag{2.238}$$

where the remainder $H_n^1(F;a,b)$ satisfies the estimation

$$|\, H_n^1(F;a,b) \,| \le \frac{\sqrt{b-a}}{\sqrt{2}}\,[T(\Upsilon,\Upsilon]^{\frac{1}{2}} \left|\int_a^b (t-a)(b-t)\left[F^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}. \tag{2.239}$$

Proof. If we apply Theorem 1.10 for $f \to \Upsilon$ and $h \to F^{(n)}$ we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \Upsilon(t)F^{(n)}(t)dt - \frac{1}{b-a}\int_a^b \Upsilon(t)dt \cdot \frac{1}{b-a}\int_a^b F^{(n)}(t)dt\right|\\
&\quad \le \frac{1}{\sqrt{2}}\,[T(\Upsilon,\Upsilon)]^{\frac{1}{2}}\,\frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)\left[F^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.
\end{aligned}$$

Therefore we have

$$\int_a^b \Upsilon(t)F^{(n)}(t)dt = \frac{F^{(n-1)}(b) - F^{(n-1)}(a)}{b-a}\int_a^b \Upsilon(t)dt + H_n^1(F;a,b),$$

where the remainder $H_n^1(F;a,b)$ satisfies the estimation (2.433). Now, from Lemma 2.4 we obtain (2.238). $\square$

Integral case of the above theorem can be given as follows.

Theorem 2.89 *Let $-\infty < \alpha \leq a_1 < a_2 \cdots < a_r \leq \beta < \infty$, $r \geq 2$ be the given points, let $F : [\alpha,\beta] \to \mathbb{R}$ be such that $F \in C^{n+1}[\alpha,\beta]$ for $n \in \mathbb{N}$, let $x : [a,b] \to \mathbb{R}$ be continuous functions such that $x([a,b]) \subseteq [\alpha,\beta]$, $\lambda : [a,b] \to \mathbb{R}$ be as defined in Remark 2.22 or in Remark 2.24, $\bar{x} = \frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$ and let the functions H_{lj}, $l = 0,\ldots,k_j$, $j = 1,\ldots,r$, ω, G, $\tilde{\Upsilon}$ and functional T be defined in (2.137), (2.138), (1.180), (2.237) and (1.6). Then we have*

$$\begin{aligned}
&\frac{\int_a^b F(x(t))d\lambda(t)}{\int_a^b d\lambda(t)} - F(\bar{x}) \\
&= \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\,d\lambda(t)}{\int_a^b d\lambda(t)} - G(\bar{x},s)\right] \sum_{j=1}^{r}\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds \\
&\quad + \frac{F^{(n-1)}(\beta) - F^{(n-1)}(\alpha)}{\beta-\alpha} \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)d\lambda(t)}{\int_a^b d\lambda(t)} - G(\bar{x},s)\right] \frac{\omega(s)}{(n-2)!}ds \\
&\quad + \tilde{H}_n^1(F;\alpha,\beta)
\end{aligned} \tag{2.240}$$

where the remainder $\tilde{H}_n^1(F;\alpha,\beta)$ satisfies the estimation

$$|\tilde{H}_n^1(F;\alpha,\beta)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}} \left[T(\tilde{\Upsilon},\tilde{\Upsilon})\right]^{\frac{1}{2}} \left|\int_\alpha^\beta (s-\alpha)(\beta-s)\left[F^{(n+1)}(s)\right]^2 ds\right|^{\frac{1}{2}}.$$

Using Theorem 1.11 we also get the following Grüss type inequality.

Theorem 2.90 *Let $-\infty < a \leq a_1 < a_2 \cdots < a_r \leq b < \infty$, $r \geq 2$ be the given points, let $F : [a,b] \to \mathbb{R}$ be such that $F \in C^{n+1}[a,b]$ for $n \in \mathbb{N}$, $F^{(n+1)} \geq 0$ on $[a,b]$ and let $\boldsymbol{x} = (x_1,\ldots,x_m)$, $\boldsymbol{w} = (w_1,\ldots,w_m)$ be m-tuples such that $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$, $\bar{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$ and let the function Υ be defined in (2.236). Then we have the representation (2.238) and the remainder $H_n^1(F;a,b)$ satisfies the bound*

$$|H_n^1(F;a,b)| \leq \frac{\|\Upsilon'\|_\infty}{2}\left\{(b-a)\left[F^{(n-1)}(b)+F^{(n-1)}(a)\right] - \left[F^{(n-2)}(b)-F^{(n-2)}(a)\right]\right\}. \tag{2.241}$$

Proof. Applying Theorem 1.11 for $f \to \Upsilon$ and $h \to F^{(n)}$ we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \Upsilon(t)F^{(n)}(t)dt - \frac{1}{b-a}\int_a^b \Upsilon(t)dt \cdot \frac{1}{b-a}\int_a^b F^{(n)}(t)dt\right| \\
&\quad \leq \frac{1}{2(b-a)}\|\Upsilon'\|_\infty \int_a^b (t-a)(b-t)F^{(n+1)}(t)dt.
\end{aligned} \tag{2.242}$$

Since

$$\begin{aligned}
\int_a^b (t-a)(b-t)F^{(n+1)}(t)dt &= \int_a^b [2t-(a+b)]F^{(n)}(t)dt \\
&= (b-a)\left[F^{(n-1)}(b)+F^{(n-1)}(a)\right] - 2\left[F^{(n-2)}(b)-F^{(n-2)}(a)\right],
\end{aligned}$$

using the identity (2.223) and (2.242) we deduce (2.241). □

Integral version of the above theorem can be given as follows.

Theorem 2.91 *Let* $-\infty < \alpha \le a_1 < a_2 \cdots < a_r \le \beta < \infty$, $r \ge 2$ *be the given points, let* $F : [\alpha, \beta] \to \mathbb{R}$ *be such that* $F \in C^{n+1}[\alpha, \beta]$ *for* $n \in \mathbb{N}$ *and* $F^{(n+1)} \ge 0$ *on* $[\alpha, \beta]$, *let* $x : [a,b] \to \mathbb{R}$ *be continuous functions such that* $x([a,b]) \subseteq [\alpha, \beta]$, $\lambda : [a,b] \to \mathbb{R}$ *be as defined in Remark 2.22 or in Remark 2.24,* $\bar{x} = \frac{\int_a^b x(t)\,d\lambda(t)}{\int_a^b d\lambda(t)}$ *and let the function* $\tilde{\Upsilon}$ *be defined in (2.237). Then we have the representation (2.240) and the remainder* $\tilde{H}_n^1(F;\alpha,\beta)$ *satisfies the bound*

$$|\,\tilde{H}_n^1(F;\alpha,\beta)\,| \le \frac{\|\Upsilon'\|_\infty}{2}\left\{(\beta-\alpha)\left[F^{(n-1)}(\beta)+F^{(n-1)}(\alpha)\right]-\left[F^{(n-2)}(\beta)-F^{(n-2)}(\alpha)\right]\right\}.$$

We also give the Ostrowsky type inequality related to the generalization of majorization inequality.

Theorem 2.92 *Let* $-\infty < a \le a_1 < a_2 \cdots < a_r \le b < \infty$ $r \ge 2$ *be the given points, let* $\boldsymbol{x} = (x_1,\ldots,x_m)$ *and* $\mathbf{w} = (w_1,\ldots,w_m)$ *be m-tuples such that* $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1,\ldots,m$, $W_m = \sum_{i=1}^m w_i$, $\bar{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i$. *Let* (p,q) *be a pair of conjugate exponents, that is* $1 \le p,q \le \infty$ *and* $\frac{1}{p}+\frac{1}{q} = 1$ *and let* $F \in C^n[a,b]$. *Also, let* H_{lj} *and* Υ *be as defined in (2.137) and (2.236) respectively.*
Then we have

$$\left|\frac{1}{W_m}\sum_{i=1}^m w_i F(x_i) - F(\bar{x}) - \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\bar{x},s)\right]\sum_{j=1}^r\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds\right| \le ||F^{(n)}||_p||\Upsilon||_q. \tag{2.243}$$

The constant on the right hand side of (2.243) is sharp for $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. Using the identity (2.223) and applying Hölder's inequality we obtain

$$\left|\frac{1}{W_m}\sum_{i=1}^m w_i F(x_i) - F(\bar{x}) - \int_a^b \left[\frac{1}{W_m}\sum_{i=1}^m w_i G(x_i,s) - G(\bar{x},s)\right]\sum_{j=1}^r\sum_{l=0}^{k_j} F^{(l+2)}(a_j)H_{lj}(s)ds\right| = \left|\int_a^b \Upsilon(t)F^{(n)}(t)dt\right| \le ||F^{(n)}||_p||\Upsilon||_q.$$

For the proof of the sharpness of the constant $||\Upsilon||_q$ let us find a function F for which the equality in (2.243) is obtained.
For $1 < p < \infty$ take F to be such that

$$F^{(n)}(t) = sgn\,\Upsilon(t)\,|\Upsilon(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $F^{(n)}(t) = sgn\,\Upsilon(t)$.
For $p = 1$ we prove that

$$\left|\int_a^b \Upsilon(t)F^{(n)}(t)dt\right| \le \max_{t\in[a,b]}|\Upsilon(t)|\left(\int_a^b \left|F^{(n)}(t)\right|dt\right) \tag{2.244}$$

is the best possible inequality. Suppose that $|\Upsilon(t)|$ attains its maximum at $t_0 \in [a,b]$. First we assume that $\Upsilon(t_0) > 0$. For ε small enough we define $F_\varepsilon(t)$ by

$$F_\varepsilon(t) = \begin{cases} 0, & a \le t \le t_0, \\ \frac{1}{\varepsilon n!}(t-t_0)^n, & t_0 \le t \le t_0+\varepsilon, \\ \frac{1}{(n-1)!}(t-t_0)^{n-1}, & t_0+\varepsilon \le t \le b. \end{cases}$$

Then for ε small enough

$$\left| \int_a^b \Upsilon(t) F^{(n)}(t)dt \right| = \left| \int_{t_0}^{t_0+\varepsilon} \Upsilon(t) \frac{1}{\varepsilon} dt \right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \Upsilon(t)dt.$$

Now from the inequality (2.244) we have

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \Upsilon(t)dt \le \Upsilon(t_0) \int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon} dt = \Upsilon(t_0).$$

Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \Upsilon(t)dt = \Upsilon(t_0)$$

the statement follows. In the case $\Upsilon(t_0) < 0$, we define $F_\varepsilon(t)$ by

$$F_\varepsilon(t) = \begin{cases} \frac{1}{(n-1)!}(t-t_0-\varepsilon)^{n-1}, & a \le t \le t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \le t \le t_0+\varepsilon, \\ 0, & t_0+\varepsilon \le t \le b, \end{cases}$$

and the rest of the proof is the same as above. □

Integral version of the above theorem can be stated as follows.

Theorem 2.93 *Let $-\infty < \alpha \le a_1 < a_2 \cdots < a_r \le \beta < \infty$, $r \ge 2$ be the given points, let $x : [a,b] \to \mathbb{R}$ be continuous functions such that $x([a,b]) \subseteq [\alpha,\beta]$, $\lambda : [a,b] \to \mathbb{R}$ be as defined in Remark 2.22 or in Remark 2.24 and $\overline{x} = \frac{\int_a^b x(t)\, d\lambda(t)}{\int_a^b d\lambda(t)}$. Let (p,q) be a pair of conjugate exponents, that is $1 \le p,q \le \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $F \in C^n[\alpha,\beta]$ and let the H_{lj} and $\tilde{\Upsilon}$ be defined in (2.137) and (2.237).*
Then we have

$$\left| \frac{\int_a^b F(x(t))\, d\lambda(t)}{\int_a^b d\lambda(t)} - F(\overline{x}) - \int_\alpha^\beta \left[\frac{\int_a^b G(x(t),s)\, d\lambda(t)}{\int_a^b d\lambda(t)} - G(\overline{x},s) \right] \sum_{j=1}^r \sum_{l=0}^{k_j} F^{(l+2)}(a_j) H_{lj}(s)ds \right|$$
$$\le ||F^{(n)}||_p ||\tilde{\Upsilon}||_q. \tag{2.245}$$

The constant on the right hand side of (2.245) is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

2.3 Majorization and Taylor's Formula

In this section, we give generalization of majorization theorems for the class of n-convex functions by using interpolation by Taylor's polynomials. We use inequalities for the Čebyšev functional to obtain bounds for the identities related to generalizations of majorization inequalities obtained by Taylor's interpolation. We also give analogous results as in the previous subsections in the form of the mean value theorems, exponential convexity and Cauchy's type of means.

2.3.1 Results Obtained by Taylor Formula

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Theorem 2.94 *Let n be a positive integer and $\phi : [a,b] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, then for all $x \in [a,b]$* ***Taylor's formula*** *at the point $c \in [a,b]$ is*

$$\phi(x) = T_{n-1}(\phi;c,x) + R_{n-1}(\phi;c,x), \tag{2.246}$$

where $T_{n-1}(\phi;c,x)$ is a Taylor's polynomial of degree n-1, i.e.

$$T_{n-1}(\phi;c,x) = \sum_{k=0}^{n-1} \frac{\phi^{(k)}(c)}{k!}(x-c)^k \tag{2.247}$$

and the remainder in the integral form is given by

$$R_{n-1}(\phi;c,x) = \frac{1}{(n-1)!}\int_c^x \phi^{(n)}(t)(x-t)^{n-1}dt.$$

Remark 2.27 *Due to absolute continuity of $\phi^{(n-1)}$ on $[\vartheta_1, \vartheta_2]$, its derivative $\phi^{(n)}$ exists as an L_1 function.*

There are two other important expressions for the remainder in Taylor's formula in terms of the magnitude of the n-th derivative of ϕ given by

$$R_{n-1}(\phi;a,x) = \frac{\phi^{(n)}(t)}{(n-1)!}(x-t)^{n-1}(x-a), \tag{2.248}$$

known as Cauchy's form and

$$R_{n-1}(\phi;a,x) = \frac{\phi^{(n)}(t)}{n!}(x-a)^n, \tag{2.249}$$

known as Lagrange's form of the remainder. The statement for the integral form of the remainder is more advanced than these.

In the rest of this subsection, we need the following real valued function of our interest defined as:

$$(x-t)_+ = \begin{cases} (x-t), & t \leq x, \\ 0, & t > x. \end{cases} \tag{2.250}$$

Next we give identities obtained by using Taylor's formula.

Theorem 2.95 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 1$ and let $\mathbf{w} = (w_1,\ldots,w_m)$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that $x_i, y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$. Then*

$$\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) = \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k\right)$$
$$+\frac{1}{(n-1)!}\int_\alpha^\beta \left[\sum_{i=1}^m w_i((x_i-t)_+)^{n-1} - \sum_{i=1}^m w_i((y_i-t)_+)^{n-1}\right]\phi^{(n)}(t)dt, \tag{2.251}$$

and

$$\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i)$$
$$= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(\beta-x_i)^k - \sum_{i=1}^m w_i(\beta-y_i)^k\right)(-1)^k$$
$$-\frac{1}{(n-1)!}\int_\alpha^\beta (-1)^{n-1}\left[\sum_{i=1}^m w_i((t-x_i)_+)^{n-1} - \sum_{i=1}^m w_i((t-y_i)_+)^{n-1}\right]\phi^{(n)}(t)dt. \tag{2.252}$$

Proof. Using Taylor's formula at point α in $\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i)$, we have

$$\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i)$$
$$= \sum_{i=1}^m w_i\left(\sum_{k=0}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(x_i-\alpha)^k + \frac{1}{(n-1)!}\int_\alpha^{x_i} \phi^{(n)}(t)(x_i-t)^{n-1}dt\right)$$
$$-\sum_{i=1}^m w_i\left(\sum_{k=0}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(y_i-\alpha)^k + \frac{1}{(n-1)!}\int_\alpha^{y_i} \phi^{(n)}(t)(y_i-t)^{n-1}dt\right)$$
$$= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k\right)$$
$$+\frac{1}{(n-1)!}\int_\alpha^{x_i}\sum_{i=1}^m w_i(x_i-t)^{n-1}\phi^{(n)}(t)dt - \frac{1}{(n-1)!}\int_\alpha^{y_i}\sum_{i=1}^m w_i(y_i-t)^{n-1}\phi^{(n)}(t)dt$$
$$= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k\right)$$
$$+\frac{1}{(n-1)!}\int_\alpha^\beta\sum_{i=1}^m w_i((x_i-t)_+)^{n-1}\phi^{(n)}(t)dt - \frac{1}{(n-1)!}\int_\alpha^\beta\sum_{i=1}^m w_i((y_i-t)_+)^{n-1}\phi^{(n)}(t)dt, \tag{2.253}$$

where

$$\int_\alpha^\beta \sum_{i=1}^m w_i((x_i-t)_+)^{n-1}\phi^{(n)}(t)dt = \int_\alpha^{x_i} \sum_{i=1}^m w_i(x_i-t)^{n-1}\phi^{(n)}(t)dt + \int_{x_i}^\beta 0,$$

and

$$\int_\alpha^\beta \sum_{i=1}^m w_i((y_i-t)_+)^{n-1}\phi^{(n)}(t)dt = \int_\alpha^{y_i} \sum_{i=1}^m w_i(y_i-t)^{n-1}\phi^{(n)}(t)dt + \int_{y_i}^\beta 0.$$

So by using above result we will get (2.251).

Similarly using Taylor's formula at point β in $\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i)$, we have

$$\begin{aligned}
&\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) \\
&\quad = \sum_{i=1}^m w_i\left(\sum_{k=0}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}(x_i-\beta)^k - \frac{1}{(n-1)!}\int_{x_i}^\beta \phi^{(n)}(t)(x_i-t)^{n-1}dt\right) \\
&\quad - \sum_{i=1}^m w_i\left(\sum_{k=0}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}(y_i-\beta)^k - \frac{1}{(n-1)!}\int_{y_i}^\beta \phi^{(n)}(t)(y_i-t)^{n-1}dt\right) \\
&\quad = \sum_{k=1}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(x_i-\beta)^k - \sum_{i=1}^m w_i(y_i-\beta)^k\right) \\
&\quad - \frac{1}{(n-1)!}\left[\int_{x_i}^\beta \sum_{i=1}^m w_i(x_i-t)^{n-1} - \int_{y_i}^\beta \sum_{i=1}^m w_i(y_i-t)^{n-1}\right]\phi^{(n)}(t)dt \\
&\quad = \sum_{k=1}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(\beta-x_i)^k - \sum_{i=1}^m w_i(\beta-y_i)^k\right)(-1)^k \\
&\quad - \frac{1}{(n-1)!}\int_\alpha^\beta (-1)^{n-1}\sum_{i=1}^m w_i((t-x_i)_+)^{n-1}\phi^{(n)}(t)dt \\
&\quad + \frac{1}{(n-1)!}\int_\alpha^\beta (-1)^{n-1}\sum_{i=1}^m w_i((t-y_i)_+)^{n-1}\phi^{(n)}(t)dt,
\end{aligned} \tag{2.254}$$

where

$$\int_\alpha^\beta (-1)^{n-1}\sum_{i=1}^m w_i((t-x_i)_+)^{n-1}\phi^{(n)}(t)dt = \int_\alpha^{x_i} o + \int_{x_i}^\beta (-1)^{n-1}\sum_{i=1}^m w_i(t-x_i)^{n-1}\phi^{(n)}(t)dt,$$

$$\int_\alpha^\beta \sum_{i=1}^m w_i((t-y_i)_+)^{n-1}\phi^{(n)}(t)dt = \int_\alpha^{y_i} o + \int_{y_i}^\beta (-1)^{n-1}\sum_{i=1}^m w_i(t-y_i)^{n-1}\phi^{(n)}(t)dt.$$

So by using above result we will get (2.252). □

Integral version of the above theorem can be stated as follows.

Theorem 2.96 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 1$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions. Then*

$$\begin{aligned}\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\ = \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\int_a^b w(\tau)\left[(x(\tau)-\alpha)^k - (y(\tau)-\alpha)^k\right]d\tau\right) \\ + \frac{1}{(n-1)!}\int_\alpha^\beta \left(\int_a^b w(\tau)\left[((x(\tau)-t)_+)^{n-1} - ((y(\tau)-t)_+)^{n-1}\right]d\tau\right)\phi^{(n)}(t)dt, \quad (2.255)\end{aligned}$$

and

$$\begin{aligned}\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\ = \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\int_a^b w(\tau)\left[(\beta - x(\tau))^k - (\beta - y(\tau))^k\right]d\tau\right)(-1)^k \\ - \frac{1}{(n-1)!}\int_\alpha^\beta (-1)^{n-1}\left(\int_a^b w(\tau)\left[((t-x(\tau))_+)^{n-1} - ((t-y(\tau))_+)^{n-1}\right]d\tau\right)\phi^{(n)}(t)dt. \\ (2.256)\end{aligned}$$

In the following theorem we obtain generalizations of majorization inequality for n-convex functions.

Theorem 2.97 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 1$ and let $\mathbf{w} = (w_1,\ldots,w_m)$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that $x_i, y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$. Then*

(i) If ϕ is n-convex function and

$$\sum_{i=1}^m w_i((x_i - t)_+)^{n-1} - \sum_{i=1}^m w_i((y_i - t)_+)^{n-1} \geq 0, \quad t \in [\alpha,\beta], \quad (2.257)$$

then

$$\begin{aligned}\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) \\ \geq \sum_{k=1}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k\right). \quad (2.258)\end{aligned}$$

(ii) If ϕ is n-convex function and

$$(-1)^{n-1}\left(\sum_{i=1}^m w_i((t-x_i)_+)^{n-1} - \sum_{i=1}^m w_i((t-y_i)_+)^{n-1}\right) \leq 0, \quad t \in [\alpha,\beta], \quad (2.259)$$

then

$$\sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i)$$
$$\geq \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^{m} w_i(\beta - x_i)^k - \sum_{i=1}^{m} w_i(\beta - y_i)^k\right)(-1)^k. \quad (2.260)$$

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)} \geq 0$ see [[144], p. 16]. Hence we can apply Theorem 2.95 to obtain (2.258) and (2.260) respectively. □

Integral version of above theorem can be stated as follows.

Theorem 2.98 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 1$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions. Then*

(i) If ϕ is n-convex function and

$$\int_a^b w(\tau)\left[((x(\tau)-t)_+)^{n-1} - ((y(\tau)-t)_+)^{n-1}\right]d\tau \geq 0, \quad t \in [\alpha,\beta], \quad (2.261)$$

then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$\geq \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\int_a^b w(\tau)\left[(x(\tau)-\alpha)^k - (y(\tau)-\alpha)^k\right]d\tau\right). \quad (2.262)$$

(ii) If ϕ is n-convex function and

$$(-1)^{n-1}\left(\int_a^b w(\tau)\left[((t-x(\tau))_+)^{n-1} - ((t-y(\tau))_+)^{n-1}\right]d\tau \leq 0, \quad t \in [\alpha,\beta], \quad (2.263)$$

then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$\geq \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\int_a^b w(\tau)\left[(\beta - x(\tau))^k - (\beta - x(\tau))^k\right]d\tau\right)(-1)^k. \quad (2.264)$$

In the following Corollary, we give generalization of Fuch's majorization theorem.

Corollary 2.23 ([53]) *Let all the assumptions of Theorem 2.95 be satisfied, $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ be decreasing m-tuples and $\mathbf{w} = (w_1,\ldots,w_m)$ be any m-tuple such that $x_i, y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$ which satisfies* (1.19) *and* (1.20). *Also, consider $\phi : [\alpha,\beta] \to \mathbb{R}$ is n-convex function, then*

(i) For $n \geq 1$, (2.258) holds. Moreover, let the inequality (2.258) be satisfied. If the function

$$F_1(x) := \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(x-\alpha)^k. \tag{2.265}$$

is convex, the right hand side of (2.258) is non negative, that is (1.21) holds.

(ii) If n is even, then (2.260) holds. Moreover, let the inequality (2.260) be satisfied. If the function

$$F_2(x) := \sum_{k=1}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!}(\beta-x)^k. \tag{2.266}$$

is convex, the right hand side of (2.260) is non negative, that is (1.21) holds.

Proof. (i) The given tuples satisfies (1.19) and (1.20) and the function $((x-t)_+)^{n-1}$ is convex for given n. Hence by virtue of Theorem 1.14, (2.257) holds. Therefore by following Theorem 2.97 we can obtain (2.258). Moreover, we can rewrite the right hand side of (2.258) in the form of the left hand side with $\phi = F_1$, where F_1 is defined in (2.265) and will be obtained after reorganization of this side. Since F_1 is assumed to be convex, therefore using the given conditions on m-tuples and by following Theorem 1.14 the non negativity of right hand side of (2.258) is immediate, that is (1.21) holds.
Similarly , we can prove the other part. □

The following generalization of integral majorization theorem holds.

Corollary 2.24 ([53]) *Let all the assumptions of Theorem 2.96 be satified and let $x,y : [a,b] \to [\alpha,\beta]$ be decreasing and $w : [a,b] \to \mathbb{R}$ be any continuous functions such that* (1.27) *and* (1.28) *hold. Also, consider $\phi : [\alpha,\beta] \to \mathbb{R}$ is n-convex function, then*

(i) For $n \geq 1$, (2.262) holds. Moreover, let the inequality (2.262) be satisfied. If the function F_1 defined in (2.265) is convex, the right hand side of (2.262) is non negative, that is (1.29) holds.

(ii) If n is even, then (2.264) holds. Moreover, let the inequality (2.264) be satisfied. If the function F_2 defined in (2.266) is convex, the right hand side of (2.264) is non negative, that is (1.29) holds.

In the sequel (see [53]), we consider above theorems to derive generalizations of the previous results. Let $\boldsymbol{w} = (w_1,\ldots,w_n)$, $\boldsymbol{x} = (x_1,\ldots,x_n)$ and $\boldsymbol{y} = (y_1,\ldots,y_n)$ be n-tuples such that $x_i, y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,n)$,denote

$$\mathfrak{R}(t) = \sum_{i=1}^{m} w_i((x_i-t)_+)^{n-1} - \sum_{i=1}^{m} w_i((y_i-t)_+)^{n-1},\ t \in [\alpha,\beta], \tag{2.267}$$

$$\mathfrak{B}(t) = (-1)^{n-1}\left(\sum_{i=1}^{m} w_i((t-x_i)_+)^{n-1} - \sum_{i=1}^{m} w_i((t-y_i)_+)^{n-1}\right),\ t \in [\alpha,\beta]. \tag{2.268}$$

Similarly for continuous functions $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$, denote

$$\hat{\mathfrak{R}}(t) = \int_a^b w(\tau)\left[((x(\tau)-t)_+)^{n-1} - ((y(\tau)-t)_+)^{n-1}\right]d\tau \geq 0,\quad t \in [\alpha,\beta], \tag{2.269}$$

$$\hat{\mathfrak{B}}(t) = (-1)^{n-1}\left(\int_a^b w(\tau)\left[((t-x(\tau))_+)^{n-1} - ((t-y(\tau))_+)^{n-1}\right]d\tau\right) \leq 0, \quad t \in [\alpha,\beta]. \tag{2.270}$$

Consider the Čebyšev functionals $T(\mathfrak{R},\mathfrak{R})$,$T(\mathfrak{B},\mathfrak{B})$, $T(\hat{\mathfrak{R}},\hat{\mathfrak{R}})$ and $T(\hat{\mathfrak{B}},\hat{\mathfrak{B}})$ given as:

$$T(\mathfrak{R},\mathfrak{R}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)dt\right)^2, \tag{2.271}$$

$$T(\mathfrak{B},\mathfrak{B}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{B}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{B}(t)dt\right)^2, \tag{2.272}$$

$$T(\hat{\mathfrak{R}},\hat{\mathfrak{R}}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \hat{\mathfrak{R}}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \hat{\mathfrak{R}}(t)dt\right)^2, \tag{2.273}$$

$$T(\hat{\mathfrak{B}},\hat{\mathfrak{B}}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \hat{\mathfrak{B}}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \hat{\mathfrak{B}}(t)dt\right)^2. \tag{2.274}$$

Theorem 2.99 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous for some $n \geq 1$ with $(.-\alpha)(\beta-.)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and let $\mathbf{w} = (w_1,\ldots,w_n)$, $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ be n-tuples such that $x_i, y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,n)$ and let the functions $\mathfrak{R}$, $\mathfrak{B}$ be defined by* (2.267), (2.268) *respectively . Then*
(i)

$$\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) = \sum_{k=1}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k\right) + \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-1)!}\int_\alpha^\beta \mathfrak{R}(t)dt + \mathfrak{K}_n^1(\alpha,\beta;\phi), \tag{2.275}$$

where the remainder $\mathfrak{K}_n^1(\alpha,\beta;\phi)$ satisfies the estimation

$$|\mathfrak{K}_n^1(\alpha,\beta;\phi)| \leq \frac{1}{(n-1)!}[T(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \tag{2.276}$$

(ii)

$$\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) = \sum_{k=1}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(\beta-x_i)^k - \sum_{i=1}^m w_i(\beta-y_i)^k\right)(-1)^k + \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-1)!}\int_\alpha^\beta \mathfrak{B}(t)dt - \mathfrak{K}_n^2(\alpha,\beta;\phi), \tag{2.277}$$

where the remainder $\mathfrak{K}_n^2(\alpha,\beta;\phi)$ satisfies the estimation

$$|\mathfrak{K}_n^2(\alpha,\beta;\phi)| \leq \frac{1}{(n-1)!}[T(\mathfrak{B},\mathfrak{B})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \tag{2.278}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

Integral case of above theorem can be given as:

Theorem 2.100 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous for some $n \geq 1$ with $(. - \alpha)(\beta - .)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions. Also let the functions $\hat{\mathfrak{R}}$, $\hat{\mathfrak{B}}$ be defined by* (2.269), (2.270) *respectively . Then*
(*i*)

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\int_a^b w(\tau)\left[(x(\tau)-\alpha)^k - (y(\tau)-\alpha)^k\right]d\tau\right)$$
$$+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-1)!}\int_\alpha^\beta \hat{\mathfrak{R}}(t)dt + \hat{\mathfrak{K}}_n^1(\alpha,\beta;\phi), \tag{2.279}$$

where the remainder $\hat{\mathfrak{K}}_n^1(\alpha,\beta;\phi)$ satisfies the estimation

$$|\hat{\mathfrak{K}}_n^1(\alpha,\beta;\phi)| \leq \frac{1}{(n-1)!}[T(\hat{\mathfrak{R}},\hat{\mathfrak{R}})]^{\frac{1}{2}} \left|\sqrt{\frac{\beta-\alpha}{2}}\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{2.280}$$

(*ii*)

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\int_a^b w(\tau)\left[(\beta - x(\tau))^k - (\beta - x(\tau))^k\right]d\tau\right)(-1)^k$$
$$+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-1)!}\int_\alpha^\beta \hat{\mathfrak{B}}(t)dt - \hat{\mathfrak{K}}_n^2(\alpha,\beta;\phi), \tag{2.281}$$

where the remainder $\hat{\mathfrak{K}}_n^2(\alpha,\beta;\phi)$ satisfies the estimation

$$|\hat{\mathfrak{K}}_n^2(\alpha,\beta;\phi)| \leq \frac{1}{(n-1)!}[T(\hat{\mathfrak{B}},\hat{\mathfrak{B}})]^{\frac{1}{2}} \sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{2.282}$$

The following Grüss type inequalities can be obtained by using Theorem 1.11:

Theorem 2.101 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ $(n \geq 1)$ is absolutely continuous function and $\phi^{(n+1)} \geq 0$ on $[\alpha,\beta]$ and let the functions $\mathfrak{R}, \mathfrak{B}$ be defined by* (2.267), (2.268) *respectively. Then, we have*
(*i*) *the representation* (2.275) *and the remainder $\mathfrak{K}_n^1(\alpha,\beta;\phi)$ satisfies the bound*

$$|\mathfrak{K}_n^1(\alpha,\beta;\phi)| \leq \frac{(\beta-\alpha)}{(n-1)!}||\mathfrak{R}'||_\infty\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{2.283}$$

(ii) The representation (2.277) *and the remainder* $\mathfrak{K}_n^2(\alpha,\beta;\phi)$ *satisfies the bound*

$$|\mathfrak{K}_n^2(\alpha,\beta;\phi)| \leq \frac{(\beta-\alpha)}{(n-1)!}||\mathfrak{B}'||_\infty \left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{2.284}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

Integral case of above theorem can be given as:

Theorem 2.102 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ $(n \geq 1)$ is absolutely continuous function and $\phi^{(n+1)} \geq 0$ on $[\alpha,\beta]$ and let the functions $\hat{\mathfrak{R}}, \hat{\mathfrak{B}}$ be defined by* (2.269), (2.270) *respectively. Then, we have*
(i) the representation (2.279) *and the remainder* $\hat{\mathfrak{K}}_n^1(\alpha,\beta;\phi)$ *satisfies the bound*

$$|\hat{\mathfrak{K}}_n^1(\alpha,\beta;\phi)| \leq \frac{(\beta-\alpha)}{(n-1)!}||\hat{\mathfrak{R}}'||_\infty \left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{2.285}$$

(ii) The representation (2.281) *and the remainder* $\hat{\mathfrak{K}}_n^2(\alpha,\beta;\phi)$ *satisfies the bound*

$$|\hat{\mathfrak{K}}_n^2(\alpha,\beta;\phi)| \leq \frac{(\beta-\alpha)}{(n-1)!}||\hat{\mathfrak{B}}'||_\infty \left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{2.286}$$

Now we intend to give the Ostrowski type inequalities related to generalizations of majorization's inequality.

Theorem 2.103 ([53]) *Suppose all the assumptions of Theorem 2.95 hold. Moreover, assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $|\phi^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ be a R-integrable function for some $n \geq 1$. Then, we have:*
(i)

$$\begin{aligned}&\left|\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) - \sum_{k=1}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k\right)\right| \\ &\quad \leq \frac{1}{(n-1)!}||\phi^{(n)}||_p \left(\int_\alpha^\beta \left|\sum_{i=1}^m w_i((x_i-t)_+)^{n-1} - \sum_{i=1}^m w_i((y_i-t)_+)^{n-1}\right|^q dt\right)^{1/q}. \end{aligned} \tag{2.287}$$

The constant on the right hand side of (2.287) *is sharp for $1 < p \leq \infty$ and the optimal for $p = 1$.*
(ii)

$$\begin{aligned}&\left|\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) - \sum_{k=1}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(\beta-x_i)^k - \sum_{i=1}^m w_i(\beta-y_i)^k\right)(-1)^k\right| \\ &\leq \frac{1}{(n-1)!}||\phi^{(n)}||_p \left(\int_\alpha^\beta \left|(-1)^{n-1}\left[\sum_{i=1}^m w_i((t-x_i)_+)^{n-1} - \sum_{i=1}^m w_i((t-y_i)_+)^{n-1}\right]\right|^q dt\right)^{1/q}. \end{aligned} \tag{2.288}$$

The constant on the right hand side of (2.288) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Integral case can be given as:

Theorem 2.104 ([53]) *Suppose all the assumptions of Theorem 2.96 hold. Moreover, assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$. *Let* $|\phi^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ *be a R-integrable function for some* $n \geq 2$. *Then, we have:*
(i)

$$\left| \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \right.$$
$$\left. - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!} \left(\int_a^b w(\tau) \left[(x(\tau)-\alpha)^k - (y(\tau)-\alpha)^k \right] d\tau \right) \right|$$
$$\leq \frac{1}{(n-1)!} ||\phi^{(n)}||_p \left(\int_\alpha^\beta \left| \int_a^b w(\tau) \left[((x(\tau)-t)_+)^{n-1} - ((y(\tau)-t)_+)^{n-1} \right] d\tau \right|^q dt \right)^{1/q}. \tag{2.289}$$

The constant on the right hand side of (2.289) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.
(ii)

$$\left| \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \right.$$
$$\left. - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!} \left(\int_a^b w(\tau) \left[(\beta - x(\tau))^k - (\beta - x(\tau))^k \right] d\tau \right) (-1)^k \right|$$
$$\leq \frac{1}{(n-1)!} ||\phi^{(n)}||_p \left(\int_\alpha^\beta \left| (-1)^{n-1} \left(\int_a^b w(\tau) \left[((t-x(\tau))_+)^{n-1} - ((t-y(\tau))_+)^{n-1} \right] d\tau \right) \right|^q dt \right)^{1/q}. \tag{2.290}$$

The constant on the right hand side of (2.290) *is sharp for* $1 < p \leq \infty$ *and the optimal for* $p = 1$.

Now, we construct several linear functionals as differences of the left hnd side and right hand side of some of the inequalities derived earlier. The obtained linear functionals will be used in the construction of new families of exponentially convex functions and some related results will be derived.

Remark 2.28 ([53]) *By virtue of Theorem 2.97 and 2.98 , we define the positive linear functionals with respect to n-convex function ϕ as follows*

$$\Omega_1(\phi) := \sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^{m} w_i(x_i-\alpha)^k - \sum_{i=1}^{m} w_i(y_i-\alpha)^k\right) \geq 0, \quad (2.291)$$

$$\Omega_2(\phi) := \sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^{m} w_i(\beta-x_i)^k - \sum_{i=1}^{m} w_i(\beta-y_i)^k\right)(-1)^k \geq 0, \quad (2.292)$$

$$\Omega_3(\phi) := \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\int_a^b w(\tau)\left[(x(\tau)-\alpha)^k - (y(\tau)-\alpha)^k\right]d\tau\right) \geq 0 \quad (2.293)$$

and

$$\Omega_4(\phi) := \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\int_a^b w(\tau)\left[(\beta-x(\tau))^k - (\beta-x(\tau))^k\right]d\tau\right)(-1)^k \geq 0. \quad (2.294)$$

The Lagrange and Cauchy type mean value theorems related to defined functionals are in the following theorems.

Theorem 2.105 ([53]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha,\beta]$. If the inequalities in* (2.258), (2.260), (2.262) *and* (2.264) *are valid, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\Omega_i(\phi) = \phi^{(n)}(\xi)\Omega_i(\varphi); \quad i = 1,\ldots,4,$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\Omega_i(\cdot)$ are defined in Remark 2.28.

Proof. Similar to the proof of Theorem 2.13 (also see Theorem 4.1 in [86] and [54])). □

Theorem 2.106 ([53]) *Let $\phi,\lambda : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi,\lambda \in C^n[\alpha,\beta]$. If the inequalities in* (2.258), (2.260), (2.262) *and* (2.264) *are valid, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\frac{\Omega_i(\phi)}{\Omega_i(\lambda)} = \frac{\phi^{(n)}(\xi)}{\lambda^{(n)}(\xi)}; \quad i = 1,\ldots,4,$$

provided that the denominators are non-zero and $\Omega_i(\cdot)$ are defined in Remark 2.28.

Proof. Similar to the proof of Theorem 2.14 (see also Corollary 4.2 in [86] and [54]). □

Theorem 2.258 enables us to define Cauchy means, because of

$$\xi_i = \left(\frac{\phi^{(n)}}{\lambda^{(n)}}\right)^{-1} \left(\frac{\Omega_i(\phi)}{\Omega_i(\lambda)}\right),$$

which show that ξ_i $(i = 1,\dots,4)$ are means of $[\alpha,\beta]$ for given functions ϕ and λ.

Next we construct the non trivial examples of n-exponentially and exponentially convex functions from positive linear functionals $\Omega_i(\cdot)$ $(i = 1,\dots,4)$. We use the idea given in [142].

Theorem 2.107 ([53]) *Let $\Theta = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0,\dots,x_n;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n \in I$. Then for the linear functionals $\Omega_i(\phi_t)$ $(i = 1,\dots,4)$ as defined in Remark 2.28, the following statements are valid for each $i = 1,\dots,4$:*

(i) The function $t \to \Omega_i(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \ge 0 \textit{ for all } m \in \mathbb{N},\ m = 1,2,\dots,n.$$

(ii) If the function $t \to \Omega_i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39. □

The following corollary is an immediate consequence of the above theorem:

Corollary 2.25 ([53]) *Let $\Theta = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\dots,x_n;\phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n \in I$. Then for the linear functional $\Omega_i(\phi_t)$ $(i = 1,\dots,4)$, the following statements hold:*

(i) The function $t \to \Omega_i(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \ge 0 \textit{ for all } m \in \mathbb{N},\ m = 1,2,\dots,n.$$

(ii) If the function $t \to \Omega_i(\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 2.26 ([53]) *Let $\Theta = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is $2-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in I$. Let $\Omega_i(\cdot)$ $(i=1,\ldots,4)$ be linear functionals, then the following statements hold:*

(i) If the function $t \mapsto \Omega_i(\phi_t)$ is continuous on J, then it is $2-$exponentially convex function on J. If $t \mapsto \Omega_i(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\Omega_i(\phi_s)]^{t-r} \leq [\Omega_i(\phi_r)]^{t-s}\,[\Omega_i(\phi_t)]^{s-r},$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \Omega_i(\phi_t)$ is strictly positive and differentiable on J, then for every $p,q,u,v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\Omega_i,\Theta) \leq \mu_{u,v}(\Omega_i,\Theta), \tag{2.295}$$

where

$$\mu_{p,q}(\Omega_i,\Theta) = \begin{cases} \left(\frac{\Omega_i(\phi_p)}{\Omega_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Omega_i(\phi_p)}{\Omega_i(\phi_p)}\right), & p = q, \end{cases} \tag{2.296}$$

for $\phi_p,\phi_q \in \Theta$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10. □

Remark 1.19 is also valid for these functionals.

Remark 2.29 ([53]) *Similar examples can be discussed as given in Section 1.4.*

2.3.2 Results Obtained by Green's Function and Taylor's Formula

In this section we utilize Taylor's theorem and Green's function (1.180) and establish generalization of majorization theorem for the class of n-convex functions. We give analogous results as in the previous subsection.

We begin with the next identities related to generalizations of majorization inequalities.

Theorem 2.108 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and let $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ be m-tuples such that x_i, $y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i=1,\ldots,m)$ and G be the Green function as defined in (1.180). Then*

$$\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(s-\alpha)^k ds$$
$$+\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\left(\int_{t}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt. \tag{2.297}$$

and

$$\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)$$
$$+\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(\beta-s)^k ds$$
$$-\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\left(\int_{\alpha}^{t}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt. \tag{2.298}$$

Proof. Using (1.181) in $\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i)$ we have

$$\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i)$$
$$= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i) + \int_{\alpha}^{\beta}\left[\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right]\phi''(s)ds. \tag{2.299}$$

Now applying Taylor's formula (2.246) to the function ϕ'' at the point α and replacing n by $n-2$ $(n\geq 3)$ we have

$$\phi''(s) = \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}(s-\alpha)^k + \frac{1}{(n-3)!}\int_{\alpha}^{s}\phi^{(n)}(t)(s-t)^{n-3}dt \tag{2.300}$$

and similarly Taylor's formula for ϕ'' at the point β and replacing n by $n-2$ $(n\geq 3)$ we have

$$\phi''(s) = \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\beta)}{k!}(s-\beta)^k - \frac{1}{(n-3)!}\int_{s}^{\beta}\phi^{(n)}(t)(s-t)^{n-3}dt \tag{2.301}$$

Using (2.300) in (2.299) we get

$$\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)$$

$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m}w_iG(x_i,s)-\sum_{i=1}^{m}w_iG(y_i,s)\right)(s-\alpha)^k ds$$
$$+\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m}w_iG(x_i,s)-\sum_{i=1}^{m}w_iG(y_i,s)\right)\left(\int_{\alpha}^{s}\phi^{(n)}(t)(s-t)^{n-3}dt\right)ds. \tag{2.302}$$

By applying Fubini's theorem in the last term of (2.302) we obtain (2.297).
Similarly using (2.301) in (2.299) and applying Fubini's theorem we obtain (2.298). □

Integral version of the above theorem can be stated as follows.

Theorem 2.109 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and let and $x, y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and G be the Green function as defined in (1.180). Then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau-\int_a^b w(\tau)\phi(y(\tau))d\tau=\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^k ds$$
$$+\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\left(\int_t^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt \tag{2.303}$$

and

$$\int_a^b w(\tau)\phi(x(\tau))d\tau-\int_a^b w(\tau)\phi(y(\tau))d\tau=\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(\beta-s)^k ds$$
$$-\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\left(\int_{\alpha}^{t}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt. \tag{2.304}$$

In the following theorem we obtain generalizations of majorization inequality for n-convex functions.

Theorem 2.110 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and let $\mathbf{w} = (w_1,\ldots,w_n)$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that x_i, $y_i \in [\alpha,\beta], w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$ and G be the Green function as defined in (1.180).*

(i) If ϕ is n-convex and

$$\int_t^{\beta}\left(\sum_{i=1}^{m}w_iG(x_i,s)-\sum_{i=1}^{m}w_iG(y_i,s)\right)(s-t)^{n-3}ds \geq 0, \quad t \in [\alpha,\beta], \tag{2.305}$$

then

$$\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i - y_i)$$
$$\geq \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s)\right)(s-\alpha)^k ds. \tag{2.306}$$

(ii) If ϕ is n-convex and

$$\int_{\alpha}^{t}\left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s)\right)(s-t)^{n-3} ds \leq 0, \quad t\in[\alpha,\beta], \tag{2.307}$$

then

$$\sum_{i=1}^{m} w_i\,\phi\,(x_i) - \sum_{i=1}^{m} w_i\,\phi\,(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i - y_i)$$
$$\geq \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s)\right)(\beta-s)^k ds. \tag{2.308}$$

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is $n-$times differentiable and $\phi^{(n)} \geq 0$ see [144, p. 16 and p. 293]. Hence, we can apply Theorem 2.108 to obtain (2.306) and (2.308) respectively. □

Integral version of the above theorem can be stated as follows.

Theorem 2.111 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and G be the Green function as defined in (1.180). Then*

(i) If ϕ is n-convex and

$$\int_{t}^{\beta}\left(\int_{a}^{b} w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau\right)(s-t)^{n-3} ds \geq 0, \quad t\in[\alpha,\beta], \tag{2.309}$$

then

$$\int_{a}^{b} w(\tau)\phi(x(\tau))d\tau - \int_{a}^{b} w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_{a}^{b} w(\tau)(x(\tau)-y(\tau))d\tau$$
$$\geq \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\int_{a}^{b} w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau\right)(s-\alpha)^k ds \tag{2.310}$$

(ii) If ϕ is n-convex and

$$\int_{\alpha}^{t}\left(\int_{a}^{b} w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds \leq 0, \quad t \in [\alpha,\beta], \quad (2.311)$$

then

$$\int_{a}^{b} w(\tau)\phi(x(\tau))d\tau - \int_{a}^{b} w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_{a}^{b} w(\tau)(x(\tau)-y(\tau))d\tau$$
$$\geq \sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\int_{a}^{b} w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(\beta-s)^k ds$$
$$(2.312)$$

The following generalization of majorization theorem holds.

Theorem 2.112 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ be two m-tuples such that $\mathbf{y} \prec \mathbf{x}$ with x_i, $y_i \in [\alpha,\beta]$ $(i = 1,\ldots,m)$ and G be the Green function as defined in (1.180).*

(i) If ϕ is n-convex, then

$$\sum_{i=1}^{m}\phi(x_i) - \sum_{i=1}^{m}\phi(y_i)$$
$$\geq \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m}G(x_i,s)-\sum_{i=1}^{m}G(y_i,s)\right)(s-\alpha)^k ds.$$
$$(2.313)$$

(ii) If the inequality (2.313) holds and the function

$$\nu_1(.) = \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta} G(.,s)(s-\alpha)^k ds \quad (2.314)$$

is convex then the right hand side of (2.313) will be non negative, that is (2.153) holds.

(iii) If n is even and ϕ is n-convex, then

$$\sum_{i=1}^{m}\phi(x_i) - \sum_{i=1}^{m}\phi(y_i)$$
$$\geq \sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m}G(x_i,s)-\sum_{i=1}^{m}G(y_i,s)\right)(\beta-s)^k ds.$$
$$(2.315)$$

(iv) If the inequality (2.315) holds and the function

$$\nu_2(.) = \sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta} G(.,s)(\beta-s)^k ds \quad (2.316)$$

is convex then the right hand side of (2.313) will be non negative, that is (2.153) holds.

(v) If n is odd and ϕ is n-convex, then

$$\sum_{i=1}^{m}\phi(x_i)-\sum_{i=1}^{m}\phi(y_i) \le \sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m}G(x_i,s)-\sum_{i=1}^{m}G(y_i,s)\right)(\beta-s)^k ds. \tag{2.317}$$

(vi) If the inequality (2.317) holds and the function ν_2 defined in (2.314), is concave then the right hand side of (2.317) will be non positive, that is reverse inequality in (2.153) holds.

Proof. (i) Since G is convex and $\boldsymbol{y} \prec \boldsymbol{x}$ therefore by Theorem 2.110 we have

$$\sum_{i=1}^{m}G(x_i,s)-\sum_{i=1}^{m}G(y_i,s)\ge 0.$$

Also $(s-t)^{n-3}\ge 0$ for $s\in[t,\beta]$. Hence (2.305) holds for $w_i=1(i=1,\dots,m)$. Using Theorem 2.110, the inequality (2.313) holds.

(ii) We can write the right hand side of (2.313) as

$$\sum_{i=1}^{m}\nu_1(x_i)-\sum_{i=1}^{m}\nu_1(y_i).$$

Now using majorization theorem we obtain that the right hand side of (2.313) is non negative.

Similarly we can prove the other parts. □

In the following theorem we give generalization of Fuch's majorization theorem.

Theorem 2.113 ([6]) *Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n\ge 3$ and let $\boldsymbol{x}=(x_1,\dots,x_m)$ and $\boldsymbol{y}=(y_1,\dots,y_m)$ be decreasing m-tuples and $\boldsymbol{w}=(w_1,\dots,w_m)$ be any m-tuple such that $x_i,\ y_i\in[\alpha,\beta], w_i\in\mathbb{R}\ (i=1,\dots,m)$ which satisfies (1.19), (1.20) and G be the Green function as defined in (1.180).*

(i) If ϕ is n-convex, then

$$\sum_{i=1}^{m}w_i\phi(x_i)-\sum_{i=1}^{m}w_i\phi(y_i) \ge \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m}w_iG(x_i,s)-\sum_{i=1}^{m}w_iG(y_i,s)\right)(s-\alpha)^k ds. \tag{2.318}$$

(ii) If the inequality (2.318) holds and the function ν_1 defined in (2.314) is convex, then the right hand side of (2.318) will be non negative, that is (2.157) holds.

(iii) If n is even and ϕ is n-convex, then

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \geq \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s) \right) (\beta - s)^k ds. \tag{2.319}$$

(iv) If the inequality (2.319) holds and the function ν_2 defined in (2.316) is convex, then the right hand side of (2.319) will be non negative, that is (2.157) holds.

(v) If n is odd and ϕ is n-convex, then

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \leq \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s) \right) (\beta - s)^k ds. \tag{2.320}$$

(vi) If the inequality (2.320) holds and the function ν_2 defined in (2.316) is concave, then the right hand side of (2.317) will be non positive, that is reverse inequality in (2.157) holds.

Proof. The proof is similar to the proof of Theorem 2.112 but using Theorem 1.12 instead of Theorem 1.14. □

The following generalization of integral majorization theorem holds.

Theorem 2.114 ([6]) *Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and let $x, y : [a,b] \to [\alpha, \beta]$ be decreasing and $w : [a,b] \to \mathbb{R}$ be any continuous functions such that (1.27), (1.28) hold and G be the Green function as defined in (1.180).*

(i) If ϕ is n-convex, then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \geq \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_{\alpha}^{\beta} \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-\alpha)^k ds. \tag{2.321}$$

(ii) If the inequality (2.321) holds and the function ν_1 defined in (2.314) is convex, then the right hand side of (2.321) will be non negative, that is (2.159) holds

(iii) If n is even and ϕ is n-convex, then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\ \geq \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau \right) (\beta - s)^k ds. \tag{2.322}$$

(iv) If the inequality (2.322) holds and the function ν_2 defined in (2.322) is convex, then the right hand side of (2.322) will be non negative, that is (2.159) holds.

(v) If n is odd and ϕ is n-convex, then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\ \leq \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau \right) (\beta - s)^k ds. \tag{2.323}$$

(vi) If the inequality (2.323) holds and the function ν_2 defined in (2.316) is concave, then the right hand side of (2.323) will be non positive, that is reverse inequality in (2.159) holds.

In the sequel we use the above theorems to obtain generalizations of the previous results.

For m-tuples $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ with $x_i,\ y_i \in [\alpha,\beta], w_i \in \mathbb{R}\ (i = 1,\ldots,m)$ and the Green function G defined by (1.180), denote

$$\mathfrak{R}(t) = \sum_{i=1}^{m} w_i \int_t^\beta (G(x_i,s) - G(y_i,s))\,(s-t)^{n-3}ds, \quad t \in [\alpha,\beta], \tag{2.324}$$

$$\mathfrak{B}(t) = \sum_{i=1}^{m} w_i \int_\alpha^t (G(x_i,s) - G(y_i,s))\,(s-t)^{n-3}ds, \quad t \in [\alpha,\beta], \tag{2.325}$$

similarly for continuous functions $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$, denote

$$\tilde{\mathfrak{R}}(t) = \int_t^\beta \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-t)^{n-3}ds, \quad t \in [\alpha,\beta], \tag{2.326}$$

$$\tilde{\mathfrak{B}}(t) = \int_\alpha^t \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-t)^{n-3}ds, \quad t \in [\alpha,\beta]. \tag{2.327}$$

Consider the Čebyšev functionals $T(\mathfrak{R},\mathfrak{R})$, $T(\mathfrak{B},\mathfrak{B})$, $T(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})$ and $T(\tilde{\mathfrak{B}},\tilde{\mathfrak{B}})$ are given by:

$$T(\mathfrak{R},\mathfrak{R}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)dt \right)^2 \tag{2.328}$$

$$T(\mathfrak{B},\mathfrak{B}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{B}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{B}(t)dt\right)^2 \tag{2.329}$$

$$T(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \tilde{\mathfrak{R}}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \tilde{\mathfrak{R}}(t)dt\right)^2 \tag{2.330}$$

$$T(\tilde{\mathfrak{B}},\tilde{\mathfrak{B}}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \tilde{\mathfrak{B}}^2(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \tilde{\mathfrak{B}}(t)dt\right)^2 \tag{2.331}$$

Theorem 2.115 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ with $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and let $\mathbf{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that x_i, $y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$ and let the functions G, $\mathfrak{R}$ and $\mathfrak{B}$ be defined by (1.180), (2.324) and (2.325) respectively. Then*

(i)

$$\begin{aligned}\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^m w_i(x_i-y_i)\\ &+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\sum_{i=1}^m w_iG(x_i,s)-\sum_{i=1}^m w_iG(y_i,s)\right)(s-\alpha)^k ds\\ &+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_\alpha^\beta \mathfrak{R}(t)dt + \kappa_n^1(\phi;\alpha,\beta).\end{aligned} \tag{2.332}$$

where the remainder $\kappa_n^1(\phi;\alpha,\beta)$ satisfies the estimation

$$\left|\kappa_n^1(\phi;\alpha,\beta)\right| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}[T(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \tag{2.333}$$

(ii)

$$\begin{aligned}\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^m w_i(x_i-y_i)\\ &+\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_\alpha^\beta\left(\sum_{i=1}^m w_iG(x_i,s)-\sum_{i=1}^m w_iG(y_i,s)\right)(\beta-s)^k ds\\ &+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-3)!}\int_\alpha^\beta \mathfrak{B}(t)dt - \kappa_n^2(\phi;\alpha,\beta).\end{aligned} \tag{2.334}$$

where the remainder $\kappa_n^2(\phi;\alpha,\beta)$ satisfies the estimation

$$\left|\kappa_n^2(\phi;\alpha,\beta)\right| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}[T(\mathfrak{B},\mathfrak{B})]^{\frac{1}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \tag{2.335}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

Integral case of the above theorem can be given as follows.

Theorem 2.116 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geq 3$ with $(\cdot - \alpha)(\beta - \cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and let $x, y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and let the functions G, $\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{B}}$ be defined by (1.180), (2.326) and (2.327) respectively. Then*

(i)

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^k ds$$
$$+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{R}}(t)dt + \tilde{\kappa}_n^1(\phi;\alpha,\beta). \tag{2.336}$$

where the remainder $\tilde{\kappa}_n^1(\phi;\alpha,\beta)$ satisfies the estimation Υ

$$\left|\tilde{\kappa}_n^1(\phi;\alpha,\beta)\right| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}\left[T(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})\right]^{\frac{1}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{2.337}$$

(ii)

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\beta)^k ds$$
$$+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-3)!}\int_\alpha^\beta \tilde{\mathfrak{B}}(t)dt - \tilde{\kappa}_n^2(\phi;\alpha,\beta). \tag{2.338}$$

where the remainder $\tilde{\kappa}_n^2(\phi;\alpha,\beta)$ satisfies the estimation

$$\left|\tilde{\kappa}_n^2(\phi;\alpha,\beta)\right| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}\left[T(\tilde{\mathfrak{B}},\tilde{\mathfrak{B}})\right]^{\frac{1}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{2.339}$$

Using Theorem 1.11 we obtain the following Grüss type inequalities.

Theorem 2.117 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ $(n \geq 3)$ is absolutely continuous function and $\phi^{(n+1)} \geq 0$ on $[\alpha,\beta]$ and let the functions $\mathfrak{R}$ and $\mathfrak{B}$ be defined by (2.324) and (2.325) respectively. Then we have* *(i) the representation* (2.332) *and the remainder* $\kappa_n^1(\phi;\alpha,\beta)$ *satisfies the bound*

$$\left|\kappa_n^1(\phi;\alpha,\beta)\right| \leq \frac{1}{(n-3)!}\|\mathfrak{R}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \tag{2.340}$$

(ii) the representation (2.334) *and the remainder* $\kappa_n^2(\phi;\alpha,\beta)$ *satisfies the bound*

$$\left|\kappa_n^2(\phi;\alpha,\beta)\right| \leq \frac{1}{(n-3)!}\|\mathfrak{B}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}.$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

Integral case of the above theorem can be given as follows.

Theorem 2.118 ([6]) *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ $(n \geq 3)$ is absolutely continuous function and $\phi^{(n+1)} \geq 0$ on $[\alpha,\beta]$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and the functions G, $\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{B}}$ be defined by (1.180), (2.326) and (2.327) respectively. Then we have*

(i) the representation (2.336) *and the remainder* $\tilde{\kappa}_n^1(\phi;\alpha,\beta)$ *satisfies the bound*

$$\left|\tilde{\kappa}_n^1(\phi;\alpha,\beta)\right| \leq \frac{1}{(n-3)!}\|\tilde{\mathfrak{R}}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \tag{2.341}$$

(ii) the representation (2.338) *and the remainder* $\tilde{\kappa}_n^2(\phi;\alpha,\beta)$ *satisfies the bound*

$$\left|\tilde{\kappa}_n^2(\phi;\alpha,\beta)\right| \leq \frac{1}{(n-3)!}\|\tilde{\mathfrak{B}}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}.$$

We present the Ostrowski type inequalities related to generalizations of majorization's inequality.

Theorem 2.119 ([6]) *Suppose that all assumptions of Theorem 2.108 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $\left|\phi^{(n)}\right|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function for some $n \geq 3$. Then we have:*

(i)

$$\begin{aligned}
&\left|\sum_{i=1}^m w_i\,\phi\left(x_i\right) - \sum_{i=1}^m w_i\,\phi\left(y_i\right) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^m w_i\left(x_i-y_i\right)\right. \\
&\left. - \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\sum_{i=1}^m w_iG(x_i,s) - \sum_{i=1}^m w_iG(y_i,s)\right)(s-\alpha)^k ds\right| \\
&\leq \frac{1}{(n-3)!}\left\|\phi^{(n)}\right\|_p \|f\|_q,
\end{aligned} \tag{2.342}$$

$$\text{where } f(t) = \int_t^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i, s) - \sum_{i=1}^{m} w_i G(y_i, s) \right) (s-t)^{n-3} ds.$$

The constant on the right-hand side of (2.342) is sharp for $1 < p \le \infty$ *and the best possible for* $p = 1$.

(ii)

$$\begin{aligned} &\left| \sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{i=1}^{m} w_i (x_i - y_i) \right. \\ &\left. - \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i, s) - \sum_{i=1}^{m} w_i G(y_i, s) \right) (\beta - s)^k ds \right| \\ &\le \frac{1}{(n-3)!} \left\| \phi^{(n)} \right\|_p \left\| \bar{f} \right\|_q, \end{aligned} \tag{2.343}$$

$$\text{where } \bar{f}(t) = \int_{\alpha}^{t} \left(\sum_{i=1}^{m} w_i G(x_i, s) - \sum_{i=1}^{m} w_i G(y_i, s) \right) (s-t)^{n-3} ds.$$

The constant on the right-hand side of (2.343) is sharp for $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Integral case can be given as:

Theorem 2.120 ([6]) *Suppose that all assumptions of Theorem 2.109 hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p, q \le \infty$, $1/p + 1/q = 1$. *Let* $\left|\phi^{(n)}\right|^p :$ $[\alpha, \beta] \to \mathbb{R}$ *be an R-integrable function for some* $n \ge 3$. *Then we have:*

(i)

$$\begin{aligned} &\left| \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \right. \\ &\left. - \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_{\alpha}^{\beta} \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-\alpha)^k ds \right| \\ &\le \frac{1}{(n-3)!} \left\| \phi^{(n)} \right\|_p \|g\|_q, \end{aligned} \tag{2.344}$$

$$\text{where } g(t) = \int_t^{\beta} \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-t)^{n-3} ds.$$

The constant on the right-hand side of (2.344) is sharp for $1 < p \le \infty$ *and the best possible for* $p = 1$.

(ii)

$$\left| \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \right.$$
$$\left. - \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau \right) (\beta-s)^k ds \right|$$
$$\leq \frac{1}{(n-3)!} \left\| \phi^{(n)} \right\|_p \|\bar{g}\|_q, \tag{2.345}$$

$$\textit{where } \bar{g}(t) = \int_\alpha^t \left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau \right) (s-t)^{n-3} ds.$$

The constant on the right-hand side of (2.345) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Motivated by inequalities (2.306), (2.308), (2.310) and (2.312), under the assumptions of Theorems 2.110 and 2.111 we define the following linear functionals:

$$F_1(\phi) = \sum_{i=1}^m w_i\,\phi(x_i) - \sum_{i=1}^m w_i\,\phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^m w_i(x_i-y_i)$$
$$- \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_\alpha^\beta \left(\sum_{i=1}^m w_i G(x_i,s) - \sum_{i=1}^m w_i G(y_i,s) \right) (s-\alpha)^k ds \tag{2.346}$$

$$F_2(\phi) = \sum_{i=1}^m w_i\,\phi(x_i) - \sum_{i=1}^m w_i\,\phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^m w_i(x_i-y_i)$$
$$- \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!} \int_\alpha^\beta \left(\sum_{i=1}^m w_i G(x_i,s) - \sum_{i=1}^m w_i G(y_i,s) \right) (\beta-s)^k ds \tag{2.347}$$

$$F_3(\phi) = \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$- \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$- \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau \right) (s-\alpha)^k ds \tag{2.348}$$

$$F_4(\phi) = \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$- \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$- \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau \right) (\beta-s)^k ds \tag{2.349}$$

Remark 2.30 ([6]) Under the assumptions of Theorems 2.110 and 2.111, it holds $\digamma_i(\phi) \geq 0$, $i = 1, \ldots, 4$ for all n-convex functions ϕ.

The Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

Theorem 2.121 ([6]) *Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha, \beta]$. If the inequalities (2.305), (2.307), (2.309) and (2.311) hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$\digamma_i(\phi) = \phi^{(n)}(\xi_i)\digamma_i(\varphi), \quad i = 1, \ldots, 4 \tag{2.350}$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\digamma_i$, $i = 1, \ldots, 4$ are defined by (2.346)-(2.349).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (see also the proof of Theorem 4.1 in [86]). □

Theorem 2.122 ([6]) *Let $\phi, \psi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi, \psi \in C^n[\alpha, \beta]$. If the inequalities (2.305), (2.307), (2.309) and (2.311) hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$\frac{\digamma_i(\phi)}{\digamma_i(\psi)} = \frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)}, \quad i = 1, \ldots, 4 \tag{2.351}$$

provided that the denominators are non-zero and $\digamma_i$, $i = 1, \ldots, 4$ are defined by (2.346)-(2.349).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.14 (see the proof of Corollary 4.2 in [86]). □

Now we construct families of n-exponentially and exponentially convex functions by using the idea given in [142].

Theorem 2.123 ([6]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is n-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1, 2, .., 4)$ as defined by $(2.346)-(2.349)$, the following statements hold:*

(i) The function $t \to \digamma_i(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1, .., t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, \ldots, n.$$

(ii) If the function $t \to \digamma_i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39. □

The following corollary is an immediate consequence of the above theorem

Corollary 2.27 ([6]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\dots,x_n;\phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1,..,4)$ as defined by (2.346) – (2.349), the following statements hold:*

(i) The function $t \to \digamma_i(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leq n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \text{ for all } m \in \mathbb{N},\ m = 1,2,\dots,n.$$

(ii) If the function $t \to \digamma_i(\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 2.28 ([6]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\dots,x_n;\phi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n \in I$. Let $\digamma_i$, $i = 1,\dots,4$ be linear functionals defined by (2.346)-(2.349). Then the following statements hold:*

(i) If the function $t \mapsto \digamma_i(\phi_t)$ is continuous on J, then it is 2-exponentially convex function on J. If $t \mapsto \digamma_i(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\digamma_i(\phi_s)]^{t-r} \leq [\digamma_i(\phi_r)]^{t-s} [\digamma_i(\phi_t)]^{s-r}, \quad i = 1,\dots,4$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \digamma_i(\phi_t)$ is strictly positive and differentiable on J, then for every $p,q,u,v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\digamma_i,\Omega) \leq \mu_{u,v}(\digamma_i,\Omega), \tag{2.352}$$

where

$$\mu_{p,q}(\digamma_i,\Omega) = \begin{cases} \left(\frac{\digamma_i(\phi_p)}{\digamma_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\digamma_i(\phi_p)}{\digamma_i(\phi_p)}\right), & p = q, \end{cases} \tag{2.353}$$

for $\phi_p, \phi_q \in \Omega$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10. □

Remark 1.19 is also valid for these functionals.

Remark 2.31 ([6]) *Similar examples can be discussed as given in Section 1.4.*

2.3.3 Results Obtained by New Green's Functions and Taylor's Formula

In this subsection, see [110], we give results related to majorization theorems that include newly defined four different Green's functions (2.47), (2.48), (2.49) and (2.50), and also associated functions (2.46), (2.51), (2.52) and (2.53), introduced in ([125]), in combination with Taylor's formula. One can see that this version of results present generalizations of majorization type theorems discussed in [6].

We star with theorem that gives equivalent statements that include arbitrary convex functions on one side and Green functions (2.47), (2.48), (2.49) and (2.50) on other side, using nice properties of Green's functions as continuity and convexity.

Theorem 2.124 *Let $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be a continuous convex function on the interval $[\vartheta_1,\vartheta_2]$ and $\boldsymbol{x}=(x_1,\ldots,x_l)$, $\boldsymbol{y}=(y_1,\ldots,y_l)$ and $\boldsymbol{p}=(p_1,\ldots,p_l)$ be l-tuples such that $x_i,y_i\in[\vartheta_1,\vartheta_2]$ and $p_i\in\mathbb{R}$ for $i=1,2,\ldots,l$, which satisfy the condition*

$$\sum_{i=1}^{l} p_i y_i = \sum_{i=1}^{l} p_i x_i. \tag{2.354}$$

If we define G_d $(d=1,2,3,4)$ as in (2.47), (2.48), (2.49) and (2.50), then we have following equivalent statements.

(i) For every continuous convex function $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$, we have

$$\sum_{i=1}^{l} p_i f(y_i) \le \sum_{i=1}^{l} p_i f(x_i). \tag{2.355}$$

(ii) For all $v\in[\vartheta_1,\vartheta_2]$, we have

$$\sum_{i=1}^{l} p_i G_d(y_i,v) \le \sum_{i=1}^{l} p_i G_d(x_i,v). \tag{2.356}$$

Moreover, if we change the sign of inequality in both inequalities (2.355) and (2.356), then the above result still holds.

Proof. The scheme of proof is similar for each $d=1,2,3,4$, therefore we will only give the proof for $d=4$.

$(i)\Rightarrow(ii)$: Let statement (i) holds. As the function $G_4:[\vartheta_1,\vartheta_2]\cdot[\vartheta_1,\vartheta_2]\to\mathbb{R}$ is convex and continuous, so it will satisfy the condition (2.355), i.e.,

$$\sum_{i=1}^{l} p_i G_4(y_i,v) \le \sum_{i=1}^{l} p_i G_4(x_i,v). \tag{2.357}$$

$(ii)\Rightarrow(i)$: Let $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be a convex function. Without loss of generality we can assume that $f\in C^2([\vartheta_1,\vartheta_2])$. Further, assume that the statement (ii) holds. Then by Lemma 2.2, we have

$$f(x_i)=f(\vartheta_1)+(\vartheta_2-\vartheta_1)f'(\vartheta_1)-(\vartheta_2-x_i)f'(\vartheta_2)+\int_{\vartheta_1}^{\vartheta_2} G_4(x_i,v)f''(v)dv, \tag{2.358}$$

$$f(y_i)=f(\vartheta_1)+(\vartheta_2-\vartheta_1)f'(\vartheta_1)-(\vartheta_2-y_i)f'(\vartheta_2)+\int_{\vartheta_1}^{\vartheta_2} G_4(y_i,v)f''(v)dv. \tag{2.359}$$

From (2.358) and (2.359), we get

$$\sum_{i=1}^{l} p_i f(x_i) - \sum_{i=1}^{l} p_i f(y_i) = -\sum_{i=1}^{l} p_i(\vartheta_2 - x_i) f'(\vartheta_2) + \sum_{i=1}^{l} p_i(\vartheta_2 - y_i) f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{l} p_i G_4(x_i, v) - \sum_{i=1}^{l} p_i G_4(y_i, v) \right] f''(v) dv. \tag{2.360}$$

Using (2.354), we have

$$\sum_{i=1}^{l} p_i f(x_i) - \sum_{i=1}^{l} p_i f(y_i) = \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{l} p_i G_4(x_i, v) - \sum_{i=1}^{l} p_i G_4(y_i, v) \right] f''(v) dv. \tag{2.361}$$

As f is convex function, therefore $f''(v) \geq 0$ for all $v \in [\vartheta_1, \vartheta_2]$. Hence using (2.356) in (2.361), we get (2.355).

Note that the condition for the existence of second derivative of f is not necessary ([144, p.172]). As it is possible to approximate uniformly a continuous convex function by convex polynomials, so we can directly eliminate this differentiablity condition. □

Remark 2.32 *To avoid many notation, we denote*

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) := \sum_{i=1}^{l} p_i f(x_i) - \sum_{i=1}^{l} p_i f(y_i).$$

If l-tuples $\mathbf{x}, \mathbf{y}, \mathbf{p}$ satisfy the assumptions of Fuchs theorem (Theorem 1.14), then

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) \geq 0$$

for continuous and convex function f. Furthre, $\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{p}, f(.)) = 0$ when f is a constant or linear function.
In particular, if $p_1 = p_2 = \cdots = p_l = 1$, then

$$\mathbb{M}(\mathbf{x}, \mathbf{y}, \mathbf{1}, f(.)) := \sum_{i=1}^{l} f(x_i) - \sum_{i=1}^{l} f(y_i) \geq 0.$$

Next we give new identities related to generalization of majorization theorems in the sense of Taylor's interpolation which will be very useful for us in deducing new results.

Theorem 2.125 *Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and let $\boldsymbol{x} = (x_1, \ldots, x_l)$, $\boldsymbol{y} = (y_1, \ldots, y_l)$ and $\boldsymbol{p} = (p_1, \ldots, p_l)$ be l-tuples such that $x_i, y_i \in [\vartheta_1, \vartheta_2]$ and $p_i \in \mathbb{R}$ for $i = 1, 2, \ldots, l$. If we define G_d $(d = 1, 2, 3, 4)$ as in (2.47), (2.48), (2.49) and (2.50), then*

$$\mathbb{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, f(.)) = f'(\xi_d) \sum_{i=1}^{l} p_i (x_i - y_i) + \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, G_d(., v)) (v - \vartheta_1)^k dv + \frac{1}{(n-3)!} \int_{\vartheta_1}^{\vartheta_2} \left(\int_{u}^{\vartheta_2} \mathbb{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, G_d(., v)) (v - u)^{n-3} dv \right) f^{(n)}(u) du, \tag{2.362}$$

and

$$\begin{aligned}
&\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) = f'(\xi_d)\sum_{i=1}^{l} p_i(x_i - y_i)\\
&+\sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(\vartheta_2 - v)^k dv\\
&-\frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\left(\int_{\vartheta_1}^{u}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv\right)f^{(n)}(u)du, \qquad (2.363)
\end{aligned}$$

where $\xi_1, \xi_4 = \vartheta_2$, $\xi_2, \xi_3 = \vartheta_1$ *and* $G_d\,(d = 1,2,3,4)$.

Proof. The scheme of proof is similar for each $d = 1,2,3,4$, therefore we will only give the proof for $d = 4$. From (2.360), we have

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) = \sum_{i=1}^{l} p_i(x_i - y_i)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_4(.,v))f''(v)dv. \qquad (2.364)$$

Now using Taylor's formula (2.246) for the function f'' at point ϑ_1 and replacing n by $n-2$ $(n \geq 3)$, we have

$$f''(v) = \sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}(v-\vartheta_1)^k + \frac{1}{(n-3)!}\int_{\vartheta_1}^{v} f^{(n)}(u)(v-u)^{n-3}du. \qquad (2.365)$$

Similarly, Taylor's formula on the function f'' at point ϑ_2 and replacing n by $n-2$ $(n \geq 3)$, we get

$$f''(v) = \sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_2)}{k!}(v-\vartheta_2)^k - \frac{1}{(n-3)!}\int_{v}^{\vartheta_2} f^{(n)}(u)(v-u)^{n-3}du. \qquad (2.366)$$

Using (2.365) in (2.364), we have

$$\begin{aligned}
&\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) = f'(\vartheta_2)\sum_{i=1}^{l} p_i(x_i - y_i)\\
&+\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_4(.,v))(v-\vartheta_1)^k dv\\
&+\frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_1}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_4(.,v))\left(\int_{\vartheta_1}^{v} f^{(n)}(u)(v-u)^{n-3}du\right)dv. \qquad (2.367)
\end{aligned}$$

Now by using Fubini's theorem in (2.367), we get (2.362). In similar way, we can find (2.363), by using (2.366) in (2.364). □

The following is an application of the previous theorem which is in fact generalization of majorization inequality for n-convex functions.

Corollary 2.29 *Let* $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous for some* $n \geq 3$ *and let* $\boldsymbol{x} = (x_1,\ldots,x_l)$, $\boldsymbol{y} = (y_1,\ldots,y_l)$ *and* $\boldsymbol{p} = (p_1,\ldots,p_l)$ *be l-tuples such that* $x_i, y_i \in [\vartheta_1, \vartheta_2]$ *and* $p_i \in \mathbb{R}$ *for* $i = 1,2,\ldots,l$. *Define* $G_d\,(d = 1,2,3,4)$ *as in* (2.47), (2.48), (2.49) *and* (2.50).

(i) If f is n-convex and

$$\int_u^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv \geq 0, u \in [\vartheta_1,\vartheta_2], \tag{2.368}$$

then

$$\begin{aligned} &\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i) \\ &\geq \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-\vartheta_1)^k dv. \end{aligned} \tag{2.369}$$

(ii) If f is n-convex and

$$\int_{\vartheta_1}^{u} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv \leq 0, u \in [\vartheta_1,\vartheta_2], \tag{2.370}$$

then

$$\begin{aligned} &\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i) \\ &\geq \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(\vartheta_2-v)^k dv, \end{aligned} \tag{2.371}$$

where $\xi_1,\xi_4=\vartheta_2$ and $\xi_2,\xi_3=\vartheta_1$.

Proof. By the n-convexity of the function f, we can assume without loss of generality that f is n-times differentiable and $f^{(n)} \geq 0$ (see [144, p.16 and p. 293]). So using (2.362) and (2.363), we can have (2.369) and (2.371) respectively. □

The following corollary gives the generalization of classical majorization theorem.

Corollary 2.30 *Let $f:[\vartheta_1,\vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 3$ and let $\boldsymbol{x}=(x_1,\ldots,x_l)$, $\boldsymbol{y}=(y_1,\ldots,y_l)$ be l-tuples such that $x_i,y_i \in [\vartheta_1,\vartheta_2]$ for $i=1,2,\ldots,l$ and $\boldsymbol{x} \succ \boldsymbol{y}$. Define G_d $(d=1,2,3,4)$ as in* (2.47), (2.48), (2.49) *and* (2.50).

(i) If f is n-convex, then

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\mathbf{1},f(.)) \geq \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\mathbf{1},G_d(.,v))(v-\vartheta_1)^k dv. \tag{2.372}$$

(ii) If inequality (2.372) *is satisfied and*

$$F_1(.) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} G_d(.,v)(v-\vartheta_1)^k dv \tag{2.373}$$

is convex then the right hand side of (2.372) *is non negative, i.e.,* (1.18) *is satisfied.*

(iii) *If f is n-convex, where n is even, then*

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\mathbf{1},f(.)) \geq \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\mathbf{1},G_d(.,v))(\vartheta_2-v)^k dv. \tag{2.374}$$

(iv) *If inequality* (2.374) *is satisfied and*

$$\digamma_2(.) = \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} G_d(.,v)(\vartheta_2-v)^k dv, \tag{2.375}$$

is convex then the right hand side of (2.374) *is non negative, i.e.,* (1.18) *is satisfied.*

(v) *If f is n-convex, where n is odd, then*

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\mathbf{1},f(.)) \leq \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\mathbf{1},G_d(.,v))(\vartheta_2-v)^k dv. \tag{2.376}$$

(vi) *If the function $\digamma_2$ which is defined in* (2.375), *is concave and the inequality* (2.376) *is satisfied, then the right hand side of* (2.376) *is non positive, i.e., reverse inequality in* (1.18) *is satisfied,*

where $\mathbf{1} = (1,1,\ldots,1)$ *is l-tuple.*

Proof. (i): Note that for $v \in [u,\vartheta_2]$, we have $(v-u)^{n-3} \geq 0$. Given that $\boldsymbol{x}$ majorizes $\boldsymbol{y}$, so (1.20) holds. Moreover G_d is continuous as well as convex, for $d = 1,2,3,4$, therefore by using Theorem 1.12, we can write

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\mathbf{1},G_d(.,v)) \geq 0.$$

Thus (2.368) holds for $p_i = 1\,(i = 1,2,\ldots,l)$. Hence by using Taylor theorem (2.246), we can deduce inequality (2.372).

(ii): Right hand side of the inequality (2.372) can be written as

$$\sum_{i=1}^{l} \digamma_1(x_i) - \sum_{i=1}^{l} \digamma_1(y_i).$$

As $\digamma_1$ is convex, so by applying majorization theorem we note that the right hand side of (2.372) is non negative.

Remaining parts can also be proved in similar way. □

The following corollary gives the generalization of Fuchs's majorization theorem.

Corollary 2.31 *Let $f : [\vartheta_1,\vartheta_2] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 3$. Let $\boldsymbol{x} = (x_1,\ldots,x_l)$, $\boldsymbol{y} = (y_1,\ldots,y_l)$ be non increasing l-tuples and $\boldsymbol{p} = (p_1,\ldots,p_l)$ be l-tuple such that $x_i, y_i \in [\vartheta_1,\vartheta_2]$ and $p_i \in \mathbb{R}$ for $i = 1,2,\ldots,l$, which satisfy conditions* (1.19) *and* (1.20). *Define G_d $(d = 1,2,3,4)$ as in* (2.47), (2.48), (2.49) *and* (2.50).

(i) If f is n-convex, then

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) \geq \sum_{k=0}^{n-3} \frac{f^{(k+2)}(\vartheta_1)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-\vartheta_1)^k dv. \tag{2.377}$$

(ii) If the inequality (2.377) is satisfied and the function F_1 which defined in (2.373), is convex then the right hand side of (2.377) is non negative, i.e., (1.21) is satisfied.

(iii) If f is n-convex, where n is even, then

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) \geq \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(\vartheta_2-v)^k dv. \tag{2.378}$$

(iv) If the inequality (2.378) is satisfied and the function F_2 which defined in (2.375), is convex then the right hand side of (2.378) is non negative, i.e., (1.21) is satisfied.

(v) If f is n-convex, where n is odd, then

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) \leq \sum_{k=0}^{n-3} \frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!} \int_{\vartheta_1}^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(\vartheta_2-v)^k dv. \tag{2.379}$$

(vi) If the function F_2 which is defined in (2.375), is concave and the inequality (2.379) is satisfied, then the right hand side of (2.379) is non positive, i.e., reverse inequality in (1.21) is satisfied.

Proof. Following the proof of Corollary 2.30, one can prove the result easily. □

We are ending this subsection with the following remark.

Remark 2.33 *We can give the majorization theorems in the integral version of Theorem 2.125, Corollary 2.29, Corollary 2.30 and Corolary 2.31 like as given in [6].*

Next, using the family of $(n+1)$-convex functions at a point, we prove that the linear functionals, deduced from the generalized identity (2.369), constructed on two different intervals have monotonicity property (see [143]).

Definition 2.2 *Let $J \subseteq \mathbb{R}$ be an interval, $\tau \in J$ and $n \in \mathbb{N}$. A function $f : J \to \mathbb{R}$ is said to be $(n+1)$-convex at point τ if there exits a constant W_τ such that the function*

$$\mathbb{F}(w) = f(w) - \frac{W_\tau}{n!} w^n \tag{2.380}$$

is n-concave on $I \cap (-\infty, \tau]$ and n-convex on $I \cap [\tau, \infty)$. A function f is said to be $(n+1)$-concave at a point τ if the function $-f$ is $(n+1)$-convex at a point τ.

It is the usual sense that a function is $(n+1)$-convex on an interval iff it is $(n+1)$-convex at every point of the interval (see [143, 125]). Pečarić et al. in [143] described necessary and sufficient conditions on two linear functionals $\Psi : C([\vartheta_1,\tau]) \to \mathbb{R}$ and $\Omega : C([\tau,\vartheta_2]) \to \mathbb{R}$ so that the inequality $\Psi(f) \leq \Omega(f)$ holds for every function f that is $(n+1)$-convex at point τ.
Now we define linear functionals $\Psi_d(f)$ and $\Omega_d(f)$ for fix $d = 1,2,3,4$ whose are deduced from the difference of left and right sides of identity (2.369), constructed on the intervals $[\vartheta_1,\tau]$ and $[\tau,\vartheta_2]$ respectively, i.e., for $\boldsymbol{x},\boldsymbol{y} \in [\vartheta_1,\tau]^l$, $\boldsymbol{p} \in \mathbb{R}^l$, $\boldsymbol{r},\boldsymbol{s} \in [\tau,\vartheta_2]^{\bar{l}}$ and $\bar{\boldsymbol{p}} \in \mathbb{R}^{\bar{l}}$ let

$$\begin{aligned}\Psi_d(f) &:= \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)\\ &-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\tau}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-\vartheta_1)^k dv, \end{aligned}\tag{2.381}$$

where $\xi_1,\xi_4=\tau$ and $\xi_2,\xi_3=\vartheta_1$ and

$$\begin{aligned}\Omega_d(f) &:= \mathbb{M}(\boldsymbol{r},\boldsymbol{s},\bar{\boldsymbol{p}},f(.)) - f'(\xi_d)\sum_{i=1}^{\bar{l}} \bar{p}_i(r_i-s_i)\\ &-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\tau)}{k!}\int_{\tau}^{\vartheta_2}\mathbb{M}(\boldsymbol{r},\boldsymbol{s},\bar{\boldsymbol{p}},G_d(.,v))(v-\tau)^k dv, \end{aligned}\tag{2.382}$$

where $\xi_1,\xi_4=\vartheta_2$ and $\xi_2,\xi_3=\tau$.
When we apply identity (2.362) to the linear functionals $\Psi_d(f)$ and $\Omega_d(f)$ for fix $d=1,2,3,4$ on the intervals $[\vartheta_1,\tau]$ and $[\tau,\vartheta_2]$ respectively, we get

$$\Psi_d(f) = \frac{1}{(n-3)!}\int_{\vartheta_1}^{\tau}\left(\int_{u}^{\tau}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv\right)f^{(n)}(u)du,\tag{2.383}$$

and

$$\Omega_d(f) = \frac{1}{(n-3)!}\int_{\tau}^{\vartheta_2}\left(\int_{u}^{\vartheta_2}\mathbb{M}(\boldsymbol{r},\boldsymbol{s},\bar{\boldsymbol{p}},G_d(.,v))(v-u)^{n-3}dv\right)f^{(n)}(u)du.\tag{2.384}$$

Now, we are in that position to state the monotonicity of linear functionals $\Psi_p(f)$ and $\Omega_p(f)$ involving $(n+1)$-convex function at a point:

Theorem 2.126 *Let $\boldsymbol{x},\boldsymbol{y} \in [\vartheta_1,\tau]^l$, $\boldsymbol{p} \in \mathbb{R}^l$, $\boldsymbol{r},\boldsymbol{s} \in [\tau,\vartheta_2]^{\bar{l}}$ and $\bar{\boldsymbol{p}} \in \mathbb{R}^{\bar{l}}$ in such a way that for $d=1,2,3,4$*

$$\int_{u}^{\tau}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv \geq 0,\ \ u\in[\vartheta_1,\tau],\tag{2.385}$$

$$\int_{u}^{\vartheta_2}\mathbb{M}(\boldsymbol{r},\boldsymbol{s},\bar{\boldsymbol{p}},G_d(.,v))(v-u)^{n-3}dv \geq 0,\ \ u\in[\tau,\vartheta_2],\tag{2.386}$$

$$\int_{\vartheta_1}^{\tau}\left(\int_u^{\tau}\mathbb{M}\left(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d\left(.,v\right)\right)\left(v-u\right)^{n-3}dv\right)du$$
$$=\int_{\tau}^{\vartheta_2}\left(\int_u^{\vartheta_2}\mathbb{M}\left(\boldsymbol{r},\boldsymbol{s},\overline{\boldsymbol{p}},G_d\left(.,v\right)\right)\left(v-u\right)^{n-3}dv\right)du, \tag{2.387}$$

where G_d $(d=1,2,3,4)$ is defined as in (2.47), (2.48), (2.49) *and* (2.50) *respectively and let linear functionals $\Psi_d(f)$ and $\Omega_d(f)$ be defined in (2.381) and (2.382). If $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ is $(n+1)$-convex at point τ, then the monotonicity of these linear functionals yields*

$$\Psi_d(f)\le\Omega_d(f),\quad for\ \ d=1,2,3,4. \tag{2.388}$$

If the inequalities in (2.385) and (2.386) are reversed, then the reverse inequality in (2.388) holds.

Proof. From Definition 2.2, we can define a function $\mathbb{F}(w)$ in such manner that $\mathbb{F}(w)$ is n-concave on $[\vartheta_1,\tau]$ and n-convex on $[\tau,\vartheta_2]$. Now by using Theorem 2.29 to the function $\mathbb{F}(w)$ on $[\vartheta_1,\tau]$ and $[\tau,\vartheta_2]$ respectively, we get

$$\Psi_d(\mathbb{F})=\Psi_d(f)-\frac{W_\tau}{n!}\Psi_d(w^n)\le 0\ \ and\ \ \Omega_d(\mathbb{F})=\Omega_d(f)-\frac{W_\tau}{n!}\Omega_d(w^n)\ge 0, \tag{2.389}$$

by fixing $d=1,2,3,4$. By putting $f=w^n$ in the identities (2.383) and (2.384) we have

$$\Psi_d(w^n)=\left(n^3-3n^2+2n\right)\int_{\vartheta_1}^{\tau}\left(\int_u^{\tau}\mathbb{M}\left(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d\left(.,v\right)\right)\left(v-u\right)^{n-3}dv\right)du \tag{2.390}$$

and

$$\Omega_d(w^n)=\left(n^3-3n^2+2n\right)\int_{\tau}^{\vartheta_2}\left(\int_u^{\vartheta_2}\mathbb{M}\left(\boldsymbol{r},\ \boldsymbol{s},\ \overline{\boldsymbol{p}},G_d\left(.,v\right)\right)\left(v-u\right)^{n-3}dv\right)du. \tag{2.391}$$

So (2.387) implies that

$$\Psi_d(w^n)=\Omega_d(w^n).$$

Therefore the required result follows from (2.389). $\square$

Remark 2.34 *As in [125], in the proof of above theorem we have shown that for $d=1,2,3,4$*

$$\Psi_d(f)\le\frac{W_\tau}{n!}\Psi_d(w^n)=\frac{W_\tau}{n!}\Omega_d(w^n)\le\Omega_d(f).$$

Moreover, if we replace condition (2.387) with the weaker condition that is $W_\tau\left(\Omega_d(w^n)-\Psi_d(w^n)\right)\ge 0$, the inequality (2.388) still holds.

Remark 2.35 *We can also give the results of this subsection by defining the linear functionals via using inequality (2.371) and the newly defined Green functions G_d for $d = 1,2,3,4$.*

In the sequel, we give the upper bounds of Grüss and Ostrowski type for given generalized results.

Let $\boldsymbol{x} = (x_1,\ldots,x_l)$, $\boldsymbol{y} = (y_1,\ldots,y_l)$ and $\boldsymbol{p} = (p_1,\ldots,p_l)$ be l-tuples such that $x_i, y_i \in [\vartheta_1,\vartheta_2]$ and $p_i \in \mathbb{R}$ for $i = 1,2,\ldots,l$. Let G_d $(d = 1,2,3,4)$ be Green's functions as defined in (2.47), (2.48), (2.49) and (2.50). For $d = 1,2,3,4$, we define

$$\Theta_d(u) = \int_u^{\vartheta_2} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv, \qquad u \in [\vartheta_1,\vartheta_2], \tag{2.392}$$

$$\Upsilon_d(u) = \int_{\vartheta_1}^{u} \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv, \qquad u \in [\vartheta_1,\vartheta_2]. \tag{2.393}$$

Now consider the following Čebyšev functionals for $d = 1,2,3,4$

$$T(\Theta_d,\Theta_d) = \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Theta_d^2(u)du - \left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du\right)^2, \tag{2.394}$$

$$T(\Upsilon_d,\Upsilon_d) = \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Upsilon_d^2(u)du - \left(\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Upsilon_d(u)du\right)^2. \tag{2.395}$$

Theorem 2.127 *Let all the assumptions of Theorem 2.125 be satisfied. Let $f : [\vartheta_1,\vartheta_2] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geq 3$ and $(\cdot - \vartheta_1)(\vartheta_2 - \cdot)[f^{(n+1)}]^2 \in L[\vartheta_1,\vartheta_2]$. If Θ_d and Υ_d are functions, defined in (2.392) and (2.393) respectively, then the following identities hold for $d = 1,2,3,4$.*

(i)

$$\begin{aligned}
\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) &= f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i) \\
&+ \sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-\vartheta_1)^k dv \\
&+ \frac{f^{(n-1)}(\vartheta_2)-f^{(n-1)}(\vartheta_1)}{(\vartheta_2-\vartheta_1)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du + \mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2),
\end{aligned} \tag{2.396}$$

where $\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)$ is the remainder which satisfies the following inequality

$$\begin{aligned}
\left|\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)\right| &\leq \frac{\sqrt{\vartheta_2-\vartheta_1}}{\sqrt{2}(n-3)!}[T(\Theta_d,\Theta_d)]^{\frac{1}{2}} \\
&\left|\int_{\vartheta_1}^{\vartheta_2}(u-\vartheta_1)(\vartheta_2-u)[f^{(n+1)}(u)]^2du\right|^{\frac{1}{2}}.
\end{aligned} \tag{2.397}$$

(ii)

$$\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) = f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)$$
$$+\sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(\vartheta_2-v)^k dv$$
$$+\frac{f^{(n-1)}(\vartheta_2)-f^{(n-1)}(\vartheta_1)}{(\vartheta_1-\vartheta_2)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Upsilon(u)du - \mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2), \tag{2.398}$$

where $\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)$ *is the remainder which satisfies the following inequality*

$$\left|\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)\right| \le \frac{\sqrt{\vartheta_2-\vartheta_1}}{\sqrt{2}(n-3)!}[T(\Upsilon_d,\Upsilon_d)]^{\frac{1}{2}}$$
$$\left|\int_{\vartheta_1}^{\vartheta_2}(u-\vartheta_1)(\vartheta_2-u)[f^{(n+1)}(u)]^2du\right|^{\frac{1}{2}}. \tag{2.399}$$

Moreover, $\xi_1,\xi_4=\vartheta_2$ *and* $\xi_2,\xi_3=\vartheta_1$.

Proof. (i): From (2.362) and (2.396), we have for fixed $d=1,2,3,4$,

$$\frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)f^{(n)}(u)du$$
$$=\frac{f^{(n-1)}(\vartheta_2)-f^{(n-1)}(\vartheta_1)}{(\vartheta_2-\vartheta_1)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du+\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2).$$

This implies

$$\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)$$
$$=\frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)f^{(n)}(u)du-\frac{1}{(\vartheta_2-\vartheta_1)(n-3)!}\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)du\int_{\vartheta_1}^{\vartheta_2}f^{(n)}(u)du,$$

which can be written in terms of Čebyšev functional as

$$\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)=\frac{(\vartheta_2-\vartheta_1)}{(n-3)!}T(\Theta_d,f^{(n)}). \tag{2.400}$$

Using Theorem (1.10), we get (2.397).
(ii): Similarly, we can prove the part (ii) by comparing (2.363) and (2.398). □

Following theorem gives Grüss type inequalities.

Theorem 2.128 *Let* $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is absolutely continuous for some* $n\ge 3$ *and* $f^{(n+1)}\ge 0$ *on* $[\vartheta_1,\vartheta_2]$. *If* Θ_d *and* Υ_d *are functions, defined in* (2.392) *and* (2.393) *respectively, then for* $d=1,2,3,4$, *(i) the remainder* $\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)$ *in* (2.396) *satisfies the following bound*

$$\left|\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2)\right| \leq \frac{1}{(n-3)!}\|\Theta_d'\|_\infty \left\{\frac{f^{(n-1)}(\vartheta_2)+f^{(n-1)}(\vartheta_1)}{2} - \frac{f^{(n-2)}(\vartheta_2)-f^{(n-2)}(\vartheta_1)}{\vartheta_2-\vartheta_1}\right\}. \tag{2.401}$$

(ii) the remainder $\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)$ *in* (2.398) *satisfies the following bound*

$$\left|\mathbb{REM}(f^{(n)},\Upsilon_d,\vartheta_1,\vartheta_2)\right| \leq \frac{1}{(n-3)!}\|\Upsilon_d'\|_\infty \left\{\frac{f^{(n-1)}(\vartheta_2)+\phi^{(n-1)}(\vartheta_1)}{2} - \frac{f^{(n-2)}(\vartheta_2)-f^{(n-2)}(\vartheta_1)}{\vartheta_2-\vartheta_1}\right\}. \tag{2.402}$$

Proof. From (2.400), we have for fixed $d = 1,2,3,4$,

$$\mathbb{REM}(f^{(n)},\Theta_d,\vartheta_1,\vartheta_2) = \frac{(\vartheta_2-\vartheta_1)}{(n-3)!}T(\Theta_d,f^{(n)}). \tag{2.403}$$

Using Theorem 1.11 on right hand side, we deduce (2.401).
(ii): Part (ii) can be proved in a similar way. □

Next theorem gives the Ostrowski type inequalities related to generalized majorization inequality.

Theorem 2.129 *Let all the assumptions of Theorem 2.125 be satisfied. Let* $f : [\vartheta_1,\vartheta_2] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous for some* $n \geq 3$. *Let* (q,q') *be a pair of conjugate exponents, that is* $1 \leq q,q' \leq \infty$ *and* $\frac{1}{q}+\frac{1}{q'} = 1$. *If* $|f^n|^q : [\vartheta_1,\vartheta_2] \to \mathbb{R}\,(n \geq 3)$ *is R-integrable function, then we have the following identities for* $d = 1,2,3,4$.

(i)

$$\begin{aligned}&\Bigg|\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) - f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)\\ &-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))\,(v-\vartheta_1)^k dv\Bigg|\\ &\leq \frac{1}{(n-3)!}\|f^{(n)}\|_q\|f\|_{q'},\end{aligned} \tag{2.404}$$

where

$$f(u) = \int_u^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))\,(v-u)^{n-3}dv.$$

Right hand side of (2.404) *is constant which is sharp for* $1 < q \leq \infty$ *and the best possible for* $q = 1$.

(ii)

$$\Big|\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.))-f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)$$
$$-\sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(\vartheta_2-v)^k dv\Big|$$
$$\leq \frac{1}{(n-3)!}\|f^{(n)}\|_q\|\bar{f}\|_{q'}, \tag{2.405}$$

where

$$\bar{f}(u)=\int_{\vartheta_1}^{u}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-u)^{n-3}dv.$$

Right hand side of (2.405) *is constant which is sharp for* $1<q\leq\infty$ *and the best possible for* $q=1$.

Moreover, $\xi_1,\xi_4=\vartheta_2$ *and* $\xi_2,\xi_3=\vartheta_1$.

Proof. By the arrangement of identity (2.362) for fixed $d=1,2,3,4$, we have the following identity:

$$\Big|\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.))-f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)$$
$$-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-\vartheta_1)^k dv\Big|$$
$$=\frac{1}{(n-3)!}\left|\int_{\vartheta_1}^{\vartheta_2}\Theta_d(u)f^{(n)}(u)du\right|, \tag{2.406}$$

classical Hölder's inequality apply to the right hand side of (2.406) implies (2.404). The proof of the sharpness of the constant $\|f\|_{q'}$ is analogous to the proof of Theorem 19 in [6]. □

Remark 2.36 *We can give the integral version of the upper bound theorems like Theorem 2.127, Theorem 2.128 and Theorem 2.129 as given in [6].*

Since the general convex functions are defined by a functional inequality, it is not surprising that this notion will lead to a number of interesting and fundamental inequalities. Now we give some essential results for general convex functions.
Suppose all the assumptions of Corollary 2.29 are satisfied. Using inequalities (2.369) and (2.371) we now define following linear functionals:

$$\mathfrak{R}_1(f)=\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.))-f'(\xi_d)\sum_{i=1}^{l}p_i(x_i-y_i)$$
$$-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(\vartheta_1)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-\alpha)^k dv, \tag{2.407}$$

and

$$\begin{aligned}\Re_2(f) &= \mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f(.)) - f^{'}(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)\\ &- \sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(\beta-v)^k dv,\end{aligned} \tag{2.408}$$

where $\xi_1,\xi_4=\vartheta_2$ and $\xi_2,\xi_3=\vartheta_1$.

Remark 2.37 *Let all the assumptions of Corollary 2.29 are satisfied. Then* $\Re_i(f)\geq 0$, $i=1,2$ *for all n-convex functions f.*

The following theorems give the Lagrange and Cauchy type mean value theorems for the functionals defined in (2.407) and (2.408).

Theorem 2.130 *Let* $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ *be such that* $f\in C^n[\vartheta_1,\vartheta_2]$. *Let the inequalities* (2.368) *and* (2.370) *hold. Let* $\Re_i(f),\ i=1,2$ *be functionals defined in* (2.407) *and* (2.408) *and also* $\psi(x)=\frac{x^n}{n!}$. *Then there exist* $\lambda_i\in[\vartheta_1,\vartheta_2]$ *such that*

$$\Re_i(f)=f^{(n)}(\lambda_i)\Re_i(\psi),\qquad i=1,2. \tag{2.409}$$

Proof. Since $f^{(n)}$ is continuous on $[\vartheta_1,\vartheta_2]$, so $m\leq f^{(n)}(x)\leq M$ for $x\in[\vartheta_1,\vartheta_2]$, where $m=\min_{x\in[\vartheta_1,\vartheta_2]} f^{(n)}(x)$ and $M=\max_{x\in[\vartheta_1,\vartheta_2]} f^{(n)}(x)$. Consider the functions f_1 and f_2 defined on I as

$$f_1(x)=\frac{Mx^n}{n!}-f(x) \quad and \quad f_2(x)=f(x)-\frac{mx^n}{n!} \;\; for\;\; x\in[\vartheta_1,\vartheta_2].$$

It is easily seen that

$$f_1^{(n)}(x)=M-f^{(n)}(x) \quad and \quad f_2^{(n)}(x)=f^{(n)}(x)-m \;\; for\;\; x\in I.$$

So, f_1 and f_2 are n-convex functions.
Now by applying f_1 for f in Corollary 2.29, we have

$$\begin{aligned}&\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},f_1(.)) - f'(\xi_d)\sum_{i=1}^{l} p_i(x_i-y_i)\\ &\geq \sum_{k=0}^{n-3}\frac{f_1^{(k+2)}(\vartheta_2)}{k!}\int_{\vartheta_1}^{\vartheta_2}\mathbb{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p},G_d(.,v))(v-\vartheta_1)^k dv,\end{aligned} \tag{2.410}$$

where $\xi_1,\xi_4=\vartheta_2$, $\xi_2,\xi_3=\vartheta_1$ and $d=1,2,3,4$. Hence, we get after some simplification

$$\Re_1(f)\leq M\Re_1(\psi). \tag{2.411}$$

Now by applying f_2 for f in Corollary 2.29 and some simplification we get

$$m\Re_1(\psi)\leq\Re_1(f). \tag{2.412}$$

If $\Re_1(\psi) = 0$, then from (2.411) and (2.412) follow that for any $\lambda_1 \in [\vartheta_1, \vartheta_2]$, (2.409) is satisfied.
If $\Re_1(\psi) > 0$, it follows from (2.411) and (2.412) that

$$m \leq \frac{\Re_1(f)}{\Re_1(\psi)} \leq M. \tag{2.413}$$

Now using the fact that for $m \leq \rho \leq M$ there exists $\lambda_1 \in [\vartheta_1, \vartheta_2]$ such that $f^{(n)}(\lambda_1) = \rho$, we get (2.409). $\square$

Corollary 2.32 *Let $f, g : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be such that $f, g \in C^n[\vartheta_1, \vartheta_2]$. Consider the inequalities* (2.368) *and* (2.370) *hold. Let $\Re_i(f),\ i = 1,2$ be functionals defined in* (2.407) *and* (2.408). *Then there exist $\lambda_i \in [\vartheta_1, \vartheta_2]$ such that*

$$\frac{\Re_i(f)}{\Re_i(g)} = \frac{f^{(n)}(\lambda_i)}{g^{(n)}(\lambda_i)}, \qquad i = 1,2, \tag{2.414}$$

provided that the denominators are non-zero.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.14 (see the proof of Corollary 4.2 in [86]). $\square$

We can define Cauchy means for $i = 1,2$ by using generalized Cauchy second mean value theorem i.e., Corollary 2.698 as

$$\lambda_i = \left(\frac{f^{(n)}}{g^{(n)}} \right)^{-1} \frac{\Re_i(f)}{\Re_i(g)},$$

which shows that λ is a mean of ϑ_1, ϑ_2 for given functions f and g.

Remark 2.38 *We can give the n-exponential convexity, exponential convexity as well as* log*-convexity from the above defined positive linear functionals $\Re_i(f),\ i = 1,2$ for both discrete as well as continuous case by using the interesting method introduced by Pečarić et al. (2013) [86, 84] (see also [6, 125]). We can also construct a large families of functions which are exponentially convex as given in [6]. From the* log*-convexity, we can get the Dresher's inequality from which we find the Cauchy means and investigate their monotonicity.*

At the end of this subsection, we explore applications of obtained generalized identities. We give the Ostrowski type of upper bounds for generalized identities in discrete case for some concrete convex functions. In fact, in first two applications we discuss about the relationship between the components of both vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. We can also give nice examples of exponentially convex function using obtained generalized result as disccused in [6].

Application 1. Let $f : (0, \infty) \to \mathbb{R}$ be function defined by $f(x) = -\log x$. Let us consider that $x = (x_1, x_2, \ldots, x_l)$ and $y = (y_1, y_2, \ldots, y_l)$ be positive l-tuples. Then the Ostrowskitype inequality (2.404) for $n = 3$ as an upper bound of our generalized result becomes

$$\left|\sum_{i=1}^{l} p_i(-logx_i) - \sum_{i=1}^{l} p_i(-\log y_i) + \frac{1}{\vartheta_2}\sum_{i=1}^{l} p_i(x_i - y_i) - \frac{1}{\vartheta_1^2}\mathbb{G}_d\right|$$
$$\leq \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'}.$$

If the majorization condition $\sum_{i=1}^{l} p_i x_i = \sum_{i=1}^{l} p_i y_i$ holds and $p_i = 1$, $(i = 1,2,\ldots,l)$ then

$$\left|\log\left(x_1^{-1}\cdot x_2^{-1}\cdots x_l^{-1}\right) + \log\left(y_1\cdot y_2\cdots y_l\right) - \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|$$
$$\leq \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'},$$

i.e.

$$\left|\log\left(\frac{\Pi_{i=1}^{l} y_i}{\Pi_{i=1}^{l} x_i}\right) - \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right| \leq \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'},$$

and if

$$\left|\log\left(\frac{\Pi_{i=1}^{l} y_i}{\Pi_{i=1}^{l} x_i}\right)\right| \geq \left|\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|,$$

then

$$\left|\log\left(\frac{\Pi_{i=1}^{l} y_i}{\Pi_{i=1}^{l} x_i}\right)\right| - \left|\frac{1}{x^2}\tilde{\mathbb{G}}_d\right| \leq \left|\ln\left(\frac{\Pi_{i=1}^{l} y_i}{\Pi_{i=1}^{l} x_i}\right) - \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|$$
$$\leq \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'},$$

$$0 \leq \left|\log\left(\frac{\Pi_{i=1}^{l} y_i}{\Pi_{i=1}^{l} x_i}\right)\right| \leq \frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'} + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d.$$

If the quotient in the left hand side is greater than equal to 1, then

$$0 \leq \Pi_{i=1}^{l} y_i \leq e^{\frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q} - \vartheta_1^{1-3q}\right)^{\frac{1}{q}} \|f\|_{q'} + \frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d} \Pi_{i=1}^{l} x_i,$$

i.e. we get relation between the elements of y and the elements of x,
here

$$\mathbb{G}_d := \int_{\vartheta_1}^{\vartheta_2}\left(\sum_{i=1}^{l} p_i G_d(x_i,v) - \sum_{i=1}^{l} p_i G_d(y_i,v)\right) dv,$$
$$\tilde{\mathbb{G}}_d := \int_{\vartheta_1}^{\vartheta_2}\left(\sum_{i=1}^{l} G_d(x_i,v) - \sum_{i=1}^{l} G_d(y_i,v)\right) dv,$$
$$f(t) = \int_{u}^{\vartheta_2}\left(\sum_{i=1}^{l} p_i G_d(x_i,v) - \sum_{i=1}^{l} p_i G_d(y_i,v)\right)(v-u)^{n-3} dv.$$

Application 2. Let $f:(0,\infty)\to\mathbb{R}$ be function defined by $f(x)=x\log x$. Let $x=(x_1,x_2,\ldots,x_l)$ and $y=(y_1,y_2,\ldots,y_l)$ be positive l-tuples. Then the Ostrowski type inequality (2.404) for $n=3$ as an upper bound of our generalized result becomes

$$\begin{aligned}&\left|\sum_{i=1}^{l} p_i x_i log x_i - \sum_{i=1}^{l} p_i y_i \log y_i + (\log\vartheta_2+1)\sum_{i=1}^{l} p_i\,(x_i-y_i)+\frac{1}{\vartheta_1^2}\mathbb{G}_d\right|\\ &\leq \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q}-\vartheta_1^{1-2q}\right)^{\frac{1}{q}}\|f\|_{q'}.\end{aligned}$$

If the majorization condition $\sum_{i=1}^{l} p_i x_i=\sum_{i=1}^{l} p_i y_i$ holds and $p_i=1$, $(i=1,2,\ldots,l)$ then

$$\begin{aligned}&\left|\log\left(x_1^{x_1}\cdot x_2^{x_2}\cdots x_l^{x_l}\right)+\log\left(y_1^{-y_1}\cdot y_2^{-y_2}\cdots y_l^{-y_l}\right)+\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|\\ &\leq \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q}-\vartheta_1^{1-2q}\right)^{\frac{1}{q}}\|f\|_{q'},\end{aligned}$$

i.e.

$$\left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)+\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|\leq \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q}-\vartheta_1^{1-2q}\right)^{\frac{1}{q}}\|f\|_{q'}$$

and if

$$\left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)\right|\geq\left|\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|,$$

then

$$\begin{aligned}\left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)\right|-\left|\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right| &\leq \left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)+\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d\right|\\ &\leq \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q}-\vartheta_1^{1-2q}\right)^{\frac{1}{q}}\|f\|_{q'},\end{aligned}$$

i.e. we get

$$0\leq\left|\log\left(\frac{\Pi_{i=1}^{l}x_i^{x_i}}{\Pi_{i=1}^{l}y_i^{y_i}}\right)\right|\leq \frac{1}{(2q-1)^{\frac{1}{q}}}\left(\vartheta_2^{1-2q}-\vartheta_1^{1-2q}\right)^{\frac{1}{q}}+\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d.$$

If the quotient in the left hand side is greater than equal to 1, then

$$0\leq\Pi_{i=1}^{l}x_i^{x_i}\leq e^{\frac{2}{(1-3q)^{\frac{1}{q}}}\left(\vartheta_2^{1-3q}-\vartheta_1^{1-3q}\right)^{\frac{1}{q}}+\frac{1}{\vartheta_1^2}\tilde{\mathbb{G}}_d}\,\Pi_{i=1}^{l}y_i^{y_i}.$$

We get another relation between the elements of x and the elements of y. Here, $\mathbb{G}_d$, $\tilde{\mathbb{G}}_d$ and $f(t)$ are defined as in Application 1.

Application 3. Let $x=(x_1,x_2,\ldots,x_l)$ and $y=(y_1,y_2,\ldots,y_l)$ be l-tuples such that $x_i,y_i\in[\vartheta_1,\vartheta_2]$ and $p=(p_1,p_2,\ldots,p_l)$ be such that $p_i\in\mathbb{R}$. Then the Ostrowski-type inequality (2.404) for $n=3$ as an upper bound of our generalized result is as follows:

- let $f(x)=e^x$, $x\in\mathbb{R}$, then

$$\begin{aligned}0 &\leq \left|\sum_{i=1}^{l} p_i e^{x_i}-\sum_{i=1}^{l} p_i e^{y_i}-e^{\vartheta_2}\sum_{i=1}^{l} p_i\left(x_i-y_i\right)-e^{\vartheta_1}\mathbb{G}_d\right| \\ &\leq \frac{1}{q}(e^{q\vartheta_2}-e^{q\vartheta_1})^{\frac{1}{q}}\parallel f\parallel_{q'},\end{aligned}$$

- let $f(x)=x^r$, $x\in[0,\infty)$, $r>1$, then

$$\begin{aligned}0 &\leq \left|\sum_{i=1}^{l} p_i x_i^r-\sum_{i=1}^{l} p_i y_i^r-r\vartheta_2^{r-1}-r(r-1)\vartheta_1^{r-2}\mathbb{G}_d\right| \\ &\leq \frac{r(r-1)(r-2)}{(rq-3q+1)^{\frac{1}{q}}}\left(\vartheta_2^{q(r-3)+1}-\vartheta_1^{q(r-3)+1}\right)^{\frac{1}{q}}\parallel f\parallel_{q'}.\end{aligned}$$

Remark 1.19 is also valid for these functionals.

Remark 2.39 ([6]) *Similar examples can be discussed as given in Section 1.4.*

2.4 Majorization and Generalized Montgomery Identity

The aim of this section is to present new generalizations of majorization theorems for n-convex functions by using Montgomery identity in combination with Green's functions. In the first subsection we obtain new generalizations using only Montgomery identity. In the second subsection we combinate Montgomery identity with nice properties of Green's function. In the next subsection, these results will be complemented with new results that include other type of Green's functions. In the last two subsection we consider generalizations of Jensen's and the Jensen-Steffensen inequality using Montgomery identity in combination with Green's functions. For obtained results in every subsection we also give new bounds for the reminders in new majorization identities by using the Čebyšev type inequalities. We give corresponding mean value theorems with connection to n-exponential convexity for functionals related to these new majorization identities.

The following theorem is a consequence of Theorem 1 in [140] (see also [144, p. 328]) and represents one more form of an integral majorization result.

Theorem 2.131 ([18]) *Let $x, y : [\alpha, \beta] \to I$ be two decreasing continuous functions and $w : [\alpha, \beta] \to \mathbb{R}$ continuous. If*

$$\int_\alpha^u w(t) y(t)\, dt \leq \int_\alpha^u w(t) x(t)\, dt, \quad \text{for each} \quad u \in (\alpha, \beta), \tag{2.415}$$

$$\text{and} \quad \int_\alpha^\beta w(t) y(t)\, dt = \int_\alpha^\beta w(t) x(t)\, dt, \tag{2.416}$$

holds, then for every continuous convex function $\phi : I \to \mathbb{R}$ the inequality

$$\int_\alpha^\beta w(t)\, \phi(y(t))\, dt = \int_\alpha^\beta w(t)\, \phi(x(t))\, dt$$

holds.

In [17], the following extension of Montgomery identity via Taylor's formula is obtained.

Proposition 2.1 *Let $n \in \mathbb{N}$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Then the following identity holds*

$$\begin{aligned} \phi(x) = {} & \frac{1}{b-a} \int_a^b \phi(t)\, dt + \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(a)}{k!\,(k+2)} \frac{(x-a)^{k+2}}{b-a} - \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(b)}{k!\,(k+2)} \frac{(x-b)^{k+2}}{b-a} \\ & + \frac{1}{(n-1)!} \int_a^b T_n(x,s)\, \phi^{(n)}(s)\, ds \end{aligned} \tag{2.417}$$

where

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1}, & a \leq s \leq x, \\ -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1}, & x < s \leq b. \end{cases} \tag{2.418}$$

We will refer to (2.417) as ***generalized Montgomery identity****.*

In case $n = 1$ the sum $\sum_{k=0}^{n-2} \cdots$ is empty, so identity (2.417) reduces to well-known ***Montgomery identity*** *(see for instance [129])*

$$\phi(x) = \frac{1}{b-a} \int_a^b \phi(t)\, dt + \int_a^b P(x,s)\, \phi'(s)\, ds$$

where $P(x,s)$ is Peano's kernel, defined by

$$P(x,s) = \begin{cases} \frac{s-a}{b-a}, & a \leq s \leq x, \\ \frac{s-b}{b-a}, & x < s \leq b. \end{cases}$$

2.4.1 Results Obtained by Montgomery Identity

In this subsection we will state our results for decreasing x and y satisfying the assumption of Theorem 2.131, but they are still valid for increasing x and y (see Theorem 1.18 and [123, p. 584]).

Theorem 2.132 ([18]) *Suppose all assumptions from the Proposition 2.1 hold. Additionally suppose that $m \in \mathbb{N}$, $x_i, y_i \in [a,b]$ and $w_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$. Then*

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \tag{2.419}$$

$$= \frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)!} \sum_{i=1}^{m} w_i \left[\phi^{(k+1)}(a)\left[(x_i-a)^{k+2}-(y_i-a)^{k+2}\right]\right.\right.$$
$$\left.\left.-\phi^{(k+1)}(b)\left[(x_i-b)^{k+2}-(y_i-b)^{k+2}\right]\right]\right]$$
$$+\frac{1}{(n-1)!}\int_a^b \left(\sum_{i=1}^{m} w_i \left(T_n(x_i,s)-T_n(y_i,s)\right)\right)\phi^{(n)}(s)\,ds$$

Proof. We take extension of Montgomery identity via Taylor's formula (2.417) to obtain

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) = \frac{1}{b-a}\int_a^b \phi(t)\,dt \sum_{i=1}^{m} w_i - \frac{1}{b-a}\int_a^b \phi(t)\,dt \sum_{i=1}^{m} w_i$$
$$+\sum_{i=1}^{m} w_i \left(\sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(a)}{k!\,(k+2)} \frac{(x_i-a)^{k+2}}{b-a} - \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(b)}{k!\,(k+2)} \frac{(x_i-b)^{k+2}}{b-a}\right)$$
$$-\sum_{i=1}^{m} w_i \left(\sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(a)}{k!\,(k+2)} \frac{(y_i-a)^{k+2}}{b-a} - \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(b)}{k!\,(k+2)} \frac{(y_i-b)^{k+2}}{b-a}\right)$$
$$+\frac{1}{(n-1)!}\sum_{i=1}^{m} w_i \int_a^b T_n(x_i,s)\,\phi^{(n)}(s)\,ds - \frac{1}{(n-1)!}\sum_{i=1}^{m} w_i \int_a^b T_n(y_i,s)\,\phi^{(n)}(s)\,ds.$$

By simplifying this expressions we obtain (2.419). □

We may state its integral version as follows:

Theorem 2.133 ([18]) *Let $x,y : [\alpha,\beta] \to [a,b]$ be two functions and $w : [\alpha,\beta] \to \mathbb{R}$ continuous function. Let $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Then for all $s \in [a,b]$ we have the following identity*

$$\int_\alpha^\beta w(t)\,\phi(x(t))\,dt - \int_\alpha^\beta w(t)\,\phi(y(t))\,dt. \tag{2.420}$$

$$= \frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)!} \int_\alpha^\beta w(t)\left[\phi^{(k+1)}(a)\left[(x(t)-a)^{k+2}-(y(t)-a)^{k+2}\right]\right.\right.$$
$$\left.\left.-\phi^{(k+1)}(b)\left[(x(t)-b)^{k+2}-(y(t)-b)^{k+2}\right]\right]dt\right]$$
$$+\frac{1}{(n-1)!}\int_a^b \left(\int_\alpha^\beta w(t)\left(T_n(x(t),s)-T_n(y(t),s)\right)dt\right)\phi^{(n)}(s)\,ds$$

where $T_n(\cdot,s)$ is as defined in Proposition 2.1.

Proof. Our required result is obtained by using extension of Montgomery identity via Taylor's formula (2.417) in the following expression

$$\int_\alpha^\beta w(t)\,\phi(x(t))\,dt - \int_\alpha^\beta w(t)\,\phi(y(t))\,dt$$

and then using Fubini's theorem. □

Now we state the main generalization of the majorization inequality using just obtained identities.

Theorem 2.134 ([18]) *Let all the assumptions of Theorem* 2.132 *hold with additional condition*

$$\sum_{i=1}^{n} w_i T_n(y_i,s) \leq \sum_{i=1}^{n} w_i T_n(x_i,s), \quad \forall\, s \in [a,b]. \tag{2.421}$$

Then for every n-convex function $\phi : I \to \mathbb{R}$ the following inequality holds

$$\begin{aligned}
&\sum_{i=1}^{m} w_i \phi\left(x_i\right) - \sum_{i=1}^{m} w_i \phi\left(y_i\right) \\
&\geq \frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)!} \sum_{i=1}^{m} w_i \left[\phi^{(k+1)}(a)\left[(x_i-a)^{k+2}-(y_i-a)^{k+2}\right]\right.\right. \\
&\left.\left.-\phi^{(k+1)}(b)\left[(x_i-b)^{k+2}-(y_i-b)^{k+2}\right]\right]\right]
\end{aligned} \tag{2.422}$$

Proof. Since the function ϕ is n-convex so without loss of generality we can assume that ϕ is n-times differentiable and therefore we have $\phi^{(n)} \geq 0$. Using this fact and (2.421) in (2.419) we easily arrive at our required result. □

Remark 2.40 ([18]) *If the reverse inequality holds in* (2.421)*, then the reverse inequality holds in* (2.422).

Now we state important consequence as follows:

Theorem 2.135 ([18]) *Suppose that all assumptions from the Theorem 2.132 hold. Additionally suppose that* $\mathbf{x}$, $\mathbf{y} \in [a,b]^m$ *are two decreasing m-tuples and* $\mathbf{w} \in \mathbb{R}^m$ *which satisfy conditions* (1.19), (1.20).

(i) If ϕ is $2n$*-convex then the following inequality holds*

$$\begin{aligned}
&\sum_{i=1}^{m} w_i \phi\left(x_i\right) - \sum_{i=1}^{m} w_i \phi\left(y_i\right) \\
&\geq \frac{1}{b-a}\left[\sum_{k=0}^{2n-2} \frac{1}{k!\,(k+2)!} \sum_{i=1}^{m} w_i \left[\phi^{(k+1)}(a)\left[(x_i-a)^{k+2}-(y_i-a)^{k+2}\right]\right.\right. \\
&\left.\left.-f^{(k+1)}(b)\left[(x_i-b)^{k+2}-(y_i-b)^{k+2}\right]\right]\right].
\end{aligned} \tag{2.423}$$

(ii) If the inequality (2.423) holds and the function F defined by

$$F(.) = \sum_{k=0}^{2n-2} \frac{\phi^{(k+1)}(a)}{k!(k+2)} \frac{(.-a)^{k+2}}{b-a} - \sum_{k=0}^{2n-2} \frac{\phi^{(k+1)}(b)}{k!(k+2)} \frac{(.-b)^{k+2}}{b-a} \tag{2.424}$$

is convex, then the right hand side of (2.423) is non-negative, that is (2.157) holds.

Proof. (i) Since

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1}, \ a \leq s \leq x \leq b, \\ \\ -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1}, \ a \leq x < s \leq b. \end{cases}$$

and

$$\frac{d^2}{dx^2} T_n(x,s) = \begin{cases} \frac{n-1}{b-a}\left[(x-s)^{n-2} + (n-2)(x-a)(x-s)^{n-3}\right], \ a \leq s \leq x \leq b, \\ \\ \frac{n-1}{b-a}\left[(x-s)^{n-2} + (n-2)(x-b)(x-s)^{n-3}\right], \ a \leq x < s \leq b. \end{cases}$$

$T_n(\cdot,s)$ is continuous for every $n \geq 2$ and convex function for even n. Thus it satisfies inequality (2.421) by weighted majorization theorem (Theorem 1.14) and hence (2.421) by Theorem 2.134 provides us (2.422) with $2n$ instead of n.

(ii) The proof is similar to the proof of Theorem 2.112 (ii). □

Remark 2.41 ([18]) *Since in the case $a \leq s \leq x \leq b$ the function $\frac{d^2}{dx^2}T_n(x,s)$ is always positive, $T_n(x,s)$ can not be concave and reverse inequalities can not be observed.*

Also, if $w_i = 1$, $i = 1,\ldots,m$ the result of the previous theorem holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y} \prec \mathbf{x}$.

Its integral analogues are given as follows:

Theorem 2.136 ([18]) *Let all the assumptions of Theorem 2.133 hold with additional condition*

$$\int_{\alpha}^{\beta} w(t)\, T_n(y(t),s)\, dt \leq \int_{\alpha}^{\beta} w(t)\, T_n(x(t),s)\, dt, \quad \forall\, s \in [a,b] \tag{2.425}$$

where $T_n(\cdot,s)$ is defined in Proposition 2.1. Then for every n-convex function $\phi : I \to \mathbb{R}$ the following inequality holds

$$\begin{aligned} &\int_{\alpha}^{\beta} w(t)\,\phi(x(t))\,dt - \int_{\alpha}^{\beta} w(t)\,\phi(y(t))\,dt \qquad (2.426) \\ &\geq \frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!(k+2)!} \int_{\alpha}^{\beta} w(t)\left[\phi^{(k+1)}(a)\left[(x(t)-a)^{k+2} - (y(t)-a)^{k+2}\right]\right.\right. \\ &\left.\left. -\phi^{(k+1)}(b)\left[(x(t)-b)^{k+2} - (y(t)-b)^{k+2}\right]\right] dt\right]. \end{aligned}$$

Proof. Since the function ϕ is n-convex so without loss of generality we can assume that ϕ is n-times differentiable and therefore we have we have $\phi^{(n)} \geq 0$. Using this fact and (2.425) in (2.420) we easily arrive at our required result. □

Remark 2.42 ([18]) *If the reverse inequality holds in* (2.425) *then the reverse inequality holds in* (2.426).

The integral version of Theorem 2.135 can be stated as follows.

Theorem 2.137 ([18]) *Suppose that all assumptions from the Theorem 2.133 hold. Additionally suppose that x and y are decreasing which satisfy conditions* (2.415), (2.416).

(i) If ϕ is $2n$-convex then the following inequality holds

$$\begin{aligned}
&\int_{\alpha}^{\beta} w(t)\,\phi(x(t))\,dt - \int_{\alpha}^{\beta} w(t)\,\phi(y(t))\,dt \\
&\geq \frac{1}{b-a}\left[\sum_{k=0}^{2n-2} \frac{1}{k!\,(k+2)!} \int_{\alpha}^{\beta} w(t)\left[\phi^{(k+1)}(a)\left[(x(t)-a)^{k+2} - (y(t)-a)^{k+2}\right]\right.\right. \\
&\left.\left.-\phi^{(k+1)}(b)\left[(x(t)-b)^{k+2} - (y(t)-b)^{k+2}\right]\right] dt\right]
\end{aligned} \tag{2.427}$$

(ii) If the inequality (2.427) holds and the function F defined by (2.424) is convex, then the right hand side of (2.427) is non-negative, that is (2.159) holds.

Now by using aforementioned results Theorem 1.10 and Theorem 1.11, we are going to obtain generalizations of the results proved in the previous.

For $m-$tuples $\mathbf{w} = (w_1, \ldots, w_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ with $x_i, y_i \in [a,b]$, $w_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ and the function T_n defined as in (2.418), denote

$$\delta(s) = \sum_{i=1}^{n} w_i T_n(x_i, s) - \sum_{i=1}^{n} w_i T_n(y_i, s), \quad s \in [a,b]. \tag{2.428}$$

Similarly for continuous functions $x, y : [\alpha, \beta] \to [a,b]$ and $w : [\alpha, \beta] \to \mathbb{R}$, denote

$$\Delta(s) = \int_{\alpha}^{\beta} w(t)\,T_n(x(t), s)\,dt - \int_{\alpha}^{\beta} w(t)\,T_n(y(t), s)\,dt \quad s \in [a,b]. \tag{2.429}$$

Hence by using these notations we define Čebyšev functionals as follows:

$$T(\delta, \delta) = \frac{1}{b-a}\int_a^b \delta^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b \delta(s)\,ds\right)^2,$$

$$T(\Delta, \Delta) = \frac{1}{b-a}\int_a^b \Delta^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b \Delta(s)\,ds\right)^2.$$

Theorem 2.138 ([18]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)[\phi^{(n+1)}]^2 \in L[a,b]$ and $x_i, y_i \in [a,b]$ and $w_i \in \mathbb{R}$ $(i = 1,2,\ldots,m)$ and let the functions T_n and δ be defined in* (2.418) *and* (2.428) *respectively. Then it holds*

$$\begin{aligned}
&\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \qquad (2.430)\\
&= \frac{1}{b-a} \sum_{i=1}^{m} w_i \left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)!} \left[\phi^{(k+1)}(a) \left[(x_i - a)^{k+2} - (y_i - a)^{k+2} \right] \right.\right.\\
&\left.\left. -\phi^{(k+1)}(b) \left[(x_i - b)^{k+2} - (y_i - b)^{k+2} \right] \right] \right]\\
&+ \frac{\left[\phi^{(n-1)}(b) - \phi^{(n-1)}(a) \right]}{(n-1)!(b-a)^2} \int_a^b \delta(s)ds + R_n^1(\phi; a,b),
\end{aligned}$$

where the remainder $R_n^1(f;a,b)$ satisfies the estimation

$$|R_n^1(\phi;a,b)| \leq \frac{1}{(n-1)!} \left(\frac{1}{2(b-a)} \left| T(\delta,\delta) \int_a^b (s-a)(b-s)[\phi^{(n+1)}(s)]^2 ds \right| \right)^{1/2}. \tag{2.431}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

Here we state the integral version of the previous theorem.

Theorem 2.139 ([18]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)[\phi^{(n+1)}]^2 \in L[a,b]$ and $x, y : [\alpha,\beta] \to [a,b]$ and $w : [\alpha,\beta] \to \mathbb{R}$ and let the functions T_n and Δ be defined in* (2.418) *and* (2.429) *respectively. Then it holds*

$$\begin{aligned}
&\int_\alpha^\beta w(t)\,\phi(x(t))\,dt - \int_\alpha^\beta w(t)\,\phi(y(t))\,dt \qquad (2.432)\\
&= \frac{1}{b-a} \left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)!} \int_\alpha^\beta w(t) \left[\phi^{(k+1)}(a) \left[(x(t) - a)^{k+2} - (y(t) - a)^{k+2} \right] \right.\right.\\
&\left.\left. -\phi^{(k+1)}(b) \left[(x(t) - b)^{k+2} - (y(t) - b)^{k+2} \right] \right] dt \right]\\
&+ \frac{\left[\phi^{(n-1)}(b) - \phi^{(n-1)}(a) \right]}{(n-1)!(b-a)^2} \int_a^b \Delta(s)ds + R_n^2(\phi; a,b),
\end{aligned}$$

where the remainder $R_n^2(\phi;a,b)$ satisfies the estimation

$$|R_n^2(\phi;a,b)| \leq \frac{1}{(n-1)!} \left(\frac{1}{2(b-a)} \left| T(\Delta,\Delta) \int_a^b (s-a)(b-s)[\phi^{(n+1)}(s)]^2 ds \right| \right)^{1/2}. \tag{2.433}$$

Proof. This results easily follows by proceeding as in the proof of previous theorem and by replacing (2.419) by (2.420). □

By using Theorem 1.11 we obtain the following Grüss type inequality.

Theorem 2.140 ([18]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$ for $n \in \mathbb{N}$ with $\phi^{(n+1)} \geq 0$ on $[a,b]$ and let the functions T and δ be defined in (1.6) and (2.428) respectively. Then we have the representation (2.430) and the remainder $R_n^1(\phi;a,b)$ satisfies the following condition*

$$|R_n^1(\phi;a,b)| \leq \frac{1}{(n-1)!}\|\delta'\|_\infty \left[\frac{\phi^{(n-1)}(b)+\phi^{(n-1)}(a)}{2} - \frac{\phi^{(n-2)}(b)-\phi^{(n-2)}(a)}{b-a}\right]. \tag{2.434}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

Integral version of the above theorem can be given as follows.

Theorem 2.141 ([18]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$ for $n \in \mathbb{N}$ with $\phi^{(n+1)} \geq 0$ on $[a,b]$ and let the functions T and Δ be defined in (1.6) and (2.429) respectively. Then we have the representation (2.432) and the remainder $R_n^2(\phi;a,b)$ satisfies the following condition*

$$|R_n^2(\phi;a,b)| \leq \frac{1}{(n-1)!}\|\Delta'\|_\infty \left\{\frac{\phi^{(n-1)}(b)+\phi^{(n-1)}(a)}{2} - \frac{\phi^{(n-1)}(b)-\phi^{(n-1)}(a)}{b-a}\right\}.$$

we state some Ostrowski type inequalities related to the generalized majorization inequalities.

Theorem 2.142 ([18]) *Let all the assumptions of Theorem 2.132 hold. Furthermore, let (p,q) be a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $\left|\phi^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$, $n > 1$. Then we have*

$$\begin{aligned} &\left|\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) - \frac{1}{b-a}\sum_{i=1}^m w_i\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)!}\left[\phi^{(k+1)}(a)\cdot\right.\right.\right. \\ &\left.\left.\left.\cdot\left[(x_i-a)^{k+2}-(y_i-a)^{k+2}\right] - \phi^{(k+1)}(b)\left[(x_i-b)^{k+2}-(y_i-b)^{k+2}\right]\right]\right]\right| \\ &\leq \frac{1}{(n-1)!}\|\phi^{(n)}\|_p \left\|\sum_{i=1}^m w_i\left(T_n(x_i,\cdot)-T_n(y_i,\cdot)\right)\right\|_q. \end{aligned} \tag{2.435}$$

The constant on the right hand side of (2.435) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Integral case of the above theorem can be given as follows.

Theorem 2.143 ([18]) *Let all the assumptions of Theorem 2.133 be hold. Furthermore, let (p,q) be a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $\left|f^{(n)}\right|^p$: $[a,b] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$. Then we have*

$$\left|\int_{\alpha}^{\beta} w(t)\,\phi(x(t))\,dt - \int_{\alpha}^{\beta} w(t)\,\phi(y(t))\,dt\right. \tag{2.436}$$
$$-\frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)!}\int_{\alpha}^{\beta} w(t)\left[\phi^{(k+1)}(a)\left[(x(t)-a)^{k+2}-(y(t)-a)^{k+2}\right]\right.\right.$$
$$\left.\left.\left.-\phi^{(k+1)}(b)\left[(x(t)-b)^{k+2}-(y(t)-b)^{k+2}\right]\right]dt\right]\right|$$
$$\leq \frac{1}{(n-1)!}\|\phi^{(n)}\|_p\left\|\int_{\alpha}^{\beta} w(t)\left(T_n(x(t),s)-T_n(y(t),s)\right)dt\right\|_q.$$

The constant on the right hand side of (2.436) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

For our next goal, we give here some constructions as follows. Under the assumptions of Theorem 2.134 using (2.422) and Theorem 2.136 using (2.425) we define the following functionals:

$$\Lambda_1(\phi) = \sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) - \frac{1}{b-a}\sum_{i=1}^{m} w_i\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)!}\left[\phi^{(k+1)}(a)\right.\right.$$
$$\left.\left.\cdot\left[(x_i-a)^{k+2}-(y_i-a)^{k+2}\right] - \phi^{(k+1)}(b)\left[(x_i-b)^{k+2}-(y_i-b)^{k+2}\right]\right]\right] \tag{A1}$$

$$\Lambda_2(\phi) = \int_{\alpha}^{\beta} w(t)\,\phi(x(t))\,dt - \int_{\alpha}^{\beta} w(t)\,\phi(y(t))\,dt$$
$$-\frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)!}\int_{\alpha}^{\beta} w(t)\left[\phi^{(k+1)}(a)\left[(x(t)-a)^{k+2}-(y(t)-a)^{k+2}\right]\right.\right.$$
$$\left.\left.-\phi^{(k+1)}(b)\left[(x(t)-b)^{k+2}-(y(t)-b)^{k+2}\right]\right]dt\right] \tag{A2}$$

Now we give mean value theorems for Λ_k, $k \in \{1,2\}$. Here $\phi_0(x) = \frac{x^n}{n!}$.

Theorem 2.144 ([18]) *Let* $\phi \in C^{(n)}[a,b]$ *and let* $\Lambda_k : C^{(n)}[a,b] \to \mathbb{R}$ *for* $k \in \{1,2\}$ *be linear functionals as defined in* (A1) *and* (A2) *respectively. Then there exist* $\xi_k \in [a,b]$ *for* $k \in \{1,2\}$ *such that*

$$\Lambda_k(\phi) = \phi^{(n)}(\xi_k)\Lambda_k(\phi_0), \quad k \in \{1,2\}. \tag{2.437}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (see also the proof of Theorem 4.1 in [86]). □

Theorem 2.145 ([18]) *Let* $f,g \in C^{(n)}[a,b]$ *and let* $\Lambda_k : C^{(n)}[a,b] \to \mathbb{R}$ *for* $k \in \{1,2\}$ *be linear functionals as defined in (A1) and (A2) respectively. Then there exist* $\xi_k \in [a,b]$ *for* $k \in \{1,2\}$ *such that*

$$\frac{\Lambda_k(f)}{\Lambda_k(g)} = \frac{f^{(n)}(\xi_k)}{g^{(n)}(\xi_k)},$$

assuming that both denominators are non-zero.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.14 (see also the proof of Corollary 4.2 in [86]). □

Remark 2.43 ([18]) *If the inverse of $\frac{f^{(n)}}{g^{(n)}}$ exists, then from the above mean value theorems we can give generalized means*

$$\xi_k = \left(\frac{f^{(n)}}{g^{(n)}}\right)^{-1}\left(\frac{\Lambda_k(f)}{\Lambda_k(g)}\right), \quad k \in \{1,2\}. \tag{2.438}$$

A number of important inequalities arise from the logarithmic convexity of some functions as one can see in [123].

Here, we get our results concerning the n-exponential convexity and exponential convexity for our functionals Λ_k, $k \in \{1,2\}$ as defined in $(A1)$ and $(A2)$. In the remaining part of this subsection I denotes an interval in $\mathbb{R}$.

Theorem 2.146 ([18]) *Let $D_1 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is n-exponentially convex in the $J-$sense on I for any n mutually distinct points $z_0, z_1, \ldots, z_n \in [a,b]$. Let Λ_k be the linear functionals for $k \in \{1,2\}$ as defined in $(A1)$ and $(A2)$. Then the following statements are valid:*

(a) *The function $t \mapsto \Lambda_k(f_t)$ is n-exponentially convex function in the $J-$sense on I.*

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is n-exponentially convex on I.*

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39 but using linear functionals $\Lambda_k (k = 1,2)$ instead of $\digamma_k (k = 1,2,..,5)$. □

As a consequence of the above theorem we give the following corollaries.

Corollary 2.33 ([18]) *Let $D_2 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is an exponentially convex in the $J-$sense on I for any n mutually distinct points $z_0, z_1, \ldots, z_n \in [a,b]$. Let Λ_k be the linear functionals for $k \in \{1,2\}$ as defined in $(A1)$ and $(A2)$. Then the following statements are valid:*

(a) *The function $t \mapsto \Lambda_k(f_t)$ is exponentially convex in the $J-$sense on I.*

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is exponentially convex on I.*

(c) *The matrix $\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m}$ is positive-semidefinite. Particularly,*

$$\det\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ where $i \in \{1, \ldots, m\}$.

Proof. Proof follows directly from Theorem 2.146 by using definition of exponential convexity and Corollary 1.1. □

Corollary 2.34 ([18]) *Let $D_3 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is $2-$exponentially convex in the $J-$sense on I for any n mutually distinct points $z_0, z_1, \ldots, z_n \in [a,b]$. Let Λ_k be the linear functionals for $k \in \{1,2\}$ as defined in $(A1)$ and $(A2)$. Then the following statements are valid:*

(a) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then it is $2-$exponentially convex on I. If the function $t \mapsto \Lambda_k(f_t)$ is additionally positive, then it is also log-convex on I. Moreover, for $r < s < t$, $r, s, t \in I$*

$$[\Lambda_k(f_s)]^{t-r} \leq [\Lambda_k(f_r)]^{t-s} [\Lambda_k(f_t)]^{s-r}. \tag{2.439}$$

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is positive and differentiable on I, then for every $s,t,u,v \in I$ such that $s \leq u$ and $t \leq v$, we have*

$$\mu_{s,t}(\Lambda_k, D_3) \leq \mu_{u,v}(\Lambda_k, D_3) \tag{2.440}$$

where $\mu_{s,t}$ is defined as

$$\mu_{s,t}(\Lambda_k, D_3) = \begin{cases} \left(\frac{\Lambda_k(f_s)}{\Lambda_k(f_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Lambda_k(f_s)}{\Lambda_k(f_s)}\right) & , \quad s = t. \end{cases} \tag{2.441}$$

for $f_s, f_t \in D_3$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10 but using linear functionals $\Lambda_k (k = 1,2)$ instead of $F_k (k = 1,2,..,5)$. □

Remark 1.19 is also valid for these functionals.

Remark 2.44 ([18]) *Similar examples can be discussed as given in Section 1.4.*

2.4.2 Results Obtained by Green's Function and Montgomery Identity

First we state some results related to weighted majorization identities and inequalities. For that we define some notations in terms of positive linear functional as follows (see [21]):

$$\Delta(p_i, x_i, y_i, f) = \sum_{i=1}^{m} p_i(f(y_i) - f(x_i)) \tag{2.442}$$

and by the notation $\Delta(p_i, x_i, y_i, G(\cdot, s))$ we would mean

$$\Delta(p_i, x_i, y_i, G(\cdot, s)) = \sum_{i=1}^{m} p_i(G(y_i, s) - G(x_i, s)) \tag{2.443}$$

where p_i, x_i, y_i and f are as defined in Theorem 1.14 and G is as defined in (1.180). Also

$$\Lambda(p,x,y,f) = \int_{\alpha}^{\beta} p(u)\,(f(y(u)) - f(x(u)))\,du \tag{2.444}$$

and by the notation $\Lambda(p,x,y,G(\cdot,s))$ we would mean

$$\Lambda(p,x,y,G(\cdot,s)) = \int_{\alpha}^{\beta} p(u)\,(G(y(u),s) - G(x(u),s))\,du \tag{2.445}$$

where p,x,y and f (but w, ϕ instead p, f) are as defined in Theorem 2.131 and G is as defined in (1.180). Also "*id*" would represent identity function *i.e.* $id(x) = x$ for all x.

Theorem 2.147 ([21]) *Let all the assumptions of Theorem* 1.14 *be valid. Also let* $f : I \to \mathbb{R}$ *be a function such that* $f^{(n-1)}$ $(n \geq 3)$ *is absolutely continuous,* $I \subset \mathbb{R}$ *an open interval,* $a,b \in I$, $a < b$, *then for all* $s \in [a,b]$, *we have the following identity*

$$\begin{aligned}\Delta(p_i,x_i,y_i,f) = \frac{f(b)-f(a)}{b-a}&\Delta(p_i,x_i,y_i,id)\\ &+\frac{f'(a)-f'(b)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds\\ +\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))&\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds\\ &+\frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds\right)dt\end{aligned} \tag{2.446}$$

where

$$\tilde{T}_{n-2}(s,t) = \begin{cases} \frac{1}{b-a}\left[\frac{(s-t)^{n-2}}{(n-2)} + (s-a)(s-t)^{n-3}\right], & a \leq t \leq s \leq b,\\ \frac{1}{b-a}\left[\frac{(s-t)^{n-2}}{(n-2)} + (s-b)(s-t)^{n-3}\right], & a \leq s < t \leq b.\end{cases}$$

and $G(\cdot,s)$ *is as defined in* (1.180). *Moreover, we also obtain the following identity*

$$\begin{aligned}\Delta(p_i,x_i,y_i,f) = \frac{f(b)-f(a)}{b-a}&\Delta(p_i,x_i,y_i,id)\\ &+\frac{f'(b)-f'(a)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds\\ +\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))&\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds\\ &+\frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))T_{n-2}(s,t)ds\right)dt\end{aligned} \tag{2.447}$$

where T_n *is as defined in Proposition* 2.1.

Proof. Using (1.181) in (2.442) and using linearity of $\Delta(p_i,x_i,y_i,f)$, we get

$$\Delta(p_i,x_i,y_i,f) = \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) + \int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))f''(s)ds. \quad (2.448)$$

Differentiating (2.417) twice with respect to s, we get

$$f''(s) = \frac{f'(a)-f'(b)}{b-a} + \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a} + \frac{1}{(n-3)!}\int_a^b \tilde{T}_{n-2}(s,t)f^{(n)}(t)dt. \quad (2.449)$$

Now using (2.449) in (2.448) we get

$$\begin{aligned}\Delta(p_i,x_i,y_i,f) = {} & \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) \\ & + \frac{f'(a)-f'(b)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds \\ & + \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \\ & + \frac{1}{(n-3)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\left(\int_a^b \tilde{T}_{n-2}(s,t)f^{(n)}(t)dt\right)ds\end{aligned}$$

and then using Fubini's theorem in the last term to get (2.446).

Also, by using formula (2.417) on the function f'', replacing n by $n-2$ ($n \geq 3$) and rearranging the indices we get

$$f''(s) = \frac{f'(b)-f'(a)}{b-a} + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a} + \frac{1}{(n-3)!}\int_a^b T_{n-2}(s,t)f^{(n)}(t)dt \quad (2.450)$$

Similarly, using (2.450) in (2.448) and applying Fubini's theorem, we get (2.447). □

Theorem 2.148 ([21]) *Let all the assumptions of Theorem* 2.147 *hold with additional condition*

$$\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds \geq 0, \quad t \in [a,b], \quad (2.451)$$

where G is as defined in (1.180) *and $\tilde{T}_n$ is defined in Theorem 2.147 . Then for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds*

$$\Delta(p_i,x_i,y_i,f) \geq \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) + \frac{f'(a)-f'(b)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds + \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds. \quad (2.452)$$

Proof. Since the function f is n-convex, so without loss of generality we can assume that $f^{(n)}$ exists so we have $f^{(n)} \geq 0$. Using this fact and (2.451) in (2.446) we easily arrive at our required result. □

Theorem 2.149 ([21]) *Let all the assumptions of Theorem* 2.147 *hold with additional condition*

$$\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))T_{n-2}(s,t)ds \geq 0, \quad \forall\, t \in [a,b] \quad (2.453)$$

where G is as defined in (1.180) *and T_n is defined in Proposition* 2.1. *Then for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds*

$$\Delta(p_i,x_i,y_i,f) \geq \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) + \frac{f'(b)-f'(a)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \quad (2.454)$$

Proof. Since the function f is n-convex, so without loss of generality we assume that $f^{(n)}$ exists and hence, we have $f^{(n)} \geq 0$. Using this fact and (2.453) in (2.447) we easily arrive at our required result. □

Now we state an important consequence.

Theorem 2.150 ([21]) *Let all the assumptions from Theorem 2.147 hold. If f is n-convex and n is even, then inequalities* (2.452) *and* (2.454) *hold.*

Proof. Since Green's function $G(s,t)$ is convex with respect to t for all $s \in [a,b]$ and $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{p} = (p_1,\ldots,p_m)$ satisfy condition (1.19) and (1.20). Therefore, from Theorem 1.14 (inequality (1.21)) we have

$$\Delta(p_i,x_i,y_i,G(\cdot,s)) \geq 0 \quad \text{for} \quad s \in [a,b]. \quad (2.455)$$

Also note that $\tilde{T}_{n-2}(s,t) \geq 0$ ($T_{n-2}(s,t) \geq 0$) if $n-2$ is even. Therefore combining this fact with (2.455) we get inequality (2.451) (inequality (2.453)). As f is n-convex, results follow from Theorem 2.148 (Theorem 2.149 respectively). □

We omit proofs of the following theorems because of similar nature as some previous.

Theorem 2.151 ([21]) *Let all the assumptions of Theorem 1.18 be valid. Let $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$, then for all $s \in [a,b]$ we have the following identity*

$$\begin{aligned}
\Lambda(p,x,y,f) &= \frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) \\
&+ \frac{f'(a)-f'(b)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds \\
&+ \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \\
&+ \frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b \Lambda(p,x,y,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds\right)dt \qquad (2.456)
\end{aligned}$$

where $\tilde{T}_n$ is as defined in Theorem 2.147 and G is as defined in (1.180). *Moreover, we also obtain the following identity*

$$\begin{aligned}
\Lambda(p,x,y,f) &= \frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) \\
&+ \frac{f'(b)-f'(a)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds \\
&+ \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \\
&+ \frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b \Lambda(p,x,y,G(\cdot,s))T_{n-2}(s,t)ds\right)dt \qquad (2.457)
\end{aligned}$$

where T_n is as defined in Proposition 2.1.

Theorem 2.152 ([21]) *Let all the assumptions of Theorem* 2.151 *hold with additional condition*

$$\int_a^b \Lambda(p,x,y,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds \geq 0, \quad t \in [a,b], \qquad (2.458)$$

where G is as defined in (1.180) *and $\tilde{T}_n$ is defined in Theorem* 2.147. *Then for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds*

$$\begin{aligned}
\Lambda(p,x,y,f) \geq \frac{f(b)-f(a)}{b-a}&\Lambda(p,x,y,id) \\
&+ \frac{f'(a)-f'(b)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds \\
+ \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b &\Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds. \qquad (2.459)
\end{aligned}$$

Theorem 2.153 ([21]) *Let all the assumptions of Theorem* 2.151 *hold with additional condition*

$$\int_a^b \Lambda(p,x,y,G(\cdot,s))T_{n-2}(s,t)ds \geq 0, \quad t \in [a,b] \qquad (2.460)$$

where G is as defined in (1.180) *and T_n is defined in Proposition* 2.1. *Then for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds*

$$\Lambda(p,x,y,f) \geq \frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) + \frac{f'(b)-f'(a)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \quad (2.461)$$

Here we have another result.

Theorem 2.154 ([21]) *Let all the assumptions from Theorem 2.151 hold. If f is n-convex and n is even, then inequalities* (2.459) *and* (2.461) *hold.*

Now by using aforementioned results, we are going to obtain generalizations of the result proved in previous.

Under the assumptions of Theorems 2.148, 2.149, 2.152 and 2.153 respectively, we define the following linear functionals:

$$\Omega_1(t) = \int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds \geq 0, \quad t\in[a,b] \quad (2.462)$$

$$\Omega_2(t) = \int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))T_{n-2}(s,t)ds \geq 0, \quad t\in[a,b] \quad (2.463)$$

$$\Omega_3(t) = \int_a^b \Lambda(p,x,y,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds, \quad t\in[a,b] \quad (2.464)$$

$$\Omega_4(t) = \int_a^b \Lambda(p,x,y,G(\cdot,s))T_{n-2}(s,t)ds, \quad t\in[a,b] \quad (2.465)$$

Hence by using these notations we may define Čebyšev functional as follows (e.g. using Ω):

$$T(\Omega_i,\Omega_i) = \frac{1}{b-a}\int_a^b \Omega^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b \Omega(s)\,ds\right)^2, \ i=1,2,3,4.$$

Theorem 2.155 ([21]) *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for $n \in \mathbb{N}$ with $(.-a)(b-.)[f^{(n+1)}]^2 \in L[a,b]$ and let p_i and x_i, $i\in\{1,\ldots,m\}$ satisfy the assumptions of Theorem 1.14 and let the functions G and Ω_1 be defined in* (1.180) *and* (2.462) *respectively. Then it holds*

$$\Delta(p_i,x_i,y_i,f) = \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) + \frac{f'(a)-f'(b)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds + \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds$$

$$+\frac{\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]}{(n-3)!(b-a)}\int_a^b \Omega_1(s)ds+R_n^1(f;a,b), \tag{2.466}$$

where the remainder $R_n^1(f;a,b)$ *satisfies the estimation*

$$|R_n^1(f;a,b)| \leq \frac{1}{(n-3)!}\left(\frac{(b-a)}{2}\left|T(\Omega_1,\Omega_1)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{1/2}. \tag{2.467}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

Theorem 2.156 ([21]) *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is an absolutely continuous function for* $n\in\mathbb{N}$ *with* $(.-a)(b-.)[f^{(n+1)}]^2\in L[a,b]$ *and let* p_i *and* x_i, $i\in\{1,\ldots,m\}$ *satisfy the assumptions of Theorem 1.14 and let the functions* G *and* Ω_2 *be defined in* (1.180) *and* (2.463), *respectively. Then it holds*

$$\Delta(p_i,x_i,y_i,f)=\frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id)$$
$$+\frac{f'(b)-f'(a)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds$$
$$+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds$$
$$+\frac{\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]}{(n-3)!(b-a)}\int_a^b \Omega_2(s)ds+R_n^2(f;a,b), \tag{2.468}$$

where the remainder $R_n^2(f;a,b)$ *satisfies the estimation*

$$|R_n^2(f;a,b)| \leq \frac{1}{(n-3)!}\left(\frac{(b-a)}{2}\left|T(\Omega_2,\Omega_2)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{1/2}. \tag{2.469}$$

Now we state some Ostrowski type inequalities related to the generalized majorization inequalities.

Theorem 2.157 ([21]) *Let all the assumptions of Theorem* 2.147 *hold. Furthermore, let* (q,r) *be a pair of conjugate exponents. Let* $f^{(n)}\in L_q[a,b]$ *for some* $n\in\mathbb{N}$. *Then we have*

$$|\Delta(p_i,x_i,y_i,f)-\frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id)$$
$$-\frac{f'(a)-f'(b)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds\Bigg|$$

$$\leq \frac{1}{(n-3)!}\|f^{(n)}\|_q \left\| \int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds \right\|_r. \quad (2.470)$$

The constant on the right hand side of (2.470) *is sharp for* $1 < q \leq \infty$ *and the best possible for* $q = 1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Theorem 2.158 ([21]) *Let all the assumptions of Theorem* 2.147 *hold. Furthermore, let* (q,r) *be a pair of conjugate exponents. Let* $f^{(n)} \in L_q[a,b]$ *for some* $n \in \mathbb{N}$. *Then we have*

$$\begin{aligned}
|\Delta(p_i,x_i,y_i,f) &- \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) \\
&- \frac{f'(b)-f'(a)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds \\
-\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}&\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \Bigg| \\
&\leq \frac{1}{(n-3)!}\|f^{(n)}\|_q \left\| \int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))T_{n-2}(s,t)ds \right\|_r. \quad (2.471)
\end{aligned}$$

The constant on the right hand side of (2.471) *is sharp for* $1 < q \leq \infty$ *and the best possible for* $q = 1$.

The integral analogues of stated results are as follows. Since proofs are of similar nature, we omit the details.

Theorem 2.159 ([21]) *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n)}$ *is an absolutely continuous function for* $n \in \mathbb{N}$ *with* $(.-a)(b-.)[f^{(n+1)}]^2 \in L[a,b]$ *and let* p, x *and* y *be as defined in Theorem 1.14 and let the functions* G, T *and* Ω_3 *be defined in* (1.180), (1.6) *and* (2.464) *respectively. Then it holds*

$$\begin{aligned}
\Lambda(p,x,y,f) &= \frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) \\
&+ \frac{f'(a)-f'(b)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds \\
&+ \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \\
&+ \frac{\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]}{(n-3)!(b-a)}\int_a^b \Omega_3(s)ds + R_n^3(f;a,b), \quad (2.472)
\end{aligned}$$

where the remainder $R_n^3(f;a,b)$ *satisfies the estimation*

$$|R_n^3(f;a,b)| \leq \frac{1}{(n-3)!}\left(\frac{(b-a)}{2}\left|T(\Omega_3,\Omega_3)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{1/2}. \quad (2.473)$$

Theorem 2.160 ([21]) *Let all the assumptions of Theorem 2.159 valid. Then it holds*

$$\begin{aligned}
\Lambda(p,x,y,f) &= \frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) \\
&+\frac{f'(b)-f'(a)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds \\
&+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds \\
&+\frac{\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]}{(n-3)!(b-a)}\int_a^b \Omega_4(s)ds+R_n^4(f;a,b),
\end{aligned} \tag{2.474}$$

where Ω_4 *is as defined in* (2.465) *and the remainder* $R_n^4(f;a,b)$ *satisfies the estimation*

$$|R_n^4(f;a,b)| \le \frac{1}{(n-3)!}\left(\frac{(b-a)}{2}\left|T(\Omega_4,\Omega_4)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{1/2}. \tag{2.475}$$

Now we state some Ostrowski type inequalities related to the generalized majorization inequalities in integral case.

Theorem 2.161 ([21]) *Let all the assumptions of Theorem* 2.151 *hold. Furthermore, let* (q,r) *be a pair of conjugate exponents. Let* $f^{(n)} \in L_q[a,b]$ *for some* $n \in \mathbb{N}$. *Then we have*

$$\begin{aligned}
&\Bigg|\Lambda(p,x,y,f)-\frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) \\
&\qquad -\frac{f'(a)-f'(b)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds \\
&-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds\Bigg| \\
&\qquad \le \frac{1}{(n-3)!}\|f^{(n)}\|_q \left\|\int_a^b \Lambda(p,x,y,G(\cdot,s))\tilde{T}_{n-2}(s,t)ds\right\|_r.
\end{aligned} \tag{2.476}$$

The constant on the right hand side of (2.476) *is sharp for* $1 < q \le \infty$ *and the best possible for* $q = 1$.

Theorem 2.162 ([21]) *Let all the assumptions of Theorem* 2.151 *hold. Furthermore, let* (q,r) *be a pair of conjugate exponents. Let* $f^{(n)} \in L_q[a,b]$ *for some* $n \in \mathbb{N}$. *Then we have*

$$\Bigg|\Lambda(p,x,y,f) - \frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds - \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds\Bigg| \leq \frac{1}{(n-3)!}\|f^{(n)}\|_q \left\| \int_a^b \Lambda(p,x,y,G(\cdot,s))T_{n-2}(s,t)ds \right\|_r. \quad (2.477)$$

The constant on the right hand side of (2.477) *is sharp for* $1 < q \leq \infty$ *and the best possible for* $q = 1$.

Under the assumptions of Theorem 2.148 using (2.452), Theorem 2.149 using (2.454), Theorem 2.152 using (2.459) and Theorem 2.153 using (2.461) we define the following functionals:

$$\Gamma_1(f) = \Delta(p_i,x_i,y_i,f) - \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds - \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds, \quad (A1)$$

$$\Gamma_2(f) = \Delta(p_i,x_i,y_i,f) - \frac{f(b)-f(a)}{b-a}\Delta(p_i,x_i,y_i,id) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))ds - \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Delta(p_i,x_i,y_i,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds, \quad (A2)$$

$$\Gamma_3(f) = \Lambda(p,x,y,f) - \frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds - \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds, \quad (A3)$$

$$\begin{aligned}\Gamma_4(f) = \Lambda(p,x,y,f) - &\frac{f(b)-f(a)}{b-a}\Lambda(p,x,y,id)\\ &- \frac{f'(a)-f'(b)}{b-a}\int_a^b \Lambda(p,x,y,G(\cdot,s))ds\\ - \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \Lambda(p,x,y,G(\cdot,s))&\frac{f^k(a)(s-a)^{k-1}-f^k(b)(s-b)^{k-1}}{b-a}ds. \quad \text{(A4)}\end{aligned}$$

Now we give mean value theorems for Γ_k, $k \in \{1,2,3,4\}$. Here $f_0(x) = \frac{x^n}{n!}$.

Theorem 2.163 ([21]) *Let $f \in C^n[a,b]$ and let $\Gamma_k : C^n[a,b] \to \mathbb{R}$ for $k \in \{1,2,3,4\}$ be linear functionals as defined in (A1), (A2), (A3) and (A4) respectively. Then there exist $\xi_k \in [a,b]$ for $k \in \{1,2,3,4\}$ such that*

$$\Gamma_k(f) = f^{(n)}(\xi_k)\Gamma_k(f_0), \quad k \in \{1,2,3,4\}. \tag{2.478}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (see also the proof of Theorem 4.1 in [86]). □

Applying Theorem 2.163 to function $\omega = \Gamma_k(h)f - \Gamma_k(f)h$, we get the following result.

Theorem 2.164 ([21]) *Let $f,h \in C^n[a,b]$ and let $\Gamma_k : C^n[a,b] \to \mathbb{R}$ for $k \in \{1,2,3,4\}$ be linear functionals as defined in (A1), (A2), (A3) and (A4) respectively. Then there exist $\xi_k \in [a,b]$ for $k \in \{1,2,3,4\}$ such that*

$$\frac{\Gamma_k(f)}{\Gamma_k(h)} = \frac{f^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

assuming that both denominators are non-zero.

Remark 2.45 ([21]) *If the inverse of $\frac{f^{(n)}}{h^{(n)}}$ exists, then from the above mean value theorems we can give generalized means*

$$\xi_k = \left(\frac{f^{(n)}}{h^{(n)}}\right)^{-1}\left(\frac{\Gamma_k(f)}{\Gamma_k(h)}\right). \tag{2.479}$$

Remark 2.46 ([21]) *Using the same method as in one of the previous section (see the same method [98]), we can construct new families of exponentially convex functions and Cauchy type means (see also [18]). Also, using the idea described in [98] we can obtain results for n-convex functions at point.*

Remark 2.47 ([21]) *Similar examples can be discussed as given in Section 1.4.*

2.4.3 Results Obtained by New Green's Functions and Montgomery Identity

In this subsection, we give necessary and sufficient conditions for majorization inequality. By utilizing Montgomery identity and n-convex functions we give generalization of majorization inequalities. We also discuss the results for majorized tuples. Upper bounds for identities related to generalized majorization type results are obtained. For some more recent results, related to generalizations and refinements of majorization theorem, see [79, 123] and some of the references in them.

In the following theorem we present the general identities for majorization difference.

Theorem 2.165 *Let $n \in \mathbb{N}$, $\psi : I \to \mathbb{R}$ be a function such that $\psi^{(n-1)}$ is absolutely continuous, I is open interval, $a, b \in I$, $a < b$. Suppose that $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$ be decreasing $m-$tuples from $[a,b]^m$ and $\mathbf{w} = (w_1, \ldots, w_m)$ be real $m-$tuple such that (1.20) holds. Let G_p, $p = 1,2,3,4$, and T_n be as defined in (2.47), (2.48), (2.49), (2.50) and (2.418) respectively. Then the following identities hold.*

(i)

$$\begin{aligned}\sum_{i=1}^{m} w_i \psi(x_i) - \sum_{i=1}^{m} w_i \psi(y_i) &= \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] \\ &\quad \cdot \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a} \right) dt \\ &\quad + \frac{1}{(n-3)!} \int_a^b \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] \tilde{T}_{n-2}(t,s)\, \psi^{(n)}(s)\, ds dt,\end{aligned} \tag{2.480}$$

where $n \geq 3$ and

$$\tilde{T}_{n-2}(t,s) = \begin{cases} \frac{1}{b-a} \left[\frac{(t-s)^{n-2}}{n-2} + (t-a)(t-s)^{n-3} \right], & a \leq s \leq t, \\ \frac{1}{b-a} \left[\frac{(t-s)^{n-2}}{n-2} + (t-b)(t-s)^{n-3} \right], & t < s \leq b. \end{cases} \tag{2.481}$$

(ii)

$$\begin{aligned}\sum_{i=1}^{m} w_i \psi(x_i) - \sum_{i=1}^{m} w_i \psi(y_i) &= \frac{\psi'(b) - \psi'(a)}{b-a} \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] dt \\ &\quad + \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] \cdot \\ &\quad + \frac{1}{(n-3)!} \int_a^b \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] T_{n-2}(t,s)\, \psi^{(n)}(s)\, ds dt,\end{aligned} \tag{2.482}$$

where $n \geq 4$.

Proof. Using (2.46), (2.51), (2.52) and (2.53) in $\sum_{i=1}^{m} w_i\psi(x_i) - \sum_{i=1}^{m} w_i\psi(y_i)$ and applying (1.20), we have

$$\sum_{i=1}^{m} w_i\psi(x_i) - \sum_{i=1}^{m} w_i\psi(y_i) = \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t)\right] \psi''(t)dt. \tag{2.483}$$

(i) Taking double derivative of (2.417) with respect to t, we get

$$\begin{aligned}\psi''(t) = &\sum_{k=1}^{n-1} \frac{k}{(k-1)!}\left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a}\right)\\ &+\frac{1}{(n-3)!}\int_a^b \tilde{T}_{n-2}(t,s)\,\psi^{(n)}(s)\,ds.\end{aligned} \tag{2.484}$$

Putting (2.484) in (2.483), we obtain (2.480).

(ii) Replacing ψ by ψ'' and then n by $n-2$ in (2.417), we have

$$\begin{aligned}\psi''(t) = &\frac{1}{b-a}\int_a^b \psi''(t)\,dt + \sum_{k=0}^{n-4} \frac{\psi^{(k+3)}(a)}{k!(k+2)}\frac{(t-a)^{k+2}}{b-a} - \sum_{k=0}^{n-4} \frac{\psi^{(k+3)}(b)}{k!(k+2)}\frac{(t-b)^{k+2}}{b-a}\\ &+\frac{1}{(n-3)!}\int_a^b T_{n-2}(t,s)\,\psi^{(n)}(s)\,ds.\end{aligned}$$

This implies that

$$\begin{aligned}\psi''(t) = &\frac{\psi'(b)-\psi'(a)}{b-a} + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!}\left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a}\right)\\ &+\frac{1}{(n-3)!}\int_a^b T_{n-2}(t,s)\,\psi^{(n)}(s)\,ds.\end{aligned} \tag{2.485}$$

Using (2.485) in (2.483), we get (2.482). □

The integral version of the above theorem is given below.

Theorem 2.166 *Let $n \in \mathbb{N}$, I be an open interval of $\mathbb{R}$, $\psi : I \to \mathbb{R}$ be a real valued function such that $\psi^{(n-1)}$ is absolutely continuous, $a,b \in I$, $a < b$. Suppose that x, y are decreasing continuous functions from $[a,b]$ to $\mathbb{R}$ and w be real valued continuous function on $[a,b]$ such that (1.175) holds. Let G_p, $p = 1,2,3,4$, T_n and $\tilde{T}_n$ be as defined in (2.47), (2.48), (2.49), (2.50), (2.418) and (2.481) respectively. Then the following identities hold:*

(i)
$$\int_a^b w(t)\psi(x(t))dt - \int_a^b w(t)\psi(y(t))dt$$
$$= \int_a^b \left[\int_a^b w(t)G_p(x(t),s)dt - \int_a^b w(t)G_p(y(t),s)dt\right] \cdot$$
$$\sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\psi^{(k)}(a)(s-a)^{k-1} - \psi^{(k)}(b)(s-b)^{k-1}}{b-a}\right) ds$$
$$+\frac{1}{(n-3)!}\int_a^b\int_a^b\left[\int_a^b w(t)G_p(x(t),s)dt - \int_a^b w(t)G_p(y(t),s)dt\right]\tilde{T}_{n-2}(s,u)\,\psi^{(n)}(u)\,dsdu, \tag{2.486}$$

where $n \geq 3$.

(ii)
$$\int_a^b w(t)\psi(x(t))dt - \int_a^b w(t)\psi(y(t))dt$$
$$= \frac{\psi'(b)-\psi'(a)}{b-a}\int_a^b \left[\int_a^b w(t)G_p(x(t),s)dt - \int_a^b w(t)G_p(y(t),s)dt\right] ds$$
$$+ \int_a^b \left[\int_a^b w(t)G_p(x(t),s)dt - \int_a^b w(t)G_p(y(t),s)dt\right] \cdot$$
$$\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\psi^{(k)}(a)(s-a)^{k-1} - \psi^{(k)}(b)(s-b)^{k-1}}{b-a}\right) ds$$
$$+\frac{1}{(n-3)!}\int_a^b\int_a^b\left[\int_a^b w(t)G_p(x(t),s)dt - \int_a^b w(t)G_p(y(t),s)dt\right]T_{n-2}(s,u)\,\psi^{(n)}(u)\,duds, \tag{2.487}$$

where $n \geq 4$.

From the above obtained identities we present the generalization of the majorization theorem.

Theorem 2.167 *Suppose all the assumptions of Theorem 2.165 hold and for any even n the function $\psi : I \to \mathbb{R}$ is n-convex. Let*

$$\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t) \geq 0, \ \textit{for}\ p = 1,2,3,4. \tag{2.488}$$

Then the following inequalities hold:

(i)
$$\sum_{i=1}^m w_i\psi(x_i) - \sum_{i=1}^m w_i\psi(y_i) \geq \int_a^b \left[\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t)\right]$$
$$\cdot \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a}\right) dt. \tag{2.489}$$

(ii)
$$\sum_{i=1}^{m} w_i\psi(x_i)-\sum_{i=1}^{m} w_i\psi(y_i)$$
$$\geq \frac{\psi'(b)-\psi'(a)}{b-a}\int_a^b\left[\sum_{i=1}^{m} w_iG_p(x_i,t)-\sum_{i=1}^{m} w_iG_p(y_i,t)\right]dt$$
$$+\int_a^b\left[\sum_{i=1}^{m} w_iG_p(x_i,t)-\sum_{i=1}^{m} w_iG_p(y_i,t)\right]\cdot$$
$$\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\psi^{(k)}(a)(t-a)^{k-1}-\psi^{(k)}(b)(t-b)^{k-1}}{b-a}\right)dt. \tag{2.490}$$

Proof.

(i) Since ψ is n-convex so without loss of generality we can assume that ψ is n-times differentiable and therefore we have $\psi^{(n)}\geq 0$. Also it is obvious that $\tilde{T}_{n-2}\geq 0$ if n is even, because

Case I: If $a\leq s\leq t$, then $t-s\geq 0$ and hence $\frac{(t-s)^{n-2}}{n-2}\geq 0$. Also $(t-a)\geq 0$ and $(t-s)^{n-3}\geq 0$. So in this case from (2.481) we have $\tilde{T}_{n-2}\geq 0$.

Case II: If $t<s\leq b$, then $(t-s)^{n-3}$ and $(s-b)$ are non-positive. As n is even so we have $(s-b)(t-s)^{n-3}\geq 0$, also $\frac{(t-s)^{n-2}}{n-2}\geq 0$. So in this case from (2.481) we have $\tilde{T}_{n-2}\geq 0$.

Now using (2.488) and the positivity of $\tilde{T}_{n-2}$ and $\psi^{(n)}$ in (2.480) we get (2.489).

(ii) Similar to part (i).

□

The integral version of the above theorem is given here.

Theorem 2.168 *Suppose all the assumptions of Theorem 2.166 hold and for any even n the function $\psi: I\to\mathbb{R}$ is n-convex. Let*
$$\int_a^b w(t)G_p(x(t),s)dt-\int_a^b w(t)G_p(y(t),s)dt\geq 0 \text{ for } p=1,2,3,4. \tag{2.491}$$
Then the following inequalities hold:

(i)
$$\int_a^b w(t)\psi(x(t))dt-\int_a^b w(t)\psi(y(t))dt\geq\int_a^b\left[\int_a^b w(t)G_p(x(t),s)dt-\int_a^b w(t)G_p(y(t),s)dt\right]$$
$$\cdot\sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{\psi^{(k)}(a)(s-a)^{k-1}-\psi^{(k)}(b)(s-b)^{k-1}}{b-a}\right)ds. \tag{2.492}$$

(ii)

$$\begin{aligned}
&\int_a^b w(t)\psi(x(t))dt - \int_a^b w(t)\psi(y(t))dt \\
&\geq \frac{\psi'(b)-\psi'(a)}{b-a}\int_a^b \left[\int_a^b w(t)G_p(x(t),s)dt - \int_a^b w(t)G_p(y(t),s)dt\right] ds. \\
&+\int_a^b \left[\int_a^b w(t)G_p(x(t),s)dt - \int_a^b w(t)G_p(y(t),s)dt\right]\cdot \\
&\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\psi^{(k)}(a)(s-a)^{k-1}-\psi^{(k)}(b)(s-b)^{k-1}}{b-a}\right) ds.
\end{aligned} \tag{2.493}$$

In the following theorem we give generalizations of majorization inequality for majorized tuples:

Theorem 2.169 *Let $n \in \mathbb{N}$, ψ be a function from an open interval I to $\mathbb{R}$ such that its $\psi^{(n-1)}$ is absolutely continuous, $a,b \in I$, $a < b$. Let $\mathbf{x} = (x_1,\dots,x_m)$, $\mathbf{y} = (y_1,\dots,y_m)$ be $m-$tuples from $[a,b]^m$ such that $\mathbf{y} \prec \mathbf{x}$. If n is even and ψ is n-convex function, then the following inequalities hold:*

(i)

$$\begin{aligned}
&\sum_{i=1}^m \psi(x_i) - \sum_{i=1}^m \psi(y_i) \geq \int_a^b \left[\sum_{i=1}^m G_p(x_i,t) - \sum_{i=1}^m G_p(y_i,t)\right] \\
&\cdot\sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{\psi^{(k)}(a)(t-a)^{k-1}-\psi^{(k)}(b)(t-b)^{k-1}}{b-a}\right) dt.
\end{aligned} \tag{2.494}$$

(ii)

$$\begin{aligned}
&\sum_{i=1}^m \psi(x_i) - \sum_{i=1}^m \psi(y_i) \geq \frac{\psi'(b)-\psi'(a)}{b-a}\int_a^b \left[\sum_{i=1}^m G_p(x_i,t) - \sum_{i=1}^m G_p(y_i,t)\right] dt \\
&+\int_a^b \left[\sum_{i=1}^m G_p(x_i,t) - \sum_{i=1}^m G_p(y_i,t)\right]\cdot \\
&\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\psi^{(k)}(a)(t-a)^{k-1}-\psi^{(k)}(b)(t-b)^{k-1}}{b-a}\right) dt.
\end{aligned} \tag{2.495}$$

Proof. Since the function $G_p(.,t)$, $p \in \{1,2,3,4\}$, $t \in [a,b]$, are convex and $\mathbf{y} \prec \mathbf{x}$ so by majorization theorem we have

$$\sum_{i=1}^m G_p(x_i,t) - \sum_{i=1}^m G_p(y_i,t) \geq 0, \quad t \in [a,b].$$

Applying Theorem 2.167 for $w_i = 1$ $(i = 1,2,\dots,m)$, we obtain (2.494) and (2.495). □

In the following theorem we give generalizations of weighted majorization theorem.

Theorem 2.170 *Let $n \in \mathbb{N}$, ψ be a function from an open interval I to $\mathbb{R}$ such that its $\psi^{(n-1)}$ is absolutely continuous, $a,b \in I$, $a < b$. Let $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ be decreasing $m-$tuples from $[a,b]^m$ and $\mathbf{w} = (w_1,\ldots,w_m)$ be nonnegative $m-$tuple such that (1.19) and (1.20) hold. If n is even and ψ is n-convex function, then (2.489) and (2.490) hold. Moreover, if (2.489) and (2.490) hold and the functions defined by*

$$L_1(.) = \int_a^b G_p(.,t) \cdot \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a} \right) dt, \tag{2.496}$$

$$L_2(.) = \frac{\psi'(b) - \psi'(a)}{b-a} \int_a^b G_p(.,t) + \int_a^b G_p(.,t) \cdot \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a} \right) dt, \quad p = 1,2,3,4, \tag{2.497}$$

are convex on $[a,b]$, then (1.21) holds in both cases.

Proof. Since the function $G_p(.,t)$, $p \in \{1,2,3,4\}$, $t \in [a,b]$, are convex and (1.19) and (1.20) hold so by Theorem 1.14, we have

$$\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t) \geq 0, \quad t \in [a,b].$$

Applying Theorem 2.167 we obtain (2.489) and (2.490).

Since (2.489) holds, the right hand side of (2.489) can be expressed as

$$\sum_{i=1}^m w_i L_1(x_i) - \sum_{i=1}^m w_i L_1(y_i).$$

Since (2.489), (2.490) hold and L_1 is convex, therefore by majorization theorem we have

$$\sum_{i=1}^m w_i L_1(x_i) - \sum_{i=1}^m w_i L_1(y_i) \geq 0,$$

i.e. the right hand side of (2.489) is nonnegative, so the inequality (1.21) immediately follows.
Similarly we obtain (1.21) by using the convexity of L_2. □

The integral version of the above theorem is given below.

Theorem 2.171 *Let $n \in \mathbb{N}$, ψ be a function from an open interval I to $\mathbb{R}$ such that $\psi^{(n-1)}$ is absolutely continuous, $a,b \in I$, $a < b$. Let x, y be decreasing functions from $[a,b]$ to I and w be nonnegative continuous function on $[a,b]$ such that (1.27) and (1.28) hold. If n is even and ψ is n-convex function, then (2.492) and (2.493) hold. Moreover, if (2.492) and (2.493) hold and the functions L_1, L_2 defined by (2.496), (2.497) respectively are convex then (1.29) holds.*

To avoid many notations under the conditions of Theorem 2.165 and the functions $Q_{1,p}$, $Q_{2,p}$, $(p = 1,2,3,4)$ from $[a,b]$ to $\mathbb{R}$ are defined by

$$Q_{1,p}(s) = \int_a^b \left[\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t) \right] \tilde{T}_{n-2}(t,s)dt, \tag{2.498}$$

$$Q_{2,p}(s) = \int_a^b \left[\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t) \right] T_{n-2}(t,s)dt. \tag{2.499}$$

Theorem 2.172 *Let $n \in \mathbb{N}, n \geq 4$, $\psi : [a,b] \to \mathbb{R}$ be such that $\psi^{(n)}$ is absolutely continuous with $(\cdot - a)(b - \cdot)(\psi^{(n+1)})^2 \in L[a,b]$ and $Q_{1,p}$, $Q_{2,p}$ $(p = 1,2,3,4)$ be defined as in (2.498), (2.499) respectively. Then*

(i) the remainder $H^1(\psi;a,b)$ defined by

$$H^1(\psi;a,b) = \sum_{i=1}^m w_i \psi(x_i) - \sum_{i=1}^m w_i \psi(y_i) - \int_a^b \left[\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t) \right] \cdot$$
$$\sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a} \right) dt$$
$$- \frac{\psi^{(n-1)}(b) - \psi^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b Q_{1,p}(s)ds, \tag{2.500}$$

satisfies the estimation

$$\left| H^1(\psi;a,b) \right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-3)!} \left| T(Q_{1,p}, Q_{1,p}) \right|^{\frac{1}{2}} \left(\int_a^b (s-a)(b-s)[\psi^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \tag{2.501}$$

(ii) the remainder $H^2(\psi;a,b)$ defined by

$$H^2(\psi;a,b) = \sum_{i=1}^m w_i \psi(x_i) - \sum_{i=1}^m w_i \psi(y_i)$$
$$- \frac{\psi'(b) - \psi'(a)}{b-a} \int_a^b \left[\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t) \right] dt$$
$$- \int_a^b \left[\sum_{i=1}^m w_i G_p(x_i,t) - \sum_{i=1}^m w_i G_p(y_i,t) \right] \cdot$$
$$\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a} \right) dt$$
$$- \frac{\psi^{(n-1)}(b) - \psi^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b Q_{2,p}(s)ds, \tag{2.502}$$

satisfies the estimation

$$\left|H^2(\psi;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-3)!}\left|T(Q_{2,p},Q_{2,p})\right|^{\frac{1}{2}}\left(\int_a^b (s-a)(b-s)[\psi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}. \tag{2.503}$$

Proof.

(i) Comparing (2.480) and (2.500) we obtain

$$H^1(\psi;a,b) = \frac{1}{(n-3)!}\int_a^b Q_{1,p}(s)\psi^{(n)}(s)ds - \frac{\psi^{(n-1)}(b)-\psi^{(n-1)}(a)}{(n-3)!\,(b-a)}\int_a^b Q_{1,p}(s)ds. \tag{2.504}$$

Applying Theorem 1.10 for $f \to Q_{1,p}$, $h \to \psi^{(n)}$ and using Čebyšev functional, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b Q_{1,p}(s)\psi^{(n)}(s)ds - \frac{1}{b-a}\int_a^b Q_{1,p}(s)ds\cdot\frac{1}{b-a}\int_a^b \psi^{(n)}(s)ds\right|\\ &\leq \frac{1}{\sqrt{2}}\left|T(Q_{1,p},Q_{1,p})\right|^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[\psi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.\end{aligned} \tag{2.505}$$

Multiplying (2.505) with $b-a$, dividing by $(n-3)!$ and using (2.576) we get (2.501).

(ii) Comparing (2.482) and (2.502) we obtain

$$H^2(\psi;a,b) = \frac{1}{(n-3)!}\int_a^b Q_{2,p}(s)\psi^{(n)}(s)ds - \frac{\psi^{(n-1)}(b)-\psi^{(n-1)}(a)}{(n-3)!\,(b-a)}\int_a^b Q_{2,p}(s)ds. \tag{2.506}$$

Applying Theorem 1.10 for $f \to Q_{2,p}$, $h \to \psi^{(n)}$ and using Čebyšev functional, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b Q_{1,p}(s)\psi^{(n)}(s)ds - \frac{1}{b-a}\int_a^b Q_{2,p}(s)ds\cdot\frac{1}{b-a}\int_a^b \psi^{(n)}(s)ds\right|\\ &\leq \frac{1}{\sqrt{2}}\left|T(Q_{2,p},Q_{2,p})\right|^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[\psi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.\end{aligned} \tag{2.507}$$

Multiplying (2.507) with $b-a$ and dividing by $(n-3)!$ and using (2.506) we get (2.503).

□

Theorem 2.173 *Let $n \in \mathbb{N}, n \geq 4$, $\psi : [a,b] \to \mathbb{R}$ be such that $\psi^{(n)}$ is monotonic nondecreasing on $[a,b]$ and $Q_{1,p}$, $Q_{2,p}$ (for $p = 1,2,3,4$) be defined as in (2.498) and (2.499) respectively. Then*

(i) the remainder $H^1(\psi;a,b)$ defined by (2.500) satisfies the estimation

$$\left|H^1(\psi;a,b)\right| \leq \frac{b-a}{(n-3)!}\|Q'_{1,p}\|_\infty \left[\frac{\psi^{(n-1)}(b)+\psi^{(n-1)}(a)}{2} - \frac{\psi^{(n-2)}(b)-\psi^{(n-2)}(a)}{b-a}\right]. \tag{2.508}$$

(ii) the remainder $H^2(\psi;a,b)$ defined by (2.502) satisfies the estimation

$$\left|H^2(\psi;a,b)\right| \leq \frac{b-a}{(n-3)!}\|Q'_{2,p}\|_\infty \left[\frac{\psi^{(n-1)}(b)+\psi^{(n-1)}(a)}{2} - \frac{\psi^{(n-2)}(b)-\psi^{(n-2)}(a)}{b-a}\right]. \tag{2.509}$$

Proof.

(i) Since (2.576) holds, then applying Theorem 1.11 for $f \to Q_{1,p}$, $h \to \psi^{(n)}$ and using Čebyšev functional, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b Q_{1,p}(s)\psi^{(n)}(s)ds - \frac{1}{b-a}\int_a^b Q_{1,p}(s)ds \cdot \frac{1}{b-a}\int_a^b \psi^{(n)}(s)ds\right|\\ &\leq \frac{1}{2(b-a)}\left\|Q'_{1,p}\right\|_\infty \int_a^b (s-a)(b-s)\psi^{(n+1)}(s)ds.\end{aligned} \tag{2.510}$$

Since

$$\begin{aligned}&\int_a^b (s-a)(b-s)\psi^{(n+1)}(s)ds = \int_a^b [2s-(a+b)]\,\psi^{(n)}(s)ds\\ &= (b-a)\left[\psi^{(n-1)}(b)+\psi^{(n-1)}(a)\right] - 2\left[\psi^{(n-2)}(b)-\psi^{(n-2)}(a)\right],\end{aligned}$$

therefore, from (2.576) and (2.579), we deduce (2.578).

(ii) Proceeding similarly as in part (i), one can obtain (2.509).

□

In the following theorems we present Ostrowski type inequalities for the generalization of majorization inequality:

Theorem 2.174 *Let $n \in \mathbb{N}, n \geq 4$, (q,r) be a pair of conjugate exponents, i.e. $1 \leq q,r \leq \infty$ and $1/q + 1/r = 1$, $\psi : [a,b] \to \mathbb{R}$ be such that $\left|\psi^{(n)}\right|^q \in L[a,b]$. Let $Q_{1,p}$ and $Q_{2,p}$ ($p = 1,2,3,4$) be defined as in (2.498) and (2.499) respectively. Then the following inequalities hold:*

(i)

$$\left| \sum_{i=1}^{m} w_i \psi(x_i) - \sum_{i=1}^{m} w_i \psi(y_i) - \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] \cdot \right.$$
$$\left. \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a} \right) dt \right|$$
$$\leq \frac{1}{(n-3)!} \left\| \psi^{(n)} \right\|_q \left\| Q_{1,p} \right\|_r$$

The constant $\left\| Q_{1,p} \right\|_r$ *is sharp for* $1 < q \leq \infty$ *and the best possible for* $q = 1$.

(ii)

$$\left| \sum_{i=1}^{m} w_i \psi(x_i) - \sum_{i=1}^{m} w_i \psi(y_i) - \frac{\psi'(b) - \psi'(a)}{b-a} \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] dt \right.$$
$$- \int_a^b \left[\sum_{i=1}^{m} w_i G_p(x_i,t) - \sum_{i=1}^{m} w_i G_p(y_i,t) \right] \cdot$$
$$\left. \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\psi^{(k)}(a)(t-a)^{k-1} - \psi^{(k)}(b)(t-b)^{k-1}}{b-a} \right) dt \right|$$
$$\leq \frac{1}{(n-3)!} \left\| \psi^{(n)} \right\|_q \left\| Q_{2,p} \right\|_r$$

The constant $\left\| Q_{2,p} \right\|_r$ *is sharp for* $1 < q \leq \infty$ *and the best possible for* $q = 1$.

Proof. The proof is similar to the proof of Theorem 12 in [2]. □

Remark 2.48 One could analogously obtain the integral variants of Theorems 2.172, 2.173 and 2.174.

2.4.4 Results Obtained for the Jensen and the Jensen-Steffensen Inequalities and their converses via Montgomery Identity

In the following theorem we give generalization of Jensen's inequality associated with Montgomery identity (see [3]).

Theorem 2.175 *Let* $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ *be such that* $f^{(2n-1)}$ *is absolutely continuous,* $I \subset \mathbb{R}$ *an open interval,* $a, b \in I$, $a < b$. *Let* $\boldsymbol{x} = (x_1, \ldots, x_m) \in [a,b]^m$ *be* $m-$*tuple and* $\boldsymbol{w} = (w_1, \ldots, w_m)$ *be positive* $m-$*tuple,* $W_m = \sum_{i=1}^{m} w_i$ *and* $\overline{x} = \frac{1}{W_m} \sum_{i=1}^{m} w_i x_i$.

(i) If $\mathbf{x}$ *is decreasing* $m-$*tuple and* $f:[a,b]\to\mathbb{R}$ *is* $2n$*-convex function, then we have*

$$\frac{1}{W_m}\sum_{i=1}^{m}w_if(x_i)-f(\bar{x})\geq\frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left(\frac{\sum\limits_{i=1}^{m}w_i(x_i-a)^{k+2}}{W_m}-(\bar{x}-a)^{k+2}\right)\right.$$
$$\left.-f^{(k+1)}(b)\left(\frac{\sum\limits_{i=1}^{m}w_i(x_i-b)^{k+2}}{W_m}-(\bar{x}-b)^{k+2}\right)\right]. \tag{2.511}$$

(ii) If the inequality (2.511) holds and the function F *defined by*

$$F(.)=\sum_{k=0}^{2n-2}\frac{f^{(k+1)}(a)}{k!\,(k+2)}\frac{(.-a)^{k+2}}{b-a}-\sum_{k=0}^{2n-2}\frac{f^{(k+1)}(b)}{k!\,(k+2)}\frac{(.-b)^{k+2}}{b-a}, \tag{2.512}$$

is convex, then the right hand side of (2.511) is non-negative and

$$f(\bar{x})\leq\frac{1}{W_m}\sum_{i=1}^{m}w_if(x_i). \tag{2.513}$$

Proof.

(i) Let k be the largest number from $\{1,\ldots,m\}$ such that $x_k\geq\bar{x}$, then as $\mathbf{x}$ is decreasing m-tuple so we have $x_l\geq\bar{x}$ for $l=1,2,\ldots,k$ and $x_l\leq\bar{x}$ for $l=k+1,k+2,\ldots,m$.

Now as $x_l\geq\bar{x}$ for $l=1,2,\ldots,k$, so we have

$$\sum_{i=1}^{l}w_i\bar{x}\leq\sum_{i=1}^{l}w_ix_i \quad \text{for } l=1,2,\ldots,k. \tag{2.514}$$

Similarly as $x_l\leq\bar{x}$ for $l=k+1,k+2,\ldots,m$, so we have

$$\sum_{i=k+1}^{j}w_ix_i\leq\sum_{i=k+1}^{j}w_i\bar{x} \quad \text{for} j=k+1,k+2,\ldots,m.$$

Hence

$$\sum_{i=1}^{j}w_ix_i=\sum_{i=1}^{m}w_ix_i-\sum_{i=j+1}^{m}w_ix_i\geq\sum_{i=1}^{m}w_i\bar{x}-\sum_{i=j+1}^{m}w_i\bar{x}=\sum_{i=1}^{j}w_i\bar{x}, \tag{2.515}$$
$$\text{for}\quad j=k,k+1,\ldots,m.$$

Using (2.514) and (2.515) we get that

$$\sum_{i=1}^{l}w_i\bar{x}\leq\sum_{i=1}^{l}w_ix_i, \text{ for all } l=1,2,\ldots,m-1$$

and obviously

$$\sum_{i=1}^{m} w_i \bar{x} = \sum_{i=1}^{m} w_i x_i.$$

Since the conditions (1.19) and (1.20) are satisfied for $\boldsymbol{x} = (x_1, \ldots, x_m), \boldsymbol{y} = (\bar{x}, \ldots, \bar{x})$, therefore using Theorem 2.135 for $\boldsymbol{y} = (\bar{x}, \ldots, \bar{x})$, we get (2.511).

(ii) We may write the right hand side of (2.511) as

$$\frac{1}{W_m} \sum_{i=1}^{m} w_i F(x_i) - F(\bar{x}).$$

Since F is convex so by Jensen's inequality, we have

$$\frac{1}{W_m} \sum_{i=1}^{m} w_i F(x_i) - F(\bar{x}) \geq 0.$$

Hence (2.513) holds.

□

In the following theorem we give integral version of Theorem 2.191.

Theorem 2.176 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $x : [\alpha, \beta] \to \mathbb{R}$ be continuous function such that $x([\alpha, \beta]) \subseteq [a,b]$, $\lambda : [\alpha, \beta] \to \mathbb{R}$ be increasing, bounded function with $\lambda(\alpha) \neq \lambda(\beta)$ and $\bar{x} = \frac{\int_\alpha^\beta x(t) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$.*

(i) If x is decreasing function and $f : [a,b] \to \mathbb{R}$ is $2n$-convex function, then we have

$$\begin{aligned} &\frac{\int_\alpha^\beta f(x(t)) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\bar{x}) \\ &\quad \geq \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{1}{k!(k+2)} \left[f^{(k+1)}(a) \left\{ \frac{\int_\alpha^\beta (x(t)-a)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\bar{x}-a)^{k+2} \right\} \right. \\ &\quad \left. - f^{(k+1)}(b) \left\{ \frac{\int_\alpha^\beta (x(t)-b)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\bar{x}-b)^{k+2} \right\} \right]. \end{aligned} \tag{2.516}$$

(ii) If the inequality (2.516) holds and the function F defined as in (2.512) is convex, then the right hand side of (2.516) is non-negative and

$$f(\bar{x}) \leq \frac{\int_\alpha^\beta f(x(t)) d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}. \tag{2.517}$$

Proof.

(i) Let γ_0 be the largest number in $[\alpha, \beta]$ such that $x(\gamma_0) \geq \bar{x}$. But x is decreasing function so we have

$$x(\gamma) \geq \bar{x} \text{ for all } \gamma \in [\alpha, \gamma_0] \text{ and } x(\gamma) \leq \bar{x} \text{ for all } \gamma \in [\gamma_0, \beta].$$

Case.1 If $x(\gamma) \geq \overline{x}$ for all $\gamma \in [\alpha, \gamma_0]$, then we may write

$$x(t) \geq \overline{x} \text{ for all } t \in [\alpha, \gamma], \gamma \in [\alpha, \gamma_0].$$

As λ is increasing so by integrating both sides with respect to λ over $[\alpha, \gamma]$, we get

$$\int_\alpha^\gamma x(t)d\lambda(t) \geq \int_\alpha^\gamma \overline{x}d\lambda(t),\ \gamma \in [\alpha, \gamma_0]. \tag{2.518}$$

Case.2 If $x(\gamma) \leq \overline{x}$ for all $\gamma \in [\gamma_0, \beta]$, then we may write

$$x(t) \leq \overline{x} \text{ for all } t \in [\gamma, \beta], \gamma \in [\gamma_0, \beta].$$

But λ is increasing so by integrating both sides with respect to λ over $[\gamma, \beta]$, we get

$$\int_\gamma^\beta x(t)d\lambda(t) \ \leq \ \int_\gamma^\beta \overline{x}d\lambda(t).$$

Therefore we have

$$\begin{aligned}\int_\alpha^\gamma x(t)d\lambda(t) &= \int_\alpha^\beta x(t)d\lambda(t) - \int_\gamma^\beta x(t)d\lambda(t) \geq \int_\alpha^\beta \overline{x}d\lambda(t) - \int_\gamma^\beta \overline{x}d\lambda(t) \\ &= \int_\alpha^\gamma \overline{x}d\lambda(t),\end{aligned}$$

i.e.

$$\int_\alpha^\gamma x(t)d\lambda(t) \geq \int_\alpha^\gamma \overline{x}d\lambda(t),\ \gamma \in [\gamma_0, \beta]. \tag{2.519}$$

From (2.518) and (2.519) we have

$$\int_\alpha^\gamma x(t)d\lambda(t) \geq \int_\alpha^\gamma \overline{x}d\lambda(t),\ \gamma \in [\alpha, \beta].$$

Also the equality

$$\int_\alpha^\beta x(t)d\lambda(t) = \int_\alpha^\beta \overline{x}d\lambda(t) \quad \text{holds.}$$

Since the conditions (1.27) and (1.28) are satisfied, therefore using Theorem 2.137 for $y(t) = \overline{x}$, we get the inequality (2.516).

(ii) We may write the right hand side of (2.516) as

$$\frac{\int_\alpha^\beta F(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - F(\overline{x}).$$

Since F is convex so by Jensen's inequality, we have

$$\frac{\int_\alpha^\beta F(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - F(\overline{x}) \geq 0.$$

Hence (2.517) holds.

$\square$

Remark 2.49 *If we take $x(t) = t$, $\lambda(t) = t$, in the inequality (2.516), then we obtain generalization of Hermite-Hadamard inequality.*

Theorem 2.177 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$ be decreasing $m-$tuple. Let $\mathbf{w} = (w_1,\ldots,w_m)$ be real $m-$tuple such that $0 \le W_k \le W_m$ $(k = 1,2,\ldots,m)$, $W_m > 0$ where $W_k = \sum_{i=1}^{k} w_i$ and $\overline{x} = \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i$.*

(i) Then for any $2n$-convex function $f : [a,b] \to \mathbb{R}$, the inequality (2.511) holds.

(ii) If the inequality (2.511) holds and the function F defined as in (2.512) is convex, then the right hand side of (2.511) is non-negative and (2.513) holds.

Proof. (i) Let k be the largest number $\{1,2,\ldots,m\}$ such that $x_k \ge \overline{x}$ then $x_l \ge \overline{x}$ for $l = 1,\ldots,k$, and we have

$$\sum_{i=1}^{l} w_i x_i - W_l x_l = \sum_{i=1}^{l-1} (x_i - x_{i+1}) W_i \ge 0$$

and so we obtain

$$\sum_{i=1}^{l} w_i \overline{x} = W_l \overline{x} \le W_l x_l \le \sum_{i=1}^{l} x_i w_i. \tag{2.520}$$

Also for $l = k+1,\ldots,m$ we have $x_{k+1} < \overline{x}$, therefore

$$x_l(W_m - W_l) - \sum_{i=l+1}^{m} w_i x_i = \sum_{i=l+1}^{m} (x_{i-1} - x_i)(W_m - W_{i-1}) \ge 0.$$

Hence, we conclude that

$$\sum_{i=l+1}^{m} w_i \overline{x} = (W_m - W_l)\overline{x} > (W_m - W_l) x_l \ge \sum_{i=l+1}^{m} w_i x_i. \tag{2.521}$$

From (2.520) and (2.521), we get

$$\sum_{i=1}^{l} w_i \overline{x} \le \sum_{i=1}^{l} x_i w_i \quad for \ \ all \ \ l = 1,2,\ldots,m-1.$$

Obviously the equality

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} x_i w_i$$

holds. Since the conditions (1.19) and (1.20) are satisfied for $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (\overline{x},\ldots,\overline{x})$,, therefore using Theorem 2.135 for $\mathbf{y} = (\overline{x},\ldots,\overline{x})$, we get (2.511).

(ii) The proof is similar to the proof of Theorem 2.191(ii). □

Theorem 2.178 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $x : [\alpha,\beta] \to \mathbb{R}$ be continuous decreasing function such that $x([\alpha,\beta]) \subseteq [a,b]$, $\lambda : [\alpha,\beta] \to \mathbb{R}$ is either continuous or of bounded variation with $\lambda(\alpha) \leq \lambda(t) \leq \lambda(\beta)$ for all $t \in [\alpha,\beta]$ and $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$.*

(i) Then for any $2n$-convex function f, the inequality (2.516) holds.

(ii) If the inequality (2.516) holds and the function F defined as in (2.512) is convex, then the right hand side of (2.516) is non-negative and (2.517) holds.

Proof. (i) Let γ_0 be the largest number in $[\alpha,\beta]$ such that $x(\gamma_0) \geq \overline{x}$. But x is decreasing function so we have

$$x(\gamma) \geq \overline{x} \text{ for all } \gamma \in [\alpha,\gamma_0] \text{ and } x(\gamma) \leq \overline{x} \text{ for all } \gamma \in [\gamma_0,\beta].$$

(a) If $x(\gamma) \geq \overline{x}$ for all $\gamma \in [\alpha,\gamma_0]$, then we may write

$$x(t) \geq \overline{x} \text{ for all } t \in [\alpha,\gamma], \gamma \in [\alpha,\gamma_0].$$

Therefore we have

$$\overline{x}\int_\alpha^\gamma d\lambda(t) \leq x(\gamma)\int_\alpha^\gamma d\lambda(t),\ \gamma \in [\alpha,\gamma_0]. \tag{2.522}$$

But

$$\int_\alpha^\gamma x(t)d\lambda(t) - x(\gamma)\int_\alpha^\gamma d\lambda(t) = -\int_\alpha^\gamma x'(t)\left(\int_\alpha^t d\lambda(x)\right)dt \geq 0. \tag{2.523}$$

From (2.522) and (2.523), we get

$$\overline{x}\int_\alpha^\gamma d\lambda(t) \leq \int_\alpha^\gamma x(t)d\lambda(t),\ \gamma \in [\alpha,\gamma_0]. \tag{2.524}$$

(b) *If* $x(\gamma) \leq \overline{x}$ for all $\gamma \in [\gamma_0,\beta]$, then we may write

$$x(t) \leq \overline{x} \text{ for all } t \in [\gamma,\beta], \gamma \in [\gamma_0,\beta],$$

therefore we have

$$\overline{x}\int_\gamma^\beta d\lambda(t) \geq x(\gamma)\int_\gamma^\beta d\lambda(t). \tag{2.525}$$

But

$$x(\gamma)\int_\gamma^\beta d\lambda(t) - \int_\gamma^\beta x(t)d\lambda(t) = -\int_\gamma^\beta x'(t)\left(\int_t^\beta d\lambda(x)\right)dt \geq 0. \tag{2.526}$$

From (2.525) and (2.526), we get

$$\overline{x}\int_\gamma^\beta d\lambda(t) \geq \int_\gamma^\beta x(t)d\lambda(t) \text{ for all } \gamma \in [\gamma_0,\beta]. \tag{2.527}$$

From (2.524) and (2.527), we get

$$\bar{x}\int_{\alpha}^{\beta} d\lambda(t) \geq \int_{\alpha}^{\beta} x(t)d\lambda(t).$$

The equality

$$\bar{x}\int_{\alpha}^{\beta} d\lambda(t) = \int_{\alpha}^{\beta} x(t)d\lambda(t),$$

obviously holds for all $\gamma \in [\alpha,\beta]$. Since the conditions (1.27) and (1.28) are satisfied, therefore using Theorem 2.137 for $y(t) = \bar{x}$, we get (2.516).

(ii) The proof is similar to the proof of Theorem 2.192 (ii). □

Theorem 2.179 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous. Let $\boldsymbol{x} = (x_1,\ldots,x_r)$ be real $r-$tuple with $x_i \in [m,M] \subseteq [a,b]$, $i = 1,2,\ldots,r$, $\boldsymbol{w} = (w_1,\ldots,w_r)$ be positive $r-$tuple, $W_r = \sum\limits_{i=1}^{r} w_i$ and $\bar{x} = \frac{1}{W_r}\sum\limits_{i=1}^{r} w_i x_i$.*

(i) Then for any $2n$-convex function $f : [a,b] \to \mathbb{R}$, the following inequality holds

$$\begin{aligned}\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) &\leq \frac{\bar{x}-m}{M-m}f(M) + \frac{M-\bar{x}}{M-m}f(m) - \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\\ &\left[f^{(k+1)}(a)\left\{\frac{\bar{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\bar{x}}{M-m}(m-a)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i-a)^{k+2}\right\}\right.\\ &\left.-f^{(k+1)}(b)\left\{\frac{\bar{x}-m}{M-m}(M-b)^{k+2} + \frac{M-\bar{x}}{M-m}(m-b)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i-b)^{k+2}\right\}\right].\end{aligned} \tag{2.528}$$

(ii) If the inequality (2.528) holds and the function F defined as in (2.512) is convex, then

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \leq \frac{\bar{x}-m}{M-m}f(M) + \frac{M-\bar{x}}{M-m}f(m).$$

Proof. (i) Putting $m = 2$, $x_1 = M$, $x_2 = m$, $w_1 = \frac{x_i-m}{M-m}$ and $w_2 = \frac{M-x_i}{M-m}$ in (2.511), we have

$$\begin{aligned}f(x_i) &\leq \frac{x_i-m}{M-m}f(M) + \frac{M-x_i}{M-m}f(m) - \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}\\ &\left[f^{(k+1)}(a)\left\{\frac{x_i-m}{M-m}(M-a)^{k+2} + \frac{M-x_i}{M-m}(m-a)^{k+2} - (x_i-a)^{k+2}\right\}\right.\\ &\left.-f^{(k+1)}(b)\left\{\frac{x_i-m}{M-m}(M-b)^{k+2} + \frac{M-x_i}{M-m}(m-b)^{k+2} - (x_i-b)^{k+2}\right\}\right].\end{aligned} \tag{2.529}$$

Multiplying (2.566) with w_i, dividing by W_r and taking the summation from $i=1$ to r, we get (2.528).

(ii) Using similar arguments as in the proof of Theorem 2.191(ii), we get the required result. □

Remark 2.50 *In Theorem 2.179, assume that* $x_0, \sum_{i=1}^{r} w_i x_i \in [m,M]$ *with* $x_0 \neq \sum_{i=1}^{r} w_i x_i$ *and* $(x_i - x_0)\big(\sum_{i=1}^{r} w_i x_i - x_i\big) \geq 0, i = 1,2,,..,r$. *If* $x_0 < \sum_{i=1}^{r} w_i x_i$, *then by taking* $m = x_0$ *and* $M = \sum_{i=1}^{r} w_i x_i$, *in inequality (2.528) we obtain the generalization of Giaccardi inequality. Similarly If* $x_0 > \sum_{i=1}^{r} w_i x_i$, *then by taking* $M = x_0$ *and* $m = \sum_{i=1}^{r} w_i x_i$, *in inequality (2.528) we obtain the generalization of Giaccardi inequality.*

Moreover, if we take $m = x_0 = 0$ *in the generalized Giaccardi inequality we obtain generalization of Jensen-Petrović's inequality.*

The integral version of the above theorem can be stated as follows.

Theorem 2.180 *Let* $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(2n-1)}$ *is absolutely continuous,* $x : [\alpha,\beta] \to \mathbb{R}$ *be continuous function such that* $x([\alpha,\beta]) \subseteq [m,M] \subseteq [a,b]$, $\lambda : [\alpha,\beta] \to \mathbb{R}$ *increasing, bounded function with* $\lambda(\alpha) \neq \lambda(\beta)$ *and* $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$.

(i) Then for any $2n$*-convex function* $f : [a,b] \to \mathbb{R}$, *the following inequality holds*

$$\frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \leq \frac{\overline{x}-m}{M-m}f(M) + \frac{M-\overline{x}}{M-m}f(m) - \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}$$
$$\left[f^{(k+1)}(a)\left\{\frac{\overline{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\overline{x}}{M-m}(m-a)^{k+2} - \frac{\int_\alpha^\beta (x(t)-a)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right\}\right.$$
$$\left.- f^{(k+1)}(b)\left\{\frac{\overline{x}-m}{M-m}(M-b)^{k+2} + \frac{M-\overline{x}}{M-m}(m-b)^{k+2} - \frac{\int_\alpha^\beta (x(t)-b)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}\right\}\right]. \tag{2.530}$$

(ii) If the inequality (2.530) holds and the function F *defined as in (2.512) is convex, then*

$$\frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \leq \frac{\overline{x}-m}{M-m}f(M) + \frac{M-\overline{x}}{M-m}f(m).$$

Corollary 2.35 *Let* $n \in \mathbb{N}$, $\mathbf{x} = (x_1,\ldots,x_r)$ *be real* $r-$*tuple with* $x_i \in [m,M]$, $\mathbf{w} = (w_1,\ldots,w_r)$ *be positive* $r-$*tuple,* $W_r = \sum_{i=1}^{r} w_i$ *and* $\overline{x} = \frac{1}{W_r}\sum_{i=1}^{r} w_i x_i$. *Then for* $2n$*-convex function* $f : [m,M] \to \mathbb{R}$, *the following inequality holds*

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \leq \frac{\overline{x}-m}{M-m}f(M)+\frac{M-\overline{x}}{M-m}f(m)-\frac{1}{M-m}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}$$
$$\left[f^{(k+1)}(m)\left\{\frac{\overline{x}-m}{M-m}(M-m)^{k+2}-\frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i-m)^{k+2}\right\}\right.$$
$$\left.-f^{(k+1)}(M)\left\{\frac{M-\overline{x}}{M-m}(m-M)^{k+2}-\frac{1}{W_r}\sum_{i=1}^{r} w_i(x_i-M)^{k+2}\right\}\right].$$

Proof. Using the inequality (2.528) for $a=m$ and $b=M$. □

Remark 2.51 *Similarly, one can also easily obtain the integral variants of Corollary 2.35.*

$$\delta(s)=\frac{1}{W_m}\sum_{i=1}^{m} w_i T_n(x_i,s)-T_n(\overline{x},s),\ s\in[a,b]. \tag{2.531}$$

Let $x:[\alpha,\beta]\to[a,b]$ be continuous function and $\lambda:[\alpha,\beta]\to\mathbb{R}$ be as in Theorem 2.178 and let $\overline{x}=\frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$, we denote

$$\triangle(s)=\frac{\int_\alpha^\beta (T_n(x(t),s)d\lambda(t)}{\int_\alpha^\beta \lambda(t)dt}-T_n(\overline{x},s),\ s\in[a,b]. \tag{2.532}$$

From Čebyšev functional we may write

$$T(\delta,\delta)=\frac{1}{b-a}\int_a^b \delta^2(s)ds-\left(\frac{1}{b-a}\int_a^b \delta(s)ds\right)^2,$$
$$T(\triangle,\triangle)=\frac{1}{b-a}\int_a^b \triangle^2(s)ds-\left(\frac{1}{b-a}\int_a^b \triangle(s)ds\right)^2.$$

Theorem 2.181 *Let $n\in\mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $(.-a)(b-.)[f^{(n+1)}]^2\in L[a,b]$. Let $x_i\in[a,b]$, $w_i\in\mathbb{R}$, $i=1,2,\ldots,m$, $W_m=\sum_{i=1}^{m} w_i\neq 0$ and $\overline{x}=\frac{1}{W_m}\sum_{i=1}^{m} w_i x_i\in[a,b]$. Let the function δ be defined as in (2.531). Then we have*

$$\frac{1}{W_m}\sum_{i=1}^{m} w_i f(x_i)-f(\overline{x})$$
$$=\frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left\{\frac{1}{W_m}\sum_{i=1}^{m} w_i(x_i-a)^{k+2}-(\overline{x}-a)^{k+2}\right\}\right.$$
$$\left.-f^{(k+1)}(b)\left\{\frac{1}{W_m}\sum_{i=1}^{m} w_i(x_i-b)^{k+2}-(\overline{x}-b)^{k+2}\right\}\right]$$
$$+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \delta(s)ds+H_n^1(f;a,b), \tag{2.533}$$

where the remainder $H_n^1(f;a,b)$ satisfies the estimation

$$\left|H_n^1(f;a,b)\right| \leq \frac{1}{(n-1)!}\left(\frac{b-a}{2}\left|T(\delta,\delta)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{\frac{1}{2}}.$$

Proof. Using Theorem 2.132 for $y_i \to \overline{x}$, we get

$$\begin{aligned}
&\frac{1}{W_m}\sum_{i=1}^{m} w_i f(x_i) - f(\overline{x}) \\
&= \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left\{\frac{1}{W_m}\sum_{i=1}^{m} w_i(x_i-a)^{k+2} - (\overline{x}-a)^{k+2}\right\}\right. \\
&\left. -f^{(k+1)}(b)\left\{\frac{1}{W_m}\sum_{i=1}^{m} w_i(x_i-b)^{k+2} - (\overline{x}-b)^{k+2}\right\}\right] \\
&+\frac{1}{(n-1)!}\int_a^b \delta(s) f^{(n)}(s)ds.
\end{aligned} \tag{2.534}$$

Now if we apply Theorem 1.10 for $f \to \delta$ and $h \to f^{(n)}$, we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \delta(s)f^{(n)}(s)ds - \left(\frac{1}{b-a}\int_a^b \delta(s)ds\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)\right| \\
&\leq \frac{1}{\sqrt{2}}[T(\delta,\delta)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore we have

$$\frac{1}{(n-1)!}\int_a^b \delta(s)f^{(n)}(s)ds = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \delta(s)ds + H_n^1(f;a,b). \tag{2.535}$$

From (2.534) and (2.535), we obtain (2.572). □

The integral version of the above theorem can be stated as follows.

Theorem 2.182 *Let $n \in \mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $(.-a)(b-.)[f^{(n+1)}]^2 \in L[a,b]$. Let $x:[\alpha,\beta]\to[a,b]$ be continuous function such that $x([\alpha,\beta]) \subseteq [a,b]$, $\lambda:[\alpha,\beta]\to\mathbb{R}$ be as defined in Theorem 2.178 and $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$. Let the function $\triangle$ be defined as in (2.532). Then we have*

$$\frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\overline{x}) \tag{2.536}$$

$$\begin{aligned}
&= \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left(\frac{1}{\int_\alpha^\beta d\lambda(t)}\int_\alpha^\beta (x(t)-a)^{k+2}d\lambda(t) - (\overline{x}-a)^{k+2}\right)\right. \\
&\left. -f^{(k+1)}(b)\left(\frac{\int_\alpha^\beta (x(t)-b)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-b)^{k+2}\right)\right] \\
&+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \triangle(s)ds + H_n^2(f;a,b),
\end{aligned} \tag{2.537}$$

where the remainder $H_n^2(f;a,b)$ satisfies the estimation

$$\left|H_n^2(f;a,b)\right| \leq \frac{1}{(n-1)!}\left(\frac{b-a}{2}\left|T(\delta,\delta)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{\frac{1}{2}}.$$

In the next theorem we obtain the Grüss type inequality.

Theorem 2.183 *Let $n \in \mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $f^{(n+1)} \geq 0$ on $[a,b]$ and let the function δ be defined as in (2.531). Then we have the representation (2.533) and the remainder $H_n^1(f;a,b)$ satisfies*

$$\left|H_n^1(f;a,b)\right| \leq \frac{1}{(n-1)!}\parallel \delta' \parallel_\infty \left[\frac{b-a}{2}\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]-\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]\right]. \tag{2.538}$$

Proof. The proof is similar to the proof of Theorem 7 in [18]. □

The integral version of the above theorem can be given as follows.

Theorem 2.184 *Let $n \in \mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $f^{(n+1)} \geq 0$ on $[a,b]$ and let the functions T and $\triangle$ be defined as in (2.533) and (2.532) respectively. Then we have the representation (2.536) and the remainder $H_n^2(f;a,b)$ satisfies*

$$\left|H_n^2(f;a,b)\right| \leq \frac{1}{(n-1)!}\parallel \triangle' \parallel_\infty \left[\frac{b-a}{2}\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]-\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]\right].$$

We present the Ostrowsky type inequalities related to the generalized of Jensen's inequality.

Theorem 2.185 *Let $n \in \mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p[a,b]$, $\boldsymbol{x}=(x_1,\ldots,x_m)\in[a,b]^m$, $\boldsymbol{w}=(w_1,\ldots,w_m)$ be real $m-$tuple, $W_m=\sum\limits_{i=1}^m w_i \neq 0$ and $\overline{x}=\frac{1}{W_m}\sum\limits_{i=1}^m w_i x_i \in [a,b]$. Let (p,q) be a pair of conjugate exponents, that is, $1\leq p,q\leq\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$\begin{aligned}
&\left|\frac{1}{W_m}\sum_{i=1}^m w_i f(x_i)-f(\overline{x})-\frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\left[f^{(k+1)}(a)\left\{\frac{\sum\limits_{i=1}^m w_i(x_i-a)^{k+2}}{W_m}-(\overline{x}-a)^{k+2}\right\}\right.\right.\\
&\qquad\left.\left.-f^{(k+1)}(b)\left\{\frac{\sum\limits_{i=1}^m w_i(x_i-b)^{k+2}}{W_m}-(\overline{x}-b)^{k+2}\right\}\right]\right|\\
&\qquad\leq \frac{1}{(n-1)!}\parallel f^{(n)}\parallel_p \left\|\frac{\sum_{i=1}^m w_i T_n(x_i,.)}{W_m}-T_n(\overline{x},.)\right\|_q.
\end{aligned} \tag{2.539}$$

The constant on the right of (2.539) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The arguments of the proof is similar to the proof of Theorem 9 in [18]. $\square$

The integral version of the above theorem given as follows

Theorem 2.186 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p[a,b]$. Let $x : [\alpha,\beta] \to \mathbb{R}$ be continuous function such that $x([\alpha,\beta]) \subset [a,b]$, $\lambda : [\alpha,\beta] \to \mathbb{R}$ be defined as Theorem 2.178 and $\overline{x} = \frac{\int_\alpha^\beta x(t)d\lambda(t)}{\int_\alpha^\beta d\lambda(t)}$. Let (p,q) be a pair of conjugate exponents, that is, $1 \leq p,q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\begin{aligned}
&\Bigg| \frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\overline{x}) - \frac{1}{b-a} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \\
&\qquad \Bigg[f^{(k+1)}(a) \left\{ \frac{\int_\alpha^\beta (x(t)-a)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-a)^{k+2} \right\} \\
&\qquad - f^{(k+1)}(b) \left\{ \frac{\int_\alpha^\beta (x(t)-b)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-b)^{k+2} \right\} \Bigg] \Bigg| \\
&\qquad \leq \| f^{(n)} \|_p \left\| \frac{\int_\alpha^\beta (T_n(x(t),.)d\lambda(t)}{\int_\alpha^\beta \lambda(t)dt} - T_n(\overline{x},.) \right\|_q .
\end{aligned} \tag{2.540}$$

The constant on the right of (2.540) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Motivated by the inequalities (2.511), (2.516), (2.528) and (2.530) we define the functionals $\Upsilon_1(f)$, $\Upsilon_2(f)$, $\Upsilon_3(f)$ and $\Upsilon_4(f)$ respectively by

$$\begin{aligned}
\Upsilon_1(f) &= \frac{1}{W_m} \sum_{i=1}^m w_i f(x_i) - f(\overline{x}) \\
&- \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{1}{k!(k+2)} \Bigg[f^{(k+1)}(a) \left\{ \frac{1}{W_m} \sum_{i=1}^m w_i (x_i - a)^{k+2} - (\overline{x}-a)^{k+2} \right\} \\
&- f^{(k+1)}(b) \left\{ \frac{1}{W_m} \sum_{i=1}^m w_i (x_i - b)^{k+2} - (\overline{x}-b)^{k+2} \right\} \Bigg].
\end{aligned} \tag{2.541}$$

$$\begin{aligned}
\Upsilon_2(f) &= \frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - f(\overline{x}) \\
&- \frac{1}{b-a} \sum_{k=0}^{2n-2} \frac{1}{k!(k+2)} \Bigg[f^{(k+1)}(a) \left\{ \frac{\int_\alpha^\beta (x(t)-a)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-a)^{k+2} \right\} \\
&- f^{(k+1)}(b) \left\{ \frac{\int_\alpha^\beta (x(t)-b)^{k+2} d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - (\overline{x}-b)^{k+2} \right\} \Bigg].
\end{aligned} \tag{2.542}$$

$$\Upsilon_3(f) = \frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) - \frac{\overline{x}-m}{M-m} f(M) - \frac{M-\overline{x}}{M-m} f(m) + \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}$$
$$\left[f^{(k+1)}(a)\left\{ \frac{\overline{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\overline{x}}{M-m}(m-a)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i (x_i-a)^{k+2} \right\}\right.$$
$$\left. - f^{(k+1)}(b)\left\{ \frac{\overline{x}-m}{M-m}(M-b)^{k+2} + \frac{M-\overline{x}}{M-m}(m-b)^{k+2} - \frac{1}{W_r}\sum_{i=1}^{r} w_i (x_i-b)^{k+2} \right\}\right]. \tag{2.543}$$

$$\Upsilon_4(f) = \frac{\int_\alpha^\beta f(x(t))d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} - \frac{\overline{x}-m}{M-m} f(M) - \frac{M-\overline{x}}{M-m} f(m) + \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{1}{k!(k+2)}$$
$$\left[f^{(k+1)}(a)\left\{ \frac{\overline{x}-m}{M-m}(M-a)^{k+2} + \frac{M-\overline{x}}{M-m}(m-a)^{k+2} - \frac{\int_\alpha^\beta (x(t)-a)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \right\}\right.$$
$$\left. - f^{(k+1)}(b)\left\{ \frac{\overline{x}-m}{M-m}(M-b)^{k+2} + \frac{M+\overline{x}}{M-m}(m-b)^{k+2} - \frac{\int_\alpha^\beta (x(t)-b)^{k+2}d\lambda(t)}{\int_\alpha^\beta d\lambda(t)} \right\}\right]. \tag{2.544}$$

Theorem 2.187 *Let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^{2n}[a,b]$. If the inequalities (2.511), (2.516) and the reverse inequalities in (2.528) and (2.530) hold, then there exist $\xi_k \in [a,b]$ for $k \in \{1,2,3,4\}$ such that*

$$\Upsilon_k(f) = f^{(2n)}(\xi_k)\Upsilon_k(f_0), \;\; k \in \{1,2,3,4\}, \tag{2.545}$$

where $f_0(x) = \frac{x^{2n}}{(2n)!}$.

Proof. The proof is similar to the proof of Theorem 11 in [18]. □

Theorem 2.188 *Let $f,g : [a,b] \to \mathbb{R}$ be such that $f,g \in C^{2n}[a,b]$. If the inequality (2.511) and (2.516) and the reverse inequality (2.528) and (2.530) hold, then there exist $\xi_k \in [a,b]$ for $k \in \{1,2,3,4\}$ such that*

$$\frac{\Upsilon_k(f)}{\Upsilon_k(g)} = \frac{f^{(2n)}(\xi_k)}{g^{(2n)}(\xi_k)},$$

provided that the denominators are non-zero.

Proof. The proof is similar to the proof of Theorem 12 in [18]. □

Remark 2.52 *If the inverse of $\frac{f^{(2n)}}{g^{(2n)}}$ exists, then from the above mean value theorem we can give the generalized means,*

$$\xi_k = \left(\frac{f^{(2n)}}{g^{(2n)}}\right)^{-1}\left(\frac{\Upsilon_k(f)}{\Upsilon_k(g)}\right), \;\; k \in \{1,2,3,4\}.$$

Theorem 2.189 *Let $H_1 = \{f_t : t \in I\}$, where I an interval in $\mathbb{R}$, be a family of functions defined on $[a,b]$ such that the function $t \to f_t[z_0, z_1, \ldots, z_{2l}]$ is n-exponentially convex in the Jensen sense on I for any $2l+1$ mutually distinct points $z_0, z_1, \ldots, z_{2l} \in [a,b]$. Let $\Upsilon_k(f)$ be the linear functionals for $k \in \{1,2,3,4\}$ as defined in (2.541), (2.542), (2.543) and (2.544). Then the following statements are valid:*

(i) The function $t \to \Upsilon_k(f_t)$ is n-exponentially convex in the Jensen sense on I.

(ii) If the function $t \to \Upsilon(f_t)$ is continuous on I, then it is n-exponentially convex on I.

Proof. The proof is similar to the proof of Theorem 13 in [18]. □

As a consequence of the above theorem we give the following corollaries.

Corollary 2.36 *Let $H_2 = \{f_t : t \in I\}$, where I an interval in $\mathbb{R}$, be a family of functions defined on the interval $[a,b]$ such that the function $t \to f_t[z_0, z_1, \ldots, z_{2l}]$ is exponentially convex in the Jensen sense on I for any $(2l+1)$ mutually distinct points $z_0, z_1, \ldots, z_{2l} \in [a,b]$. Let $\Upsilon_k(f_t)$ be linear functionals for $k \in \{1,2,3,4\}$ as defined in (2.541), (2.542), (2.543) and (2.544). Then the following statements are valid:*

(i) The function $t \to \Upsilon_k(f_t)$ is exponentially convex in the Jensen sense on I.

(ii) If the function $t \to \Upsilon_k(f_t)$ is continuous on I, then it is exponentially convex on I.

Proof. The proof follows directly from Theorem 2.189 by using the definition of exponential convexity. □

Corollary 2.37 *Let $H_3 = \{f_t : t \in I\}$, where I an interval in $\mathbb{R}$, be a family of functions defined on $[a,b]$ such that the function $t \to f_t[z_0, z_1, \ldots, z_{2l}]$ is $2-$exponentially convex in the Jensen sense on I for any $2l+1$ mutually distinct points $z_0, z_1, \ldots, z_{2l} \in [a,b]$. Let Υ_k be linear functionals for $k \in \{1,2,3,4\}$ as defined in (2.541), (2.542), (2.543) and (2.544). Then the following statements are valid:*

(i) If the function $t \to \Upsilon_k(f_t)$ is continuous on I, then it is $2-$exponentially convex on I. If $t \to \Upsilon_k(f_t)$ is additionally positive, then it is also log-convex on I. Furthermore, for every choice $r,s,t \in I$, such that $r < s < t$, it holds

$$[\Upsilon_k(f_s)]^{t-r} \leq [\Upsilon_k(f_r)]^{t-s} [\Upsilon_k(f_t)]^{s-r}.$$

(ii) If the function $t \to \Upsilon_k(f_t)$ is positive and differentiable on I, then for all $r,s,u,v \in I$ such that $r \leq u$, $s \leq v$, we have

$$\mu_{r,s}(\Upsilon_k, H_3) \leq \mu_{u,v}(\Upsilon_k, H_3), \tag{2.546}$$

where

$$\mu_{r,s}(\Upsilon_k, H_3) = \begin{cases} \left(\frac{\Upsilon_k(f_r)}{\Upsilon_k(f_s)}\right)^{\frac{1}{r-s}}, & r \neq s, \\ \exp\left(\frac{\frac{d}{dr}(\Upsilon_k(f_r))}{\Upsilon_k(f_r)}\right), & r = s. \end{cases} \tag{2.547}$$

Proof. The arguments of the proof is similar to the proof of Corollary 6 in [18]. □

Remark 2.53 *Note that the results from Theorem 2.189, Corollary 2.36 and Corollary 2.37 still hold when any two(all) points $z_0,\ldots,z_{2l}\in[a,b]$ coincide for a family of differentiable (2l times differentiable) functions f_t such that the function $t\to f_t[z_0,\ldots,z_{2l}]$ is an n-exponentially convex, exponentially convex and 2−exponentially convex in the Jensen sense, respectively.*

We denote

$$A_k=\frac{1}{W_m}\sum_{i=1}^{m}w_i(x_i-a)^{k+2}-(\overline{x}-a)^{k+2},B_k=\frac{1}{W_m}\sum_{i=1}^{m}w_i(x_i-b)^{k+2}-(\overline{x}-b)^{k+2},$$
$$C_k=\frac{\overline{x}-m}{M-m}(M-a)^{k+2}+\frac{M-\overline{x}}{M-m}(m-a)^{k+2}-\frac{1}{W_r}\sum_{i=1}^{r}w_i(x_i-a)^{k+2},$$
$$D_k=\frac{\overline{x}-m}{M-m}(M-b)^{k+2}+\frac{M-\overline{x}}{M-m}(m-b)^{k+2}-\frac{1}{W_r}\sum_{i=1}^{r}w_i(x_i-b)^{k+2}.$$

where $x_i,w_i,\overline{x}$ are as defined in Theorem 2.191.

Example 2.3 *Let us consider a family of functions*

$$\Omega_1=\{f_t:\mathbb{R}\to\mathbb{R}:t\in\mathbb{R}\}$$

defined by

$$f_t(x)=\begin{cases}\frac{e^{tx}}{t^{2n}}\ , & t\neq 0,\\ \frac{x^{2n}}{(2n)!}\ , & t=0.\end{cases}$$

Since $\frac{d^{2n}f_t}{dx^{2n}}(x)=e^{tx}>0$, the function f_t is 2n-convex on $\mathbb{R}$ for every $t\in\mathbb{R}$ and $t\to\frac{d^{2n}f_t}{dx^{2n}}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 2.189 we also have that $t\to f_t[z_0,\ldots,z_{2n}]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 2.36 we conclude that $t\to\Upsilon_k(f_t)$, $k\in\{1,2,3,4\}$ are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous so they are exponentially convex. For this family of functions, $\mu_{s,q}(\Upsilon_k,\Omega_1)$, from (2.547), $k=1$, becomes

$$\mu_{s,q}(\Upsilon_1,\Omega_1)=\left(\frac{\Upsilon_1(f_s)}{\Upsilon_1(f_q)}\right)^{\frac{1}{s-q}}\ ,\ q\neq s,$$

$$\mu_{s,q}(\Upsilon_1,\Omega_1)=\left(\left(\frac{q}{s}\right)^{2n}\frac{\frac{1}{W_m}\sum\limits_{i=1}^{m}w_ie^{sx_i}-e^{s\overline{x}}-K_1}{\frac{1}{W_m}\sum\limits_{i=1}^{m}w_ie^{qx_i}-e^{q\overline{x}}-K_2}\right)^{\frac{1}{s-q}},\qquad s\neq q,s,q\neq 0$$

$$\mu_{s,s}(\Upsilon_1,\Omega_1)=\exp\left(\frac{\frac{1}{W_m}\sum\limits_{i=1}^{m}w_ix_ie^{sx_i}-\overline{x}e^{s\overline{x}}-K_3}{\frac{1}{W_m}\sum\limits_{i=1}^{m}w_ie^{sx_i}-e^{s\overline{x}}-K_1}-\frac{2n}{s}\right),s\neq 0.$$

$$\mu_{0,0}(\Upsilon_1,\Omega_1)=\exp\left(\frac{1}{2n+1}\,\frac{\frac{1}{W_m}\sum\limits_{i=1}^{m}w_ix_i^{2n+1}-\overline{x}^{2n+1}-K_4}{\frac{1}{W_m}\sum\limits_{i=1}^{m}w_ix_i^{2n}-\overline{x}^{n}-K_5}\right),$$

where

$$K_1=\frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{s^{k+1}}{k!(k+2)}\left[e^{as}A_k-e^{bs}B_k\right],$$

$$K_2=\frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{q^{k+1}}{k!(k+2)}\left[e^{aq}A_k-e^{bq}B_k\right],$$

$$K_3=\frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{s^{k}}{k!(k+2)}\left[(as+k+1)e^{sa}A_k-(bs+k+1)e^{sb}B_k\right],$$

$$K_4=\frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{2n(2n-1)\dots(2n-k)}{k!(k+2)}\left[a^{2n-k}A_k-b^{2n-k}B_k\right],$$

$$K_5=\frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{2n(2n-1)..(2n-k)}{k!(k+2)}\left[a^{2n-k-1}A_k-b^{2n-k-1}B_k\right].$$

Similarly we can give $\mu_{s,q}(\Upsilon_k,\Omega_k)$ for $k=2,3,4$.
Now, using (2.546) $\mu_{s,q}(\Upsilon_k,\Omega_k)$ is monotonic function in parameters s and q. Using Corollary 2.37 and Theorem 2.188 it follows that :

$$M_{s,q}(\Upsilon_k,\Omega_1)=\ln\mu_{s,q}(\Upsilon_k,\Omega_1),\ k=1,2,3,4$$

satisfy

$$a\le M_{s,q}(\Upsilon_k,\Omega_1)\le b,\quad k=1,2,3,4.$$

This shows that $M_{s,q}(\Upsilon_k,\Omega_1)$ is a mean for $k=1,2,3,4$.

Example 2.4 *Let*

$$\Omega_2=\{g_t:(0,\infty)\to(0,\infty):t\in(0,\infty)\}$$

be a family of functions defined by

$$g_t(x)=\begin{cases}\frac{t^{-x}}{(-\ln t)^{2n}}, & t\neq 1;\\[2ex] \frac{x^{2n}}{(2n)!}, & t=1.\end{cases}$$

Since $\frac{d^{2n}g_t}{dx^{2n}}(x)=t^{-x}$ is the Laplace transform of a non-negative function (see [172]) it is exponentially convex. Obviously g_t is $2n$-convex function for every $t>0$.
For this family of functions, $\mu_{s,q}(\Upsilon_1,\Omega_2)$, from (2.547), becomes

$$\mu_{s,q}(\Upsilon_1,\Omega_2) = \left(\left(\frac{\ln q}{\ln s}\right)^{2n} \frac{\frac{1}{W_m}\sum_{i=1}^{m} w_i s^{-x_i} - s^{-\bar{x}} - L_1}{\frac{1}{W_m}\sum_{i=1}^{m} w_i q^{-x_i} - q^{-\bar{x}} - L_2} \right)^{\frac{1}{s-q}}, \ s \neq q;$$

$$\mu_{s,s}(\Upsilon_1,\Omega_2) = \exp\left(\frac{\bar{x}s^{-\bar{x}-1} - \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i s^{-x_i-1} - L_3}{\frac{1}{W_m}\sum_{i=1}^{m} w_i s^{-x_i} - s^{-\bar{x}} - L_1} - \frac{2n}{s\ln s} \right), \ s \neq 1.$$

$$\mu_{1,1}(\Upsilon_1,\Omega_2) = \exp\left(-\frac{1}{2n+1} \frac{\frac{1}{W_m}\sum_{i=1}^{m} w_i x_i^{2n+1} - \bar{x}^{2n+1} - L_4}{\frac{1}{W_m}\sum_{i=1}^{m} w_i x_i^{2n} - \bar{x}^{2n} - L_5} \right), \ s = 1.$$

where

$$L_1 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{(-\ln s)^{k+1}}{k!(k+2)}\left[s^{-a}A_k - s^{-b}B_k\right], \ L_2 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{(-\ln q)^{k+1}}{k!(k+2)}\left[q^{-a}A_k - q^{-b}B_k\right],$$

$$L_3 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{(-\ln s)^{k}}{k!(k+2)}\left[(a\ln s - k - 1)s^{-a-1}A_k - (b\ln s - k - 1)s^{-b-1}B_k\right],$$

$$L_4 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{2n(2n-1)\dots(2n-k)}{k!(k+2)}\left[a^{2n-k}A_k - b^{2n-k}B_k\right],$$

$$L_5 = \frac{1}{b-a}\sum_{k=0}^{2n-2}\frac{2n(2n-1)..(2n-k)}{k!(k+2)}\left[a^{2n-k-1}A_k - b^{2n-k-1}B_k\right].$$

Similarly we can give $\mu_{s,q}(\Upsilon_k,\Omega_2)$ for $k = 2,3,4$.
Now, using (2.546) it is monotonic function in parameters s and q. Using Corollary 2.37 and Theorem 2.188 it follows that :

$$M_{s,q}(\Upsilon_k,\Omega_2) = \ln \mu_{s,q}(\Upsilon_k,\Omega_2), \ k = 1,2,3,4$$

satisfy

$$a \leq M_{s,q}(\Upsilon_k,\Omega_2) \leq b, \quad k = 1,2,3,4.$$

This shows that $M_{s,q}(\Upsilon_k,\Omega_2)$ is a mean for $k = 1,2,3,4$. Because of the inequality (2.546), this mean is also monotonic.

2.4.5 Results Obtained for the Jensen and the Jensen-Steffensen Inequalities and their Converses via Green's Function and Montgomery Identity

To make the calculations simple we use the following notations:

$$\triangle(w_i,x_i,y_i,f) = \sum_{i=1}^{m} w_i f(y_i) - \sum_{i=1}^{m} w_i f(x_i), \tag{2.548}$$

$$\nabla(\overline{x},f) := \frac{1}{W_m}\sum_{i=1}^{m} w_i f(x_i) - f(\overline{x}),$$

where w_i, x_i, and f are as defined in Theorem 1.14, and also

$$\Lambda(w,x,y,f) = \int_a^b w(u)f(y(u))du - \int_a^b w(u)f(x(u))du. \tag{2.549}$$

If $W = \int_a^b w(t)dt$ and $\overline{x} = \frac{\int_a^b x(t)w(t)dt}{W}$, we denote

$$\Upsilon(\overline{x},f) := \frac{1}{W}\int_a^b w(t)f(x(t))dt - f(\overline{x}),$$

where w, x and f (ϕ instead f) are as given in Theorem 2.131.

The following generalizations of majorization theorem by Montgomery identity and Green's function are given in [21].

Theorem 2.190 ([21]) *Suppose all the assumptions of Theorem 2.147 are valid. Also let $n \in N$, $f : I \to \mathbb{R}$ be function such that $f^{(n-1)}$ $(n > 3)$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$, n is even, f is n-convex and $G(.,s)$ be as defined in (1.180). Then for all $s \in [a,b]$, the following inequalities hold:*

(i)

$$\begin{aligned}\triangle(w_i,x_i,y_i,f) \geq \frac{f(b)-f(a)}{b-a}\triangle(w_i,x_i,y_i,id) + \frac{f'(a)-f'(b)}{b-a}\cdot \\ \int_a^b \triangle(w_i,x_i,y_i,G(.,s))ds + \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \triangle(w_i,x_i,y_i,G(.,s))\cdot \\ \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds,\end{aligned} \tag{2.550}$$

(ii)

$$\begin{aligned}\triangle(w_i,x_i,y_i,f) \geq \frac{f(b)-f(a)}{b-a}\triangle(w_i,x_i,y_i,id) + \frac{f'(b)-f'(a)}{b-a}\cdot \\ \int_a^b \triangle(w_i,x_i,y_i,G(.,s))ds + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \triangle(w_i,x_i,y_i,G(.,s))\cdot \\ \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds.\end{aligned} \tag{2.551}$$

Theorem 2.191 *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Let $\boldsymbol{x} = (x_1,\ldots,x_m)$ be $m-$tuple with $x_i \in [a,b]$ and $\boldsymbol{w} = (w_1,\ldots,w_m)$ be positive real $m-$tuple, $W_m = \sum_{i=1}^{m} w_i$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^{m} w_i x_i$ and G be the Green function as defined in (1.180).*

(i) If $\boldsymbol{x}$ is decreasing $m-$tuple and $f : [a,b] \to \mathbb{R}$ is $2n$-convex function then the following inequalities hold:

$$\nabla(\overline{x}, f) \geq \frac{f'(a) - f'(b)}{b-a}\int_a^b \nabla(\overline{x}, G(.,s))ds + \sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\int_a^b \nabla(\overline{x}, G(.,s))\cdot \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds, \tag{2.552}$$

$$\nabla(\overline{x}, f) \geq \frac{f'(b) - f'(a)}{b-a}\int_a^b \nabla(\overline{x}, G(.,s))ds + \sum_{k=3}^{2n-2}\frac{k-2}{(k-1)!}\int_a^b \nabla(\overline{x}, G(.,s))\cdot \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds. \tag{2.553}$$

(ii) If the inequalities (2.552) and (2.553) hold and the functions L_1 and L_2 defined by

$$L_1(.) = \frac{f'(a) - f'(b)}{b-a}\int_a^b G(.,s)ds + \sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\int_a^b G(.,s)\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds, \tag{2.554}$$

$$L_2(.) = \frac{f'(b) - f'(a)}{b-a}\int_a^b G(.,s)ds + \sum_{k=3}^{2n-2}\frac{k-2}{(k-1)!}\int_a^b G(.,s)\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds, \tag{2.555}$$

are convex, then the right hand sides of (2.552) and (2.553) are non-negative and

$$f(\overline{x}) \leq \frac{1}{W_m}\sum_{i=1}^{m} w_i f(x_i), \tag{2.556}$$

holds in both cases.

Proof.

(i) Let k be the largest number from $\{1,\ldots,m\}$ such that $x_k \geq \overline{x}$, then as $\boldsymbol{x}$ is decreasing m-tuple so we have $x_l \geq \overline{x}$ for $l = 1,2,\ldots,k$

and $x_l \leq \bar{x}$ for $l = k+1, k+2, \ldots, m$.
Now as $x_l \geq \bar{x}$ for $l = 1, 2, \ldots, k$, so we have

$$\sum_{i=1}^{l} w_i \bar{x} \leq \sum_{i=1}^{l} w_i x_i \quad \text{for } l = 1, 2, \ldots, k. \tag{2.557}$$

Similarly as $x_l \leq \bar{x}$ for $l = k+1, k+2, \ldots, m$, so we have

$$\sum_{i=k+1}^{j} w_i x_i \leq \sum_{i=k+1}^{j} w_i \bar{x} \ \text{ for } j = k+1, k+2, \ldots, m.$$

Hence

$$\sum_{i=1}^{j} w_i x_i = \sum_{i=1}^{m} w_i x_i - \sum_{i=j+1}^{m} w_i x_i \geq \sum_{i=1}^{m} w_i \bar{x} - \sum_{i=j+1}^{m} w_i \bar{x} = \sum_{i=1}^{j} w_i \bar{x}, \tag{2.558}$$
$$\text{for } \ j = k+1, k+2, \ldots, m.$$

Using (2.557) and (2.558) we get that

$$\sum_{i=1}^{l} w_i \bar{x} \leq \sum_{i=1}^{l} w_i x_i, \ \text{for all } l = 1, 2, \ldots, m-1$$

and obviously

$$\sum_{i=1}^{m} w_i \bar{x} = \sum_{i=1}^{m} w_i x_i.$$

The conditions (1.19) and (1.20) are satisfied for $\bar{\boldsymbol{x}} = (\bar{x}, \ldots, \bar{x})$ and $\boldsymbol{y} = (x_1, \ldots, x_m)$. Also

$$\nabla(\bar{\boldsymbol{x}}, id) = 0,$$

therefore substituting $\boldsymbol{y} = (x_1, \ldots, x_m)$ and $\boldsymbol{x} = (\bar{x}, \ldots, \bar{x})$ in Theorem 2.190 (i) we get (2.552).
Proceeding similarly and using Theorem 2.190(ii), we obtain (2.553).

(ii) We may write the right hand side of (2.552) as

$$\frac{1}{W_m} \sum_{i=1}^{m} w_i L_1(x_i) - L_1(\bar{x}).$$

Since L_1 is convex so by Jensen's inequality, we have

$$\frac{1}{W_m} \sum_{i=1}^{m} w_i L_1(x_i) - L_1(\bar{x}) \geq 0.$$

Hence (2.556) holds. Analogously, we obtain (2.556) for L_2.

□

In the following theorem we give integral version of Theorem 2.191.

Theorem 2.192 *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $x : [a,b] \to \mathbb{R}$ be continuous function such that $x([a,b]) \subseteq I$, $w : [a,b] \to \mathbb{R}$ be positive continuous function with $w(a) \neq w(b)$, $W = \int_a^b w(t)dt$, $\bar{x} = \frac{\int_a^b x(t)w(t)dt}{W}$ and G be the Green function as defined in (1.180).*

(i) If x is decreasing and $f : [a,b] \to \mathbb{R}$ is $2n$-convex functions, then the following inequalities hold:

$$\Upsilon(\bar{x}, f) \geq \frac{f'(a) - f'(b)}{b-a} \int_a^b \Upsilon(\bar{x}, G(.,s))ds + \sum_{k=2}^{2n-2} \frac{k}{(k-1)!} \int_a^b \Upsilon(\bar{x}, G(.,s)) \cdot \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds, \tag{2.559}$$

$$\Upsilon(\bar{x}, f) \geq \frac{f'(b) - f'(a)}{b-a} \int_a^b \Upsilon(\bar{x}, G(.,s))ds + \sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!} \int_a^b \Upsilon(\bar{x}, G(.,s)) \cdot \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \tag{2.560}$$

(ii) If the inequalities (2.559) and (2.560) hold and the functions L_1 and L_2 defined as in (2.554) and (2.555) respectively are convex, then the right hand sides of (2.559) and (2.560) are non-negative and

$$f(\bar{x}) \leq \frac{\int_a^b w(t) f(x(t))\, dt}{W}, \tag{2.561}$$

holds in both cases.

Remark 2.54 *If we take $x(t) = t$, $w(t) = 1$, in the inequality (2.559) and (2.560) then we obtain the generalizations of Hermite-Hadamard inequality.*

Theorem 2.193 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $\mathbf{x} = (x_1, \ldots, x_m) \in [a,b]^m$ be decreasing $m-$tuple. Let $\mathbf{w} = (w_1, \ldots, w_m)$ be real $m-$tuple such that $0 \leq W_k \leq W_m$ $(k = 1, 2, \ldots, m)$, $W_m > 0$ where $W_k = \sum_{i=1}^{k} w_i$, $\bar{x} = \frac{1}{W_m} \sum_{i=1}^{m} w_i x_i$ and G be the Green function as defined in (1.180).*

(i) Then for $2n$-convex function f, the inequalities (2.552) and (2.553) hold.

(ii) If the inequalities (2.552) and (2.553) hold and the functions L_1 and L_2 defined as in (2.554) and (2.555) are convex, then the right hand sides of (2.552) and (2.553) are non-negative and (2.556) holds.

Proof. (i) Let k be the largest number $\{1, 2, \ldots, m\}$ such that $x_k \geq \bar{x}$ then $x_l \geq \bar{x}$ for $l = 1, \ldots, k$, and we have

$$\sum_{i=1}^{l} w_i x_i - W_l x_l = \sum_{i=1}^{l-1} (x_i - x_{i+1}) W_i \geq 0$$

and so we obtain

$$\sum_{i=1}^{l} w_i \overline{x} = W_l \overline{x} \leq W_l x_l \leq \sum_{i=1}^{l} x_i w_i. \tag{2.562}$$

Also for $l = k+1,\ldots,m$ we have $x_{k+1} < \overline{x}$, therefore

$$x_l(W_m - W_l) - \sum_{i=l+1}^{m} w_i x_i = \sum_{i=l+1}^{m} (x_{i-1} - x_i)(W_m - W_{i-1}) \geq 0.$$

Hence, we conclude that

$$\sum_{i=l+1}^{m} w_i \overline{x} = (W_m - W_l)\overline{x} > (W_m - W_l)x_l \geq \sum_{i=l+1}^{m} w_i x_i. \tag{2.563}$$

From (2.562) and (2.563), we get

$$\sum_{i=1}^{l} w_i \overline{x} \leq \sum_{i=1}^{l} x_i w_i \quad for \;\; all \;\; l = 1,2,\ldots,m-1.$$

Obviously the equality

$$\sum_{i=1}^{m} w_i \overline{x} = \sum_{i=1}^{m} x_i w_i$$

holds. The conditions (1.19) and (1.20) are satisfied.

Also

$$\nabla(\overline{x}, id) = 0,$$

therefore using Theorem 2.190 (i) for $\mathbf{y} = (x_1,\ldots,x_m)$ and $\mathbf{x} = (\overline{x},\ldots,\overline{x})$, we get (2.552). Proceeding similarly using Theorem 2.190(ii), we obtain (2.553).

(ii) The proof is similar to the proof of Theorem 2.191(ii). □

The integral version of the above theorem is given here.

Theorem 2.194 *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Let $x : [a,b] \to \mathbb{R}$ be continuous decreasing function such that $x([a,b]) \subseteq I$, $w : [a,b] \to \mathbb{R}$ is either continuous or of bounded variation with $w(a) \leq w(t) \leq w(b)$ for all $t \in [a,b]$, $\overline{x} = \frac{\int_a^b x(t)w(t)d(t)}{\int_a^b w(t)dt}$ and G be the Green function as defined in (1.180).*

(i) Then for any $2n$-convex function f, the inequalities (2.559) and (2.560) hold.

(ii) If the inequalities (2.559) and (2.560) hold and the functions L_1 and L_2 defined as in (2.554) and (2.555) respectively, are convex, then the right hand sides of (2.559) and (2.560) are non-negative and (2.561) holds.

Theorem 2.195 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous. Let $\mathbf{x} = (x_1,\ldots,x_r)$ be real $r-$tuple with $x_i \in [m,M] \subseteq [a,b]$, $i = 1,2,\ldots,r$, $\mathbf{w} = (w_1,\ldots,w_r)$ be positive $r-$tuple, $W_r = \sum_{i=1}^{r} w_i$, $\overline{x} = \frac{1}{W_r}\sum_{i=1}^{r} w_i x_i$ and G be the Green function as defined in (1.180).*

(i) Then for any $2n$-convex function $f:[a,b]\to\mathbb{R}$, the following inequalities hold:

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \leq \frac{\overline{x}-m}{M-m}f(M)+\frac{M-\overline{x}}{M-m}f(m)$$
$$+\frac{f'(b)-f'(a)}{b-a}\int_a^b\left[\frac{\overline{x}-m}{M-m}G(M,s)+\frac{M-\overline{x}}{M-m}G(m,s)-\frac{1}{W_r}\sum_{i=1}^{r}w_iG(x_i,s)\right]ds$$
$$-\sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\int_a^b\left[\frac{\overline{x}-m}{M-m}G(M,s)+\frac{M-\overline{x}}{M-m}G(m,s)-\frac{1}{W_r}\sum_{i=1}^{r}w_iG(x_i,s)\right]\cdot$$
$$\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds, \tag{2.564}$$

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \leq \frac{\overline{x}-m}{M-m}f(M)+\frac{M-\overline{x}}{M-m}f(m)$$
$$+\frac{f'(a)-f'(b)}{b-a}\int_a^b\left[\frac{\overline{x}-m}{M-m}G(M,s)+\frac{M-\overline{x}}{M-m}G(m,s)-\frac{1}{W_r}\sum_{i=1}^{r}w_iG(x_i,s)\right]ds$$
$$-\sum_{k=3}^{2n-2}\frac{k-2}{(k-1)!}\int_a^b\left[\frac{\overline{x}-m}{M-m}G(M,s)+\frac{M-\overline{x}}{M-m}G(m,s)-\frac{1}{W_r}\sum_{i=1}^{r}w_iG(x_i,s)\right]\cdot$$
$$\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds. \tag{2.565}$$

(ii) If the inequalities (2.564) and (2.565) hold and the functions L_1 and L_2 defined as in (2.554) and (2.555) respectively, are convex then the inequality

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i f(x_i) \leq \frac{\overline{x}-m}{M-m}f(M)+\frac{M-\overline{x}}{M-m}f(m),$$

holds in both cases.

Proof.

(i) Putting $m=2$, $x_1=M$, $x_2=m$, $w_1=\frac{x_i-m}{M-m}$ and $w_2=\frac{M-x_i}{M-m}$ in (2.552), we have

$$f(x_i)\leq\frac{x_i-m}{M-m}f(M)+\frac{M-x_i}{M-m}f(m)+\frac{f'(b)-f'(a)}{b-a}\int_a^b\left[\frac{x_i-m}{M-m}G(M,s)+\right.$$
$$\left.\frac{M-x_i}{M-m}G(m,s)-G(x_i,s)\right]ds-\sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\int_a^b\left[\frac{x_i-m}{M-m}G(M,s)+\right.$$
$$\left.\frac{M-x_i}{M-m}G(m,s)-G(x_i,s)\right]\cdot\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds. \tag{2.566}$$

Multiplying (2.566) with w_i, dividing by W_r and taking the summation from $i=1$ to r, we get (2.564). Proceeding similarly we obtain (2.565).

(ii) Using similar arguments as in the proof of Theorem 2.191(ii), we get the required result. □

Remark 2.55 *In Theorem 2.195, assume that* $x_0, \sum_{i=1}^{r} w_i x_i \in [m,M]$ *with* $x_0 \neq \sum_{i=1}^{r} w_i x_i$ *and* $(x_i - x_0)\left(\sum_{i=1}^{r} w_i x_i - x_i\right) \geq 0, i = 1,2,,..,r.$ *If* $x_0 < \sum_{i=1}^{r} w_i x_i$, *then by taking* $m = x_0$ *and* $M = \sum_{i=1}^{r} w_i x_i$, *in inequalities (2.564) and (2.565), we obtain the generalizations of Giaccardi inequality. Similarly if* $x_0 > \sum_{i=1}^{r} w_i x_i$, *then by taking* $M = x_0$ *and* $m = \sum_{i=1}^{r} w_i x_i$, *in inequalities (2.564) and (2.565), we obtain the generalizations of Giaccardi inequality. Moreover, if we take* $m = x_0 = 0$ *in the generalized Giaccardi inequalities we obtain generalizations of Jensen-Petrović's inequalities.*

The integral version of the above theorem can be stated as follows.

Theorem 2.196 *Let* $n \in \mathbb{N}$, $f : [\alpha,\beta] \to \mathbb{R}$ *be such that* $f^{(2n-1)}$ *is absolutely continuous,* $x : [a,b] \to \mathbb{R}$ *be continuous function such that* $x([a,b]) \subseteq [m,M] \subseteq [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ *be positive bounded function with* $w(a) \neq w(b)$, $W = \int_a^b w(t)dt$, $\overline{x} = \frac{\int_a^b x(t)w(t)dt}{W}$ *and* G *be the Green function as defined in (1.180).*

(i) Then for any $2n$*-convex function* $f : [a,b] \to \mathbb{R}$*, the following inequalities hold:*

$$\begin{aligned}
&\frac{\int_a^b f(x(t))w(t)dt}{W} \leq \frac{\overline{x}-m}{M-m}f(M) + \frac{M-\overline{x}}{M-m}f(m) + \frac{f'(b)-f'(a)}{b-a} \cdot \\
&\int_a^b \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W}\int_a^b w(t)\,G(x(t),s)dt\right] ds \\
&- \sum_{k=2}^{2n-2} \frac{k}{(k-1)!}\int_a^b \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W}\int_a^b w(t)\,G(x(t),s)dt\right] \cdot \\
&\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds,
\end{aligned} \tag{2.567}$$

$$\begin{aligned}
&\frac{\int_a^b f(x(t))w(t)dt}{W} \leq \frac{\overline{x}-m}{M-m}f(M) + \frac{M-\overline{x}}{M-m}f(m) + \frac{f'(a)-f'(b)}{b-a} \cdot \\
&\int_a^b \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W}\int_a^b w(t)\,G(x(t),s)dt\right] ds \\
&- \sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!}\int_a^b \left[\frac{\overline{x}-m}{M-m}G(M,s) + \frac{M-\overline{x}}{M-m}G(m,s) - \frac{1}{W}\int_a^b w(t)\,G(x(t),s)dt\right] \cdot \\
&\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds.
\end{aligned} \tag{2.568}$$

(ii) If the inequalities (2.567) and (2.568) hold and the functions L_1 and L_2 defined as in (2.554) and (2.555) are convex, then the inequality

$$\frac{\int_a^b f(x(t))w(t)dt}{W} \le \frac{\overline{x}-m}{M-m}f(M)+\frac{M-\overline{x}}{M-m}f(m),$$

holds in both cases.

Let $\boldsymbol{w}=(w_1,\ldots,w_m)$ and $\boldsymbol{x}=(x_1,\ldots,x_m)$ be $m-$tuples with $x_i\in[a,b]$, $w_i\in\mathbb{R}$ $i=1,\ldots,m$, $\overline{x}=\frac{1}{W_m}\sum_{i=1}^{m}w_ix_i\in[a,b]$, $W_m\neq 0$ and the function T_n be defined as in (2.418). We denote

$$\tilde{T}_{n-2}(t,s)=\begin{cases}\frac{1}{b-a}\left[\frac{(t-s)^{n-2}}{n-2}+(t-a)(t-s)^{n-3}\right], & a\le s\le t,\\ \\ \frac{1}{b-a}\left[\frac{(t-s)^{n-2}}{n-2}+(t-b)(t-s)^{n-3}\right], & t<s\le b.\end{cases} \tag{2.569}$$

$$\overline{h}(t)=\int_a^b \nabla(\overline{x},G(.,s))\tilde{T}_{n-2}(s,t)ds. \tag{2.570}$$

$$\aleph(t)=\int_a^b \nabla(\overline{x},G(.,s))T_{n-2}(s,t)ds. \tag{2.571}$$

Theorem 2.197 *Let $n\in\mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $(.-a)(b-.)[f^{(n+1)}]^2\in L[a,b]$. Let $x_i\in[a,b]$, $w_i\in\mathbb{R}$, $i=1,2,\ldots,m$, $W_m=\sum_{i=1}^{m}w_i\neq 0$ and $\overline{x}=\frac{1}{W_m}\sum_{i=1}^{m}w_ix_i\in[a,b]$. Let the functions T_n, $\tilde{T}_n$, T, $\overline{h}$ and $\aleph$ be as defined in (2.418), (2.569), (1.6), (2.570) and (2.571) respectively. Then*

(i) the remainder $R_n^1(\overline{x},f)$ defined by

$$\begin{aligned}R_n^1(\overline{x},f)&=\nabla(\overline{x},f)-\frac{f'(a)-f'(b)}{b-a}\int_a^b\nabla(\overline{x},G(.,s))ds\\&-\sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\int_a^b\nabla(\overline{x},G(.,s))\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\\&-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b\overline{h}(s)ds,\end{aligned} \tag{2.572}$$

satisfies the estimation

$$\left|R_n^1(\overline{x},f)\right|\le\frac{1}{(n-3)!}\left(\frac{b-a}{2}\left|T(\overline{h},\overline{h})\int_a^b(t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|\right)^{\frac{1}{2}}. \tag{2.573}$$

(ii) The remainder $R_n^2(\bar{x}, f)$ defined by

$$R_n^2(\bar{x}, f) = \nabla(\bar{x}, f) - \frac{f'(b) - f'(a)}{b-a} \int_a^b \nabla(\bar{x}, G(.,s))ds$$
$$- \sum_{k=3}^{2n-2} \frac{k-2}{(k-1)!} \int_a^b \nabla(\bar{x}, G(.,s)) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$- \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b \aleph(s)ds, \tag{2.574}$$

satisfies the estimation

$$\left|R_n^2(\bar{x}, f)\right| \leq \frac{1}{(n-3)!} \left(\frac{b-a}{2} \left| T(\aleph, \aleph) \int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt \right| \right)^{\frac{1}{2}}.$$

Proof.

(i) Using (1.181) and (2.417) in the expression $\nabla(\bar{x}, f)$, we obtain

$$\nabla(\bar{x}, f) = \frac{f'(a) - f'(b)}{b-a} \int_a^b \nabla(\bar{x}, G(.,s))ds + \sum_{k=2}^{2n-2} \frac{k}{(k-1)!} \cdot$$
$$\int_a^b \nabla(\bar{x}, G(.,s)) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$+ \frac{1}{(n-3)!} \int_a^b \bar{h}(t) f^{(n)}(t)dt. \tag{2.575}$$

Comparing (2.572) and (2.575), we obtain

$$R_n^1(\bar{x}, f) = \frac{1}{(n-3)!} \int_a^b \bar{h}(t) f^{(n)}(t)dt - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b \bar{h}(t)dt. \tag{2.576}$$

Now applying Theorem 1.10 for $f \to \bar{h}$ and $h \to f^{(n)}$ and using Čebyšev functional we obtain

$$\left| \frac{1}{b-a} \int_a^b \bar{h}(s) f^{(n)}(t)dt - \left(\frac{1}{b-a} \int_a^b \bar{h}(t)dt \right) \left(\frac{1}{b-a} \int_a^b f^{(n)}(t)dt \right) \right|$$
$$\leq \frac{1}{\sqrt{2}} [T(\bar{h}, \bar{h})]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt \right)^{\frac{1}{2}}. \tag{2.577}$$

Multiplying (2.577) with $(b-a)$ and dividing by $(n-3)!$ and using (2.576), we obtain (2.573).

(ii) Similar to the proof of (i).

□

Theorem 2.198 *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous with $f^{(n+1)} \geq 0$ on $[a,b]$ and let T_n, $\tilde{T}_n$, T, $\overline{h}$ and $\aleph$ be as defined in (2.418), (2.569), (1.6), (2.570) and (2.571) respectively. Then*

(i) the remainder $R_n^1(\overline{x}, f)$ defined by (2.572) satisfies the estimation

$$\left|R_n^1(\overline{x}, f)\right| \leq \frac{\|\overline{h}'\|_\infty}{(n-3)!}\left[\frac{(b-a)\left(f^{(n-1)}(b)+f^{(n-1)}(a)\right)}{2} - \left\{f^{(n-2)}(b)-f^{(n-2)}(a)\right\}\right]. \tag{2.578}$$

(ii) the remainder $R_n^2(\overline{x}, f)$ defined by (2.574) satisfies the estimation

$$\left|R_n^2(\overline{x}, f)\right| \leq \frac{\|\aleph'\|_\infty}{(n-3)!}\left[\frac{(b-a)\left(f^{(n-1)}(b)+f^{(n-1)}(a)\right)}{2} - \left\{f^{(n-2)}(b)-f^{(n-2)}(a)\right\}\right]$$

Proof.

(i) Since (2.576) holds and applying Theorem 1.11 for $f \to \overline{h}$ and $g \to f^{(n)}$ and using Čebyšev functional, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \overline{h}(t)f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b \overline{h}(t)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\ &\leq \frac{1}{2(b-a)}\left\|\overline{h}'\right\|_\infty \int_a^b (t-a)(b-t)f^{(n+1)}(t)dt.\end{aligned} \tag{2.579}$$

Since

$$\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt = (b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - 2\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right].$$

Therefore, from (2.576) and (2.579), we deduce (2.578).

(ii) Similar to the proof of (i).

□

Now we present the Ostrowski type inequalities related to the generalized Jensen's inequalities.

Theorem 2.199 *Let (p,q) be a pair of conjugate exponents, that is, $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q} = 1$. Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p[a,b]$, $\boldsymbol{x} = (x_1,\ldots,x_m) \in [a,b]^m$ be $m-$tuple, $\boldsymbol{w} = (w_1,\ldots,w_m)$ be real $m-$tuple, $W_m = \sum_{i=1}^m w_i \neq 0$, $\overline{x} = \frac{1}{W_m}\sum_{i=1}^m w_i x_i \in [a,b]$ and G be the Green function as defined in (1.180). Then the following inequalities hold:*

(i)

$$\left|\nabla(\bar{x},f)-\frac{f'(a)-f'(b)}{b-a}\int_a^b \nabla(\bar{x},G(.,s))ds-\sum_{k=2}^{2n-2}\frac{k}{(k-1)!}\cdot\right.$$
$$\left.\int_a^b \nabla(\bar{x},G(.,s))\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\right|$$
$$\leq \frac{1}{(n-3)!}\parallel f^{(n)}\parallel_p \left|\left|\nabla\left(\bar{x},G(.,s)\right)\tilde{T}_{n-2}(s,t)\right|\right|_q, \tag{2.580}$$

(ii)

$$\left|\nabla(\bar{x},f)-\frac{f'(b)-f'(a)}{b-a}\int_a^b \nabla(\bar{x},G(.,s))ds-\sum_{k=3}^{2n-2}\frac{k-2}{(k-1)!}\cdot\right.$$
$$\left.\int_a^b \nabla(\bar{x},G(.,s))\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\right|$$
$$\leq \frac{1}{(n-3)!}\parallel f^{(n)}\parallel_p ||\nabla(\bar{x},G(.,s))T_{n-2}(s,t)||_q. \tag{2.581}$$

The constants on the right of (2.580) and (2.581) are sharp for $1<p\leq\infty$ and the best possible for $p=1$.

Proof. The arguments of the proof is similar to the proof of Theorem 9 in [18]. □

Remark 2.56 One can also easily obtain the integral variants of Theorems 2.197, 2.198 and 2.199.

2.5 Majorization and Fink's Identity

In this section, applying the following identity, known as Fink's identity, to the majorization differences from majorization theorems, discrete and integral form, we give new identities which we use to obtain many significant results.

The following theorem is proved by A. M. Fink in [73] and it is known as **Fink's identity**.

Theorem 2.200 *Let $a,b\in\mathbb{R}$, $f:[a,b]\to\mathbb{R}$, $n\geq 1$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then*

$$f(x)= \frac{n}{b-a}\int_a^b f(t)\,dt$$
$$-\sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a}\right)$$

$$+\frac{1}{(n-1)!\,(b-a)}\int_a^b (x-t)^{n-1}\,k^{[a,b]}(t,x)\,f^{(n)}(t)\,dt, \tag{2.582}$$

where

$$k^{[a,b]}(t,x)=\begin{cases} t-a, & a\le t\le x\le b,\\ t-b, & a\le x<t\le b.\end{cases} \tag{2.583}$$

The organization of the section is in the following way. In the first subsection, using new identities obtained by Fink's identity in combination with the n-convexity of the function f, we present new refinements and generalizations of the weighted majoriazation inequality for the two decreasing m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ as well as a refinement of the integral majoriazation inequality for the two decreasing functions φ and ψ. We also present a refinement of the majoriazation-type inequality for the two majorized m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$. We study the functionals defined as the difference between the right-hand and the left-hand side of the generalized inequalities. We present some interesting results by using Čebyšev functional and the Grüss type inequalities along with some results relating to the Ostrowski-type inequality. Our objective is to study the properties of functionals, such as n-exponential and logarithmic convexity. Furthermore, we prove monotonicity property of the generalized Cauchy means obtained via these functionals. Finally, we give several examples of the families of functions for which the obtained results can be applied. In the second subsection we present analogous results that include Green's functions.

2.5.1 Results Obtained by Fink's Identity

We start with the following identities obtained by applying Fink's Identity.

Theorem 2.201 ([93]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\ge 1$, $f^{(n-1)}$ is absolutely continuous. Let $x_i,y_i\in[a,b]$, $p_i\in\mathbb{R}$ $(i=1,\ldots,m)$ and let $k^{[a,b]}(t,x)$ be the same as defined in* (2.583). *Then we have*

$$\begin{aligned}
&\sum_{i=1}^m p_i f(x_i)-\sum_{i=1}^m p_i f(y_i)\\
&\quad=\sum_{k=1}^{n-1}\left(\frac{n-k}{k!\,(b-a)}\right)\left(f^{(k-1)}(a)\left(\sum_{i=1}^m p_i(y_i-a)^k-\sum_{i=1}^m p_i(x_i-a)^k\right)\right.\\
&\quad\left.-f^{(k-1)}(b)\left(\sum_{i=1}^m p_i(y_i-b)^k-\sum_{i=1}^m p_i(x_i-b)^k\right)\right)+\frac{1}{(n-1)!\,(b-a)}\cdot\\
&\int_a^b f^{(n)}(t)\left(\sum_{i=1}^m p_i(x_i-t)^{n-1}k^{[a,b]}(t,x_i)-\sum_{i=1}^m p_i(y_i-t)^{n-1}k^{[a,b]}(t,y_i)\right)dt.
\end{aligned} \tag{2.584}$$

Proof. By using (1.181) for $x = x_i$ and $y = y_i$ in the majorization difference, we have

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) =$$
$$\sum_{i=1}^{m} p_i \sum_{k=1}^{n-1} \left(\frac{n-k}{k!}\right) \cdot$$
$$\left(\frac{f^{(k-1)}(a)\left((y_i-a)^k-(x_i-a)^k\right)-f^{(k-1)}(b)\left((y_i-b)^k-(x_i-b)^k\right)}{b-a}\right)$$
$$-\sum_{i=1}^{m} p_i \left(\frac{\int_a^b f^{(n)}(t)\left((y_i-t)^{n-1}k^{[a,b]}(t,y_i)-(x_i-t)^{n-1}k^{[a,b]}(t,x_i)\right)dt}{(n-1)!\,(b-a)}\right).$$

Now applying Fubini's theorem, we have (2.584). □

The following theorem is the integral version of Theorem 2.201.

Theorem 2.202 ([93]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and let $k^{[a,b]}(t,x)$ be the same as defined in* (2.583). *Let $p:[c,d]\to\mathbb{R}$ and $\varphi,\psi:[c,d]\to[a,b]$ be continuous functions. Then we have*

$$\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz = \sum_{k=1}^{n-1}\left(\frac{n-k}{k!\,(b-a)}\right)$$
$$\cdot\left(f^{(k-1)}(a)\left(\int_c^d p(z)(\psi(z)-a)^k\,dz - \int_c^d p(z)(\varphi(z)-a)^k\,dz\right)\right.$$
$$\left.-f^{(k-1)}(b)\left(\int_c^d p(z)(\psi(z)-b)^k\,dz - \int_c^d p(z)(\varphi(z)-b)^k\,dz\right)\right)$$
$$+\frac{1}{(n-1)!\,(b-a)}\int_a^b f^{(n)}(t)\left(\int_c^d p(z)(\varphi(z)-t)^{n-1}k^{[a,b]}(t,\varphi(z))\,dz\right.$$
$$\left.-\int_c^d p(z)(\psi(z)-t)^{n-1}k^{[a,b]}(t,\psi(z))\,dz\right)dt. \tag{2.585}$$

Proof. By using (1.181) for $x=\varphi(z)$ and $y=\psi(z)$ in the integral majorization difference $\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz$, and after simplification we have (2.585). □

Theorem 2.203 ([93]) *Let all the assumptions of Theorem 2.201 be satisfied and let for $n\geq 1$*

$$\sum_{i=1}^{m} p_i (x_i-t)^{n-1} k^{[a,b]}(t,x_i) \geq \sum_{i=1}^{m} p_i (y_i-t)^{n-1} k^{[a,b]}(t,y_i), \tag{2.586}$$

holds. If f is n-convex, then we have

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \geq$$
$$\sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \left(f^{(k-1)}(a) \left(\sum_{i=1}^{m} p_i (y_i - a)^k - \sum_{i=1}^{m} p_i (x_i - a)^k \right) \right.$$
$$\left. - f^{(k-1)}(b) \left(\sum_{i=1}^{m} p_i (y_i - b)^k - \sum_{i=1}^{m} p_i (x_i - b)^k \right) \right). \tag{2.587}$$

If opposite inequality holds in (2.586)*, then* (2.587) *holds in the reverse direction.*

Proof. Since $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $f^{(n)}$ exists almost everywhere. As f is n-convex, applying Definition 1.19, we have, $f^{(n)}(x) \geq 0$ for all $x \in [a,b]$. Now by using $f^{(n)} \geq 0$ and (2.586) in (2.584), we have (2.587). □

An integral version of the previous theorem states as follows.

Theorem 2.204 ([93]) *Let all the assumptions of Theorem 2.202 be satisfied and let for $n \geq 1$*

$$\int_c^d p(z)(\varphi(z) - t)^{n-1} k^{[a,b]}(t, \varphi(z))\, dz$$
$$\geq \int_c^d p(z)(\psi(z) - t)^{n-1} k^{[a,b]}(t, \psi(z))\, dz, \tag{2.588}$$

holds. If f is n-convex, then we have

$$\int_c^d p(z) f(\varphi(z))\, dz - \int_c^d p(z) f(\psi(z))\, dz \geq \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right)$$
$$\cdot \left(f^{(k-1)}(a) \left(\int_c^d p(z)(\psi(z) - a)^k\, dz - \int_c^d p(z)(\varphi(z) - a)^k \right) dz \right.$$
$$\left. - f^{(k-1)}(b) \left(\int_c^d p(z)(\psi(z) - b)^k\, dz - \int_c^d p(z)(\varphi(z) - b)^k \right) dz \right). \tag{2.589}$$

If opposite inequality holds in (2.588)*, then* (2.589) *holds in the reverse direction.*

Proof. The idea of the proof is the same as that of Theorem 2.203. □

The following corollary presents a refinement of the weighted majorization-type inequality for the two decreasing m-tuples $\boldsymbol{x}$ and $\mathbf{y}$.

Corollary 2.38 ([93]) *Let all the assumptions of Theorem 2.201 be satisfied and let $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be two decreasing real m-tuples such that* (1.19) *and* (1.20) *hold.*

(*i*) *Let n be even and $n > 3$. If the function $f : [a,b] \to \mathbb{R}$ is n-convex, then* (2.587) *holds.*

(ii) *Let the inequality* (2.587) *be satisfied and let* $F : [a,b] \to \mathbb{R}$ *be a function defined by*

$$F(x) = \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \left((x-b)^k f^{(k-1)}(b) - (x-a)^k f^{(k-1)}(a) \right). \quad (2.590)$$

If F *as a convex function, then the right hand side of* (2.587) *is non-negative and we have*

$$\sum_{i=1}^{m} p_i f(x_i) \geq \sum_{i=1}^{m} p_i f(y_i). \quad (2.591)$$

Proof.

(i) For

$$\eta(x) := (x-t)^{n-1} k^{[a,b]}(t,x) = \begin{cases} (x-t)^{n-1}(t-a), & a \leq t \leq x \leq b, \\ (x-t)^{n-1}(t-b), & a \leq x < t \leq b, \end{cases}$$

we have,

$$\eta''(x) := \begin{cases} (n-1)(n-2)(x-t)^{n-3}(t-a), & a \leq t \leq x \leq b, \\ (n-1)(n-2)(x-t)^{n-3}(t-b), & a \leq x < t \leq b, \end{cases}$$

showing that η is convex for even n, where $n > 3$. As $\mathbf{x}$ and $\mathbf{y}$ are decreasing real m-tuples such that (1.19) and (1.20) hold, by using the convex function $\eta(x) := (x-t)^{n-1} k^{[a,b]}(t,x)$ in (1.21), we obtain (2.586) for even n, where $n > 3$. Now as f is n-convex for even n, by applying Theorem 2.203, we have (2.587).

(ii) It is easy to see that (2.587) is equivalent to

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \geq \sum_{i=1}^{m} p_i F(x_i) - \sum_{i=1}^{m} p_i F(y_i).$$

As (1.19) and (1.20) hold, by replacing the convex function F by the convex function ϑ in Theorem 1.14 (1.21), the non-negativity of the right hand side of (2.587) is immediate and we have (2.618).

□

An integral version of Corollary 2.38, provides a refinement of the integral majorization-type inequality for the two decreasing functions φ and ψ as follows:

Corollary 2.39 ([93]) *Let all the assumptions of Theorem 2.202 be satisfied and let* $\varphi, \psi : [c,d] \to [a,b]$ *be two decreasing functions such that* (1.27) *and* (1.28) *hold.*

(i) *Let* n *be even and* $n > 3$. *If the function* $f : [a,b] \to \mathbb{R}$ *is* n*-convex, then* (2.589) *holds.*

(ii) *Let the inequality* (2.589) *be satisfied and let* $\overline{F}$ *be a function defined by*

$$\overline{F}(\varphi(z)) = \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \left((\varphi(z)-b)^k f^{(k-1)}(b) - (\varphi(z)-a)^k f^{(k-1)}(a) \right).$$

If $\overline{F}$ is a convex function, then the right hand side of (2.589) *is non-negative and we have*

$$\int_c^d p(z) f(\varphi(z))\, dz \geq \int_c^d p(z) f(\psi(z))\, dz.$$

Proof. It is easy to see that (2.589) is equivalent to

$$\begin{aligned}&\int_c^d p(z) f(\varphi(z))\, dz - \int_c^d p(z) f(\psi(z))\, dz \\ &\geq \int_c^d p(z) \overline{F}(\varphi(z))\, dz - \int_c^d p(z) \overline{F}(\psi(z))\, dz.\end{aligned}$$

The proof is analogous to the proof of Corollary 2.38 but we apply Theorem 1.18 and Theorem 2.204 instead of Theorem 1.14 and Theorem 2.203. □

For the two m-tuples $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \succ \mathbf{y}$, the following corollary presents a refinement of the majorization-type inequality.

Corollary 2.40 ([93]) *Let all the assumptions of Theorem 2.201 be satisfied and let $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ be two real m-tuples such that $\boldsymbol{x} \succ \boldsymbol{y}$.*

(i) *Let n be even and $n > 3$. If the function $f : [a,b] \to \mathbb{R}$ is n-convex, then we have*

$$\begin{aligned}&\sum_{i=1}^m f(x_i) - \sum_{i=1}^m f(y_i) \\ &\geq \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)}\right) \left(f^{(k-1)}(a) \left(\sum_{i=1}^m (y_i - a)^k - \sum_{i=1}^m (x_i - a)^k\right)\right. \\ &\left. - f^{(k-1)}(b) \left(\sum_{i=1}^m (y_i - b)^k - \sum_{i=1}^m (x_i - b)^k\right)\right). \qquad (2.592)\end{aligned}$$

(ii) *Let the inequality* (2.592) *be satisfied and let $F(x)$ be the same as defined in* (2.590). *If F is a convex function, then the right hand side of* (2.592) *is non-negative and we have the following inequality*

$$\sum_{i=1}^m f(x_i) \geq \sum_{i=1}^m f(y_i). \qquad (2.593)$$

Proof.

(i) As $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ be two real m-tuples such that $\boldsymbol{x} \succ \boldsymbol{y}$ and as $\eta(x)$ is convex for even n, where $n > 3$, by applying Theorem 1.12 (1.18) for the convex function $\eta(x)$, we have

$$\sum_{i=1}^m (x_i - t)^{n-1} k^{[a,b]}(t, x_i) \geq \sum_{i=1}^m (y_i - t)^{n-1} k^{[a,b]}(t, y_i),$$

which is equivalent to (2.586) for each $p_i = 1$ $(i = 1, \ldots, m)$. Now as f is n-convex for even n, where $n > 3$, we apply Theorem 2.203 for each $p_i = 1$ $(i = 1, \ldots, m)$ and (2.592) is immediate.

(ii) It is easy to see that (2.592) is equivalent to

$$\sum_{i=1}^{m} f(x_i) - \sum_{i=1}^{m} f(y_i) \geq \sum_{i=1}^{m} F(x_i) - \sum_{i=1}^{m} F(y_i).$$

As $\boldsymbol{x} \succ \boldsymbol{y}$, by replacing the convex function F by the convex function ϑ in (1.18), the non-negativity of the right hand side of (2.592) is immediate and we have (2.593).

□

Consider the inequalities (2.587) and (2.589) and define linear functionals

$$\begin{aligned}\Phi_1(f) = \sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \\ \cdot \left(f^{(k-1)}(a) \left(\sum_{i=1}^{m} p_i (y_i - a)^k - \sum_{i=1}^{m} p_i (x_i - a)^k \right) \right. \\ \left. - f^{(k-1)}(b) \left(\sum_{i=1}^{m} p_i (y_i - b)^k - \sum_{i=1}^{m} p_i (x_i - b)^k \right) \right),\end{aligned} \tag{2.594}$$

and

$$\begin{aligned}\Phi_2(f) = \int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \\ \cdot \left(f^{(k-1)}(a) \left(\int_c^d p(z) (\psi(z) - a)^k\,dz - \int_c^d p(z) (\varphi(z) - a)^k \right) dz \right. \\ \left. - f^{(k-1)}(b) \left(\int_c^d p(z) (\psi(z) - b)^k\,dz - \int_c^d p(z) (\varphi(z) - b)^k \right) dz \right),\end{aligned} \tag{2.595}$$

where $f : [a,b] \to \mathbb{R}$ is such that for $n \geq 1$, $f^{(n-1)}$ is absolutely continuous, $x_i, y_i \in [a,b]$, $p_i \in \mathbb{R}$ $(i = 1, \ldots, m)$; and $\varphi, \psi : [c,d] \to [a,b]$ and $p : [c,d] \to \mathbb{R}$ are continuous functions. If the function f is n-convex defined on $[a,b]$, then by the assumptions of Theorems 2.203 and 2.204, we have $\Phi_i(f) \geq 0$, $i = 1,2$.

Now, we give mean value theorems for the functionals Φ_i, $i = 1,2$. These theorems enable us to define various classes of means that can be expressed in terms of linear functionals.

First, we state the Lagrange-type mean value theorem related to the functionals Φ_i, $i = 1,2$.

Theorem 2.205 ([93]) *Let $f : [a,b] \to \mathbb{R}$ be such that for $n \geq 1$, $f^{(n-1)}$ is absolutely continuous. Let $x_i, y_i \in [a,b]$, $p_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ and let $\varphi, \psi : [c,d] \to [a,b]$ and $p : [c,d] \to \mathbb{R}$ be continuous functions. Suppose that for $n \geq 1$,* (2.586) *and* (2.588) *hold, where $k^{[a,b]}(t,x)$ be the same as defined in* (2.583)*. If $f \in C^n([a,b])$ and if Φ_1 and Φ_2 are linear functionals as defined in* (2.594) *and* (2.595) *respectively, then there exist $\xi_1, \xi_2 \in [a,b]$ such that*

$$\Phi_i(f) = f^{(n)}(\xi_i)\,\Phi_i(f_0), \qquad i = 1,2,$$

holds, where $f_0(x) = \frac{x^n}{n!}$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (analogous to the proof of Theorem 2.2 in [142]). □

The following theorem is a new analogue of the classical Cauchy mean value theorem, related to the functionals Φ_i $(i=1,2)$ and it can be proven by following the proof of Theorem 2.4 in [142].

Theorem 2.206 ([93]) *Let all the assumptions of Theorem 2.205 be satisfied and let $f,k \in C^n([a,b])$. Then there exist $\xi_i \in [a,b]$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(k)} = \frac{f^{(n)}(\xi_i)}{k^{(n)}(\xi_i)}, \qquad i=1,2, \tag{2.596}$$

holds, provided that the denominators are non-zero.

Remark 2.57 ([93]) *(i) By taking $f(x)=x^s$ and $k(x)=x^q$ in (2.596), where $s,q \in \mathbb{R}\setminus\{0,1,\ldots,n-1\}$ are such that $s \neq q$, we have*

$$\xi_i^{s-q} = \frac{q(q-1)\ldots(q-(n-1))\,\Phi_i(x^s)}{s(s-1)\ldots(s-(n-1))\,\Phi_i(x^q)}, \qquad i=1,2.$$

(ii) If the inverse of the function $f^{(n)}/k^{(n)}$ exists, then (2.596) gives

$$\xi_i = \left(\frac{f^{(n)}}{k^{(n)}}\right)^{-1}\left(\frac{\Phi_i(f)}{\Phi_i(k)}\right), \qquad i=1,2.$$

Now, we present some interesting results by using Čebyšev functional and Grüss-type inequalities.

Let us denote

$$\zeta(t) = \sum_{i=1}^{m} p_i (x_i - t)^{n-1} k^{[a,b]}(t,x_i) - \sum_{i=1}^{m} p_i (y_i - t)^{n-1} k^{[a,b]}(t,y_i), \tag{2.597}$$

and

$$\begin{aligned} \hat{\zeta}(t) &= \int_c^d p(z)(\varphi(z)-t)^{n-1} k^{[a,b]}(t,\varphi(z))\,dz \\ &\quad - \int_c^d p(z)(\psi(z)-t)^{n-1} k^{[a,b]}(t,\psi(z))\,dz, \end{aligned} \tag{2.598}$$

where $x_i, y_i, t \in [a,b]$, $p_i \in \mathbb{R}$ $(i=1,\ldots,m)$, $\varphi, \psi : [c,d] \to [a,b]$ and $p:[c,d]\to\mathbb{R}$ are continuous functions and $k^{[a,b]}(t,.)$ is the same as defined in (2.583).

Theorem 2.207 ([93]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n \geq 1$, $f^{(n)}$ is absolutely continuous with $(\cdot - a)(b-\cdot)\left(f^{(n+1)}\right)^2 \in L[a,b]$. Let $x_i, y_i \in [a,b]$ and $p_i \in \mathbb{R}$ $(i=1,\ldots,m)$. If ζ and T are defined as in (2.597) and (1.6), then*

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) =$$
$$\sum_{k=1}^{n-1}\left(\frac{n-k}{k!\,(b-a)}\right)\left(\begin{array}{c} f^{(k-1)}(a)\left(\sum_{i=1}^{m} p_i (y_i-a)^k - \sum_{i=1}^{m} p_i (x_i-a)^k\right) \\ -f^{(k-1)}(b)\left(\sum_{i=1}^{m} p_i (y_i-b)^k - \sum_{i=1}^{m} p_i (x_i-b)^k\right)\end{array}\right)$$
$$+\frac{1}{(n-1)!\,(b-a)}\left[f^{(n-1)};a,b\right]\int_a^b \zeta(t)\,dt + G_n(f;a,b), \tag{2.599}$$

where

$$\left[f^{(n-1)};a,b\right] = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}, \tag{2.600}$$

is the divided difference and the remainder $G_n(f;a,b)$ satisfies the estimation

$$|G_n(f;a,b)| \le \frac{[T(\zeta(t),\zeta(t))]^{\frac{1}{2}}}{(n-1)!\sqrt{2}} \cdot \frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)\left(f^{(n+1)}(t)\right)^2 dt\right)^{\frac{1}{2}}. \tag{2.601}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

The following theorem is the integral version of Theorem 2.207.

Theorem 2.208 ([93]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\ge 1$, $f^{(n)}$ is absolutely continuous with $(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2 \in L[a,b]$. Let $p:[c,d]\to\mathbb{R}$ and $\varphi,\psi:[c,d]\to[a,b]$ be continuous functions. If $\hat{\zeta}$ and T are defined as in (2.598) and (1.6), then*

$$\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz =$$
$$\sum_{k=1}^{n-1}\left(\frac{n-k}{k!\,(b-a)}\right)\Bigg(f^{(k-1)}(a)\left(\int_c^d p(z)(\psi(z)-a)^k dz - \int_c^d p(z)(\varphi(z)-a)^k dz\right)$$
$$-f^{(k-1)}(b)\left(\int_c^d p(z)(\psi(z)-b)^k dz - \int_c^d p(z)(\varphi(z)-b)^k dz\right)\Bigg)$$
$$+\frac{1}{(n-1)!\,(b-a)}\left[f^{(n-1)};a,b\right]\int_a^b \hat{\zeta}(t)\,dt + \hat{G}_n(f;a,b), \tag{2.602}$$

where $\left[f^{(n-1)};a,b\right]$ is the same as defined in (2.600) and the remainder $\hat{G}_n(f;a,b)$ satisfies the estimation

$$\left|\hat{G}_n(f;a,b)\right| \le \frac{\left[T\left(\hat{\zeta}(t),\hat{\zeta}(t)\right)\right]^{\frac{1}{2}}}{(n-1)!\sqrt{2}} \cdot \frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)\left(f^{(n+1)}(t)\right)^2 dt\right)^{\frac{1}{2}}.$$

Proof. The proof is analogous to the proof of Theorem 2.207. We apply Theorem 1.10 for $f \to \hat{\zeta}$ and $h \to f^{(n)}$ and get the desired results. □

Theorem 2.209 ([93]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n)}$ is absolutely continuous and let $f^{(n+1)}\geq 0$ on $[a,b]$. Let ζ be the same as defined in* (2.597) *respectively. Then we have the representation* (2.599) *and the remainder $G_n(f;a,b)$ satisfies the estimation*

$$|G_n(f;a,b)| \leq \frac{\|\zeta'(t)\|_\infty}{(n-1)!}\left(\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\left[f^{(n-2)};a,b\right]\right). \tag{2.603}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

An integral version of Theorem 2.209 states that:

Theorem 2.210 ([93]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n)}$ is absolutely continuous and let $f^{(n+1)}\geq 0$ on $[a,b]$. Let $\hat{\zeta}$ be the same as defined in* (2.598).

Then we have the representation (2.602) *and the remainder $\hat{G}_n(f;a,b)$ satisfies the estimation*

$$\left|\hat{G}_n(f;a,b)\right| \leq \frac{\|\hat{\zeta}'(t)\|_\infty}{(n-1)!}\left(\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\left[f^{(n-2)};a,b\right]\right).$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.209. We apply Theorem 1.11 for $g \to \hat{\zeta}$ and $h \to f^{(n)}$ and get the desired results. □

An Ostrowski-type inequality related to the generalization of the majorization inequality states that:

Theorem 2.211 ([93]) *Let all the assumptions of Theorem 2.201 be satisfied. Let (p,q) be a pair of conjugate exponents, that is, $p,q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $|f^{(n)}|^p:[a,b]\to\mathbb{R}$ be an $\mathbb{R}$-integrable function for some $n\geq 2$. Then we have*

$$\begin{aligned}\Bigg|\sum_{i=1}^m p_i f(x_i) - \sum_{i=1}^m p_i f(y_i) - \sum_{k=1}^{n-1}\left(\frac{n-k}{k!(b-a)}\right)\\ \cdot\left(f^{(k-1)}(a)\left(\sum_{i=1}^m p_i(y_i-a)^k-\sum_{i=1}^m p_i(x_i-a)^k\right)\right.\\ \left.-f^{(k-1)}(b)\left(\sum_{i=1}^m p_i(y_i-b)^k-\sum_{i=1}^m p_i(x_i-b)^k\right)\right)\Bigg|\\ \leq\left(\int_a^b\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b\left|\bar{\zeta}(t)\right|^q dt\right)^{\frac{1}{q}},\end{aligned} \tag{2.604}$$

where,

$$\bar{\zeta}(t) := \frac{\sum_{i=1}^m p_i(x_i-t)^{n-1}k^{[a,b]}(t,x_i)-\sum_{i=1}^m p_i(y_i-t)^{n-1}k^{[a,b]}(t,y_i)}{(n-1)!(b-a)}.$$

The constant $\left(\int_a^b\left|\bar{\zeta}(t)\right|^q dt\right)^{\frac{1}{q}}$ is sharp for $1<p\leq\infty$ and best possible for $p=1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. $\square$

The following theorem is the integral version of Theorem 2.211.

Theorem 2.212 ([93]) *Let all the assumptions of Theorem 2.202 be satisfied. Let (p,q) be a pair of conjugate exponents, that is, $p,q \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $|f^{(n)}|^p : [a,b] \to \mathbb{R}$ be an $\mathbb{R}$-integrable function for some $n \geq 2$. Then we have*

$$\Bigg| \int_c^d p(z) f(\varphi(z))\, dz - \int_c^d p(z) f(\psi(z))\, dz - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \cdot$$
$$\left(f^{(k-1)}(a) \left(\int_c^d p(z)(\psi(z)-a)^k dz - \int_c^d p(z)(\varphi(z)-a)^k dz \right) \right.$$
$$\left. - f^{(k-1)}(b) \left(\int_c^d p(z)(\psi(z)-b)^k dz - \int_c^d p(z)(\varphi(z)-b)^k dz \right) \right) \Bigg|$$
$$\leq \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^b \left| \ddot{\zeta}(t) \right|^q dt \right)^{\frac{1}{q}},$$

where,

$$\ddot{\zeta}(t) := \frac{\int_c^d p(z)(\varphi(z)-t)^{n-1} k^{[a,b]}(t,\varphi(z))\, dz - \int_c^d p(z)(\psi(z)-t)^{n-1} k^{[a,b]}(t,\psi(z))\, dz}{(n-1)!\,(b-a)}.$$

The constant $\left(\int_a^b \left| \ddot{\zeta}(t) \right|^q dt \right)^{\frac{1}{q}}$ *is sharp for* $1 < p \leq \infty$ *and best possible for* $p = 1$.

Proof. The proof is analogous to the proof of Theorem 2.211 but we use identity (2.585) instead of using (2.584). $\square$

Next, we study the n-exponential convexity and log-convexity of the functions associated with the linear functionals Φ_i $(i=1,2)$ as defined in (2.594) and (2.595). In the remaining results of this section I denotes an interval in $\mathbb{R}$.

Theorem 2.213 ([93]) *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is n-exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0,\ldots,z_n \in [a,b]$. Let Φ_i $(i=1,2)$ be linear functionals as defined in (2.594) and (2.595). Then the following statements hold:*

(i) *The function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i \left(f_{\frac{s_j+s_k}{2}} \right) \right]_{j,k=1}^m$ is positive semi-definite for all $m \in \mathbb{N}$, $m \leq n$ and $s_1,\ldots,s_m \in I$. Particularly,*

$$\det \left[\Phi_i \left(f_{\frac{s_j+s_k}{2}} \right) \right]_{j,k=1}^m \geq 0, \ \forall\, m \in \mathbb{N},\, m \leq n.$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is n-exponentially convex on I.*

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39 but using linear functionals Φ_k $(k = 1,2)$ instead of $\digamma_k (k = 1,2,..,5)$. □

The following corollary is an immediate consequence of the above theorem.

Corollary 2.41 ([93]) *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0,\ldots,z_n \in [a,b]$. Let Φ_i $(i = 1,2)$ be linear functionals as defined in (2.594) and (2.595). Then the following statements hold:*

(i) *The function $s \mapsto \Phi_i(f_s)$ is exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi-definite for all $m \in \mathbb{N}$, $m \leq n$ and $s_1,\ldots,s_m \in I$. Particularly,*

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \geq 0, \textit{for all } m \in \mathbb{N},\ m \leq n.$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is exponentially convex on I.*

Corollary 2.42 ([93]) *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is 2-exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0,\ldots,z_n \in [a,b]$. Let Φ_i $(i = 1,2)$ be linear functionals as defined in (2.594) and (2.595). Further, assume that $\Phi_i(f_s)$ $(i = 1,2)$ is strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(i) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is 2-exponentially convex on I and so it is log-convex on I and for $r,s,t \in I$ such that $r < t < s$, we have*

$$[\Phi_i(f_t)]^{s-r} \leq [\Phi_i(f_r)]^{s-t}[\Phi_i(f_s)]^{t-r}, \qquad i = 1,2. \tag{2.605}$$

If $r < s < t$ or $t < r < s$, then opposite inequalities hold in (2.633).

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is differentiable on I, then for every $s,q,u,v \in I$ such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Phi_i,\Omega) \leq \mu_{u,v}(\Phi_i,\Omega), \qquad i = 1,2, \tag{2.606}$$

where

$$\mu_{s,q}(\Phi_i,\Omega) = \begin{cases} \left(\dfrac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\dfrac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)}\right), & s = q, \end{cases} \tag{2.607}$$

for $f_s, f_q \in \Omega$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10 but using linear functionals Φ_k $(k = 1,2)$ instead of $\digamma_k (k = 1,2,..,5)$. □

Remark 1.19 [93] is also valid for these functionals.

Now, we present several families of functions which fulfil the conditions of Theorem 2.226, Corollaries 2.45 and 2.46, and so the results of these theorem and corollaries can be applied to them.

Example 2.5 ([93]) *Consider the family of functions*

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)\dots(s-(n-1))}, & s \neq 0,1,\dots,n-1, \\ \frac{x^j \ln x}{(-1)^{n-1-j} j!(n-1-j)!}, & s = j = 0,1,\dots,n-1. \end{cases}$$

Here, $\frac{d^n}{dx^n} f_s(x) = x^{s-n} = e^{(s-n)\ln x} > 0$, which shows that f_s is n-convex for $x > 0$ and $s \mapsto \frac{d^n}{dx^n} f_s(x)$ is exponentially convex by definition.
In order to prove that the function $s \mapsto [z_0,\dots,z_n; f_s]$ is exponentially convex, it is enough to show that

$$\Sigma_{j,k=1}^n \varsigma_j \varsigma_k \left[z_0,\dots,z_n; f_{\frac{s_j+s_k}{2}}\right] = \left[z_0,\dots,z_n; \Sigma_{j,k=1}^n \varsigma_j \varsigma_k f_{\frac{s_j+s_k}{2}}\right] \geq 0, \tag{2.608}$$

for all $n \in \mathbb{N}$, $\varsigma_j, s_j \in \mathbb{R}$, $j = 1,\dots,n$. By Definition 1.39, (2.608) will hold if $\Lambda(x) := \Sigma_{j,k=1}^n \varsigma_j \varsigma_k f_{\frac{s_j+s_k}{2}}(x)$ is n-convex. Since $s \mapsto \frac{d^n}{dx^n} f_s(x)$ is exponentially convex, that is

$$\sum_{j,k=1}^n \varsigma_j \varsigma_k f^{(n)}_{\frac{s_j+s_k}{2}} \geq 0, \quad \textit{for all } n \in \mathbb{N}, \varsigma_j, s_j \in \mathbb{R}, j = 1,\dots,n,$$

showing the n-convexity of Λ and so (2.608) holds. Now as the function $s \mapsto [z_0,\dots,z_n; f_s]$ is exponentially convex, $s \mapsto [z_0,\dots,z_n; f_s]$ is exponentially convex in the Jensen sense and by using Corollary 2.45, we have $s \mapsto \Phi_i(f_s)$ $(i = 1,2)$ is exponentially convex in the Jensen sense. Since these mappings are continuous, so $s \mapsto \Phi_i(f_s)$ $(i = 1,2)$ is exponentially convex.
In this case, $\mu_{s,q}(\Phi_i,\Omega)$ $(i = 1,2)$ defined in (2.607) becomes

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{(-1)^{n-1}(n-1)!\Phi_i(f_0 f_s)}{\Phi_i(f_s)} + \Sigma_{k=0}^{n-1} \frac{1}{k-s}\right), & s = q \neq 0,1,\dots,n-1, \\ \exp\left(\frac{(-1)^{n-1}(n-1)!\Phi_i(f_0 f_s)}{2\Phi_i(f_s)} + \Sigma_{\substack{k=0 \\ k\neq s}}^{n-1} \frac{1}{k-s}\right), & s = q = 0,1,\dots,n-1. \end{cases}$$

In particular for $i = 1$, we have

$$\Phi_1(f_s) = \sum_{i=1}^m p_i f_s(x_i) - \sum_{i=1}^m p_i f_s(y_i)$$
$$-\sum_{k=1}^{n-1} \left(\frac{n-k}{k!(b-a)}\right) \left(\begin{array}{l} f_s^{(k-1)}(a) \left(\Sigma_{i=1}^m p_i (y_i - a)^k - \Sigma_{i=1}^m p_i (x_i - a)^k\right) \\ -f_s^{(k-1)}(b) \left(\Sigma_{i=1}^m p_i (y_i - b)^k - \Sigma_{i=1}^m p_i (x_i - b)^k\right) \end{array}\right)$$

$$\textit{and}\quad \Phi_1(f_0 f_s)A = \sum_{i=1}^{m} p_i x_i^s \ln x_i - \sum_{i=1}^{m} p_i y_i^s \ln y_i$$

$$-\sum_{k=1}^{n-1}\left(\frac{n-k}{k!\,(b-a)}\right)\left(\begin{array}{c} B_{k,s}(a)\left(\sum_{i=1}^{m} p_i (y_i-a)^k - \sum_{i=1}^{m} p_i (x_i-a)^k\right) \\ -B_{k,s}(b)\left(\sum_{i=1}^{m} p_i (y_i-b)^k - \sum_{i=1}^{m} p_i (x_i-b)^k\right)\end{array}\right),$$

where $A = (-1)^{n-1}(n-1)!\prod_{i=0}^{n-1}(s-i)$ *such that* $s \neq 0,1,\dots,n-1$ *and* $B_{k,s}(x) = x^{s-(k-1)}\left(\prod_{i=0}^{k-1}(s-i)\ln x + \sum_{i=0}^{k-1}\prod_{\substack{j=0\\ j\neq i}}^{k-1}(s-j)\right)$.
If Φ_i $(i=1,2)$ *is positive, then Theorem* 2.206 *applied for* $f = f_s \in \Omega_2$ *and* $k = f_q \in \Omega_2$ *yields that there exists* $\xi_i \in [a,b]$ *such that*

$$\xi_i^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}, \qquad i = 1,2.$$

Since the function $\xi_i \mapsto \xi_i^{s-q}$ *is invertible for* $s \neq q$*, we have*

$$a \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq b, \qquad i = 1,2,$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega_2)$ *is continuous, symmetric and monotonous* (*by* (2.634)), *shows that* $\mu_{s,q}(\Phi_i,\Omega_2)$ *is a mean.*

Remark 2.58 ([93]) *Similar examples can be discussed as given in Section 1.4. We can also give particular cases for* Φ_i $(i=1,2)$ *as given in example* 2.5.

2.5.2 Results Obtained by Green's Function and Fink's Identity

We start with identities that include Green's function.

Theorem 2.214 ([94]) *Let* $f : [a,b] \to \mathbb{R}$ *be such that for* $n \geq 3$, $f^{(n-1)}$ *is absolutely continuous. Let* $x_i, y_i \in [a,b]$, $p_i \in \mathbb{R}$ $(i = 1,\dots,m)$ *and let* $k^{[a,b]}(t,x)$ *be the same as defined in* (2.583)*. If* G *is the Green function as defined in* (1.180)*, then we have*

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) = \frac{f(b)-f(a)}{b-a}\sum_{i=1}^{m} p_i (x_i - y_i)$$
$$+\sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!\,(b-a)}\right)\int_a^b\left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s)\right)$$
$$\cdot\left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right)ds + \frac{1}{(n-3)!\,(b-a)}$$
$$\cdot\int_a^b f^{(n)}(t)\left(\int_a^b\left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s)\right)(s-t)^{n-3}k^{[a,b]}(t,s)\,ds\right)dt. \tag{2.609}$$

Proof. Using (1.181) in the majorization difference, we have

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) = \frac{f(b)-f(a)}{b-a} \cdot \sum_{i=1}^{m} p_i (x_i - y_i) + \int_a^b \left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s) \right) f''(s)\, ds. \tag{2.610}$$

By taking Fink's identity, it is easy to see that

$$f''(x) = \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!} \right) \left(\frac{f^{(k+1)}(b)(x-b)^k - f^{(k+1)}(a)(x-a)^k}{b-a} \right) + \frac{1}{(n-3)!(b-a)} \int_a^b (x-t)^{n-3} k^{[a,b]}(t,x) f^{(n)}(t)\, dt, \tag{2.611}$$

and by using (2.611) in (2.610), we have

$$\begin{aligned}
&\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) = \frac{f(b)-f(a)}{b-a} \cdot \sum_{i=1}^{m} p_i (x_i - y_i) + \\
&\int_a^b \left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s) \right) \cdot \\
&\sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!} \right) \left(\frac{f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k}{b-a} \right) ds + \\
&\int_a^b \left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s) \right) \cdot \\
&\frac{1}{(n-3)!(b-a)} \left(\int_a^b (s-t)^{n-3} k^{[a,b]}(t,s) f^{(n)}(t)\, dt \right) ds.
\end{aligned}$$

Now by interchanging the integral and summation in the second term and by applying Fubini's theorem in the last term, we have (2.609). □

The following theorem is the integral version of Theorem 2.214.

Theorem 2.215 ([94]) *Let $f : [a,b] \to \mathbb{R}$ be such that for $n \geq 3$, $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and let $k^{[a,b]}(t,x)$ be the same as defined in* (2.583). *Let $p : [c,d] \to \mathbb{R}$ and $\varphi, \psi : [c,d] \to [a,b]$ be continuous functions. If G is the Green function as defined in* (1.180)*, then we have*

$$\begin{aligned}
&\int_c^d p(z)\, f(\varphi(z))\, dz - \int_c^d p(z)\, f(\psi(z))\, dz \\
&= \frac{f(b)-f(a)}{b-a} \cdot \int_c^d p(z)(\varphi(z)-\psi(z))\, dz + \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!\,(b-a)} \right) \cdot \\
&\int_a^b \left(\int_c^d p(z)\, G(\varphi(z), s)\, dz - \int_c^d p(z)\, G(\psi(z), s)\, dz \right) \cdot \\
&\left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds + \frac{1}{(n-3)!\,(b-a)} \cdot \\
&\int_a^b f^{(n)}(t) \left(\int_a^b \left(\begin{array}{c} \int_c^d p(z)\, G(\varphi(z), s)\, dz \\ -\int_c^d p(z)\, G(\psi(z), s)\, dz \end{array} \right) (s-t)^{n-3} k^{[a,b]}(t,s)\, ds \right) dt.
\end{aligned} \tag{2.612}$$

Proof. By using (1.181) for $x = \varphi(z)$ and $y = \psi(z)$ in the integral majorization difference $\int_c^d p(z)\, f(\varphi(z))\, dz - \int_c^d p(z)\, f(\psi(z))\, dz$ and by applying (2.611), the inequality (2.612) is immediate. □

Next we give generalized majorization theorems, in discrete and integral form, obtain by using the previous identities.

Theorem 2.216 ([94]) *Let all the assumptions of Theorem 2.214 be satisfied and let for $n \geq 3$, the inequality*

$$\int_a^b \left(\sum_{i=1}^m p_i G(x_i, s) - \sum_{i=1}^m p_i G(y_i, s) \right) (s-t)^{n-3} k^{[a,b]}(t,s)\, ds \geq 0 \tag{2.613}$$

holds. If f is n-convex, then we have

$$\begin{aligned}
&\sum_{i=1}^m p_i f(x_i) - \sum_{i=1}^m p_i f(y_i) \geq \frac{f(b)-f(a)}{b-a} \cdot \sum_{i=1}^m p_i (x_i - y_i) \\
&+ \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!\,(b-a)} \right) \int_a^b \left(\sum_{i=1}^m p_i G(x_i, s) - \sum_{i=1}^m p_i G(y_i, s) \right) \\
&\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds.
\end{aligned} \tag{2.614}$$

If the opposite inequality holds in (2.613)*, then* (2.614) *holds in the reverse direction.*

Proof. Since $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $f^{(n)}$ exists almost everywhere. As f is n-convex, applying Definition 1.19, we have, $f^{(n)}(x) \geq 0$ for all $x \in [a,b]$. Now by using $f^{(n)} \geq 0$ and (2.613) in (2.609), we have (2.614). □

An integral version of the previous theorem states as follows.

Theorem 2.217 ([94]) *Let all the assumptions of Theorem 2.215 be satisfied and let for $n \geq 3$, the inequality*

$$\int_a^b \left(\int_c^d p(z) G(\varphi(z), s) \, dz - \int_c^d p(z) G(\psi(z), s) \, dz \right) (s-t)^{n-3} k^{[a,b]}(t,s) \, ds \geq 0 \tag{2.615}$$

holds. If f is n-convex, then we have

$$\begin{aligned} &\int_c^d p(z) f(\varphi(z)) \, dz - \int_c^d p(z) f(\psi(z)) \, dz \\ &\geq \frac{f(b)-f(a)}{b-a} \cdot \int_c^d p(z) (\varphi(z) - \psi(z)) \, dz + \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!(b-a)} \right) \\ &\cdot \int_a^b \left(\int_c^d p(z) G(\varphi(z), s) \, dz - \int_c^d p(z) G(\psi(z), s) \, dz \right) \\ &\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds. \end{aligned} \tag{2.616}$$

If the opposite inequality holds in (2.615)*, then* (2.616) *holds in the reverse direction.*

Proof. The idea of the proof is the same as that of the proof of Theorem 2.203. By using $f^{(n)} \geq 0$ and (2.615) in (2.612), we have (2.616). □

The following corollary presents a refinement of the weighted majorization-type inequality for the two decreasing m-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$.

Corollary 2.43 ([94]) *Let all the assumptions of Theorem 2.214 be satisfied and let $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ be two decreasing real m-tuples such that* (1.19) *and* (1.20) *hold.*

(*i*) *Let n be even and $n > 3$. If the function $f : [a,b] \to \mathbb{R}$ is n-convex, then we have*

$$\begin{aligned} &\sum_{i=1}^m p_i f(x_i) - \sum_{i=1}^m p_i f(y_i) \geq \\ &\sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!(b-a)} \right) \int_a^b \left(\sum_{i=1}^m p_i G(x_i, s) - \sum_{i=1}^m p_i G(y_i, s) \right) \\ &\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds. \end{aligned} \tag{2.617}$$

(*ii*) *Let the inequality* (2.617) *be satisfied and and let $F : [a,b] \to \mathbb{R}$ be a function defined by*

$$\begin{aligned} F(x) = &\left(\frac{f(b)-f(a)}{b-a} \right) x + \int_a^b G(x,s) \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!(b-a)} \right) \\ &\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds. \end{aligned}$$

If F is a convex function, then the right hand side of (2.617) *is non-negative.*

$$\sum_{i=1}^{m} p_i f(x_i) \geq \sum_{i=1}^{m} p_i f(y_i). \tag{2.618}$$

Proof.

(i) As $\boldsymbol{x}$ and $\boldsymbol{y}$ are decreasing real m-tuples such that (1.19) and (1.20) hold, by using the convex function $G(x,s)$ in (1.21), we obtain

$$\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s) \geq 0.$$

For $a \leq s \leq t$, it is easy to see that

$$\int_a^t \left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s) \right) (s-t)^{n-3} k^{[a,b]}(t,s)\, ds \geq 0 \tag{2.619}$$

holds for even n, where $n > 3$ and

$$\int_a^t \left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s) \right) (s-t)^{n-3} k^{[a,b]}(t,s)\, ds \leq 0$$

holds for odd n, where $n > 3$. Now, for $t \leq s \leq b$ and for $n > 3$, the inequality

$$\int_t^b \left(\sum_{i=1}^{m} p_i G(x_i,s) - \sum_{i=1}^{m} p_i G(y_i,s) \right) (s-t)^{n-3} k^{[a,b]}(t,s)\, ds \geq 0 \tag{2.620}$$

holds. From (2.619) and (2.620), we obtain (2.613) for even n, where $n > 3$. Now as f is n-convex for even n, where $n > 3$, by applying Theorem 2.203 combine together with (1.20), we have (2.617).

(ii) By using (1.20), it is easy to see that (2.617) is equivalent to

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \geq \sum_{i=1}^{m} p_i F(x_i) - \sum_{i=1}^{m} p_i F(y_i).$$

As (1.19) and (1.20) hold, by replacing the convex function F by the convex function ϑ in Theorem 1.14 (1.21), the non-negativity of the right hand side of (2.617) is immediate and we have (2.618).

□

An integral version of Corollary 2.43, provides a refinement of the integral majorization-type inequality for the two decreasing functions φ and ψ as follows:

Corollary 2.44 ([94]) *Let all the assumptions of Theorem 2.215 be satisfied and let $\varphi, \psi : [c,d] \to [a,b]$ be two decreasing functions such that* (1.27) *and* (1.28) *hold.*

(i) *Let n be even and $n > 3$. If the function $f : [a,b] \to \mathbb{R}$ is n-convex, then we have*

$$\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz \geq \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!\,(b-a)} \right)$$
$$\cdot \int_a^b \left(\int_c^d p(z) G(\varphi(z), s)\,dz - \int_c^d p(z) G(\psi(z), s)\,dz \right)$$
$$\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds. \tag{2.621}$$

(ii) *Let the inequality* (2.621) *be satisfied and let $\bar{F}$ be a function defined by*

$$\bar{F}(\varphi(z)) = \left(\frac{f(b)-f(a)}{b-a} \right) \varphi(z) + \int_a^b G(\varphi(z), s) \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!\,(b-a)} \right)$$
$$\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds.$$

If $\bar{F}$ is a convex function, then the right hand side of (2.621) *is non-negative and we have*

$$\int_c^d p(z) f(\varphi(z))\,dz \geq \int_c^d p(z) f(\psi(z))\,dz.$$

Proof. By using (1.28), it is easy to see that (2.621) is equivalent to

$$\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz$$
$$\geq \int_c^d p(z) \bar{F}(\varphi(z))\,dz - \int_c^d p(z) \bar{F}(\psi(z))\,dz.$$

The proof is analogous to the proof of Corollary 2.43 but we apply Theorem 1.18 and Theorem 2.204 instead of Theorem 1.14 and Theorem 2.203. □

Consider the inequalities (2.614) and (2.616) and define linear functionals

$$\Phi_1(f) = \sum_{i=1}^m p_i f(x_i) - \sum_{i=1}^m p_i f(y_i) - \frac{f(b)-f(a)}{b-a} \sum_{i=1}^m p_i (x_i - y_i)$$
$$- \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!\,(b-a)} \right) \int_a^b \left(\sum_{i=1}^m p_i G(x_i, s) - \sum_{i=1}^m p_i G(y_i, s) \right)$$
$$\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds, \tag{2.622}$$

and

$$\Phi_2(f) = \int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz$$
$$-\frac{f(b)-f(a)}{b-a}\int_c^d p(z)(\varphi(z)-\psi(z))\,dz - \sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!(b-a)}\right)$$
$$\cdot \int_a^b \left(\int_c^d p(z) G(\varphi(z),s)\,dz - \int_c^d p(z) G(\psi(z),s)\,dz\right)$$
$$\cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds, \tag{2.623}$$

where $f:[a,b]\to\mathbb{R}$ is such that for $n\geq 1$, $f^{(n-1)}$ is absolutely continuous, $x_i, y_i \in [a,b]$, $p_i \in \mathbb{R}$ $(i=1,\ldots,m)$; $\varphi,\psi:[c,d]\to[a,b]$ and $p:[c,d]\to\mathbb{R}$ are continuous functions and G is the Green function as defined in (1.180). If the function f is n-convex defined on $[a,b]$, then by the assumptions of Theorems 2.203 and 2.204, we have $\Phi_i(f)\geq 0$, where $i=1,2$.

Now, we give mean value theorems for the functionals Φ_i, where $i=1,2$. These theorems enable us to define various classes of means that can be expressed in terms of linear functionals.
First, we state the Lagrange-type mean value theorem related to the functionals Φ_i, where $i=1,2$.

Theorem 2.218 ([94]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n-1)}$ is absolutely continuous. Let $x_i, y_i\in[a,b]$, $p_i\in\mathbb{R}$ $(i=1,\ldots,m)$; $\varphi,\psi:[c,d]\to[a,b]$ and $p:[c,d]\to\mathbb{R}$ be continuous functions and let G be the Green function as defined in* (1.180). *Suppose that for $n\geq 1$,* (2.613) *and* (2.615) *hold, where $k^{[a,b]}(t,x)$ is the same as defined in* (2.583). *If $f\in C^n([a,b])$ and if Φ_1 and Φ_2 are linear functionals as defined in* (2.622) *and* (2.623) *respectively, then there exist $\xi_1,\xi_2\in[a,b]$ such that*

$$\Phi_i(f) = f^{(n)}(\xi_i)\,\Phi_i(f_0), \qquad i=1,2,$$

holds, where $f_0(x)=\frac{x^n}{n!}$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (analogous to the proof of Theorem 2.2 in [142]). □

The following theorem is a new analogue of the classical Cauchy mean value theorem, related to the functionals Φ_i $(i=1,2)$ and it can be proven by following the proof of Theorem 2.4 in [142].

Theorem 2.219 ([94]) *Let all the assumptions of Theorem* 2.218 *be satisfied and let $f,k\in C^n([a,b])$. Then there exist $\xi_i\in[a,b]$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(k)} = \frac{f^{(n)}(\xi_i)}{k^{(n)}(\xi_i)}, \qquad i=1,2, \tag{2.624}$$

holds, provided that the denominators are non-zero.

Remark 2.59 ([94]) *(i) By taking $f(x)=x^s$ and $k(x)=x^q$ in (2.624), where $s,q\in\mathbb{R}\setminus\{0,1,\ldots,n-1\}$ are such that $s\neq q$, we have*

$$\xi_i^{s-q}=\frac{q(q-1)\ldots(q-(n-1))\,\Phi_i(x^s)}{s(s-1)\ldots(s-(n-1))\,\Phi_i(x^q)},\qquad i=1,2.$$

(ii) If the inverse of the function $f^{(n)}/k^{(n)}$ exists, then (2.624) gives

$$\xi_i=\left(\frac{f^{(n)}}{k^{(n)}}\right)^{-1}\left(\frac{\Phi_i(f)}{\Phi_i(k)}\right),\qquad i=1,2.$$

In this subsection (see [94]) we present some interesting results by using Čebyšev functional and the Grüss type inequalities.

Let us denote

$$\zeta(t)=\int_a^b\left(\sum_{i=1}^m p_iG(x_i,s)-\sum_{i=1}^m p_iG(y_i,s)\right)(s-t)^{n-3}k^{[a,b]}(t,s)\,ds, \tag{2.625}$$

and

$$\hat{\zeta}(t)=\int_a^b\left(\int_c^d p(z)\,G(\varphi(z),s)\,dz-\int_c^d p(z)\,G(\psi(z),s)\,dz\right)(s-t)^{n-3}k^{[a,b]}(t,s)\,ds, \tag{2.626}$$

where $x_i,y_i,s,t\in[a,b]$, $p_i\in\mathbb{R}$ $(i=1,\ldots,m)$; $\varphi,\psi:[c,d]\to[a,b]$ and $p:[c,d]\to\mathbb{R}$ are continuous functions, G is the Green function as defined in (1.180) and $k^{[a,b]}(t,.)$ is the same as defined in (2.583).

Theorem 2.220 ([94]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n)}$ is absolutely continuous with $(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2\in L[a,b]$. Let $x_i,y_i\in[a,b]$ and $p_i\in\mathbb{R}$ $(i=1,\ldots,m)$. If G, T and ζ are the same as defined in (1.180), (1.6) and (2.625) respectively, then we have*

$$\begin{aligned}
&\sum_{i=1}^m p_if(x_i)-\sum_{i=1}^m p_if(y_i)=\frac{f(b)-f(a)}{b-a}\cdot\sum_{i=1}^m p_i(x_i-y_i)+\\
&\sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!\,(b-a)}\right)\int_a^b\left(\sum_{i=1}^m p_iG(x_i,s)-\sum_{i=1}^m p_iG(y_i,s)\right)\cdot\\
&\left(f^{(k+1)}(b)(s-b)^k-f^{(k+1)}(a)(s-a)^k\right)ds+\\
&\frac{1}{(n-3)!\,(b-a)}\left[f^{(n-1)};a,b\right]\int_a^b\zeta(t)\,dt+G_n(f;a,b),
\end{aligned} \tag{2.627}$$

where

$$\left[f^{(n-1)};a,b\right]=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}, \tag{2.628}$$

is the divided difference and the remainder $G_n(f;a,b)$ satisfies the estimation

$$|G_n(f;a,b)| \leq \frac{[T(\zeta(t),\zeta(t))]^{\frac{1}{2}}}{(n-3)!\sqrt{2}} \cdot \frac{1}{\sqrt{b-a}} \left(\int_a^b (t-a)(b-t)\left(f^{(n+1)}(t)\right)^2 dt\right)^{\frac{1}{2}}. \tag{2.629}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

The following theorem is the integral version of Theorem 2.220.

Theorem 2.221 ([94]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n)}$ is absolutely continuous with $(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2 \in L[a,b]$. Let $p:[c,d]\to\mathbb{R}$ and $\varphi,\psi:[c,d]\to[a,b]$ be continuous functions. If G, T and $\hat{\zeta}$ are the same as defined in* (1.180), (1.6) *and* (2.626) *respectively, then we have*

$$\begin{aligned}
&\int_c^d p(z)f(\varphi(z))\,dz - \int_c^d p(z)f(\psi(z))\,dz \\
&= \frac{f(b)-f(a)}{b-a}\cdot\int_c^d p(z)(\varphi(z)-\psi(z))\,dz + \sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!(b-a)}\right)\cdot \\
&\int_a^b\left(\int_c^d p(z)G(\varphi(z),s)\,dz - \int_c^d p(z)G(\psi(z),s)\,dz\right)\cdot \\
&\left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right)ds + \\
&\frac{1}{(n-3)!(b-a)}\left[f^{(n-1)};a,b\right]\int_a^b \hat{\zeta}(t)\,dt + \hat{G}_n(f;a,b),
\end{aligned} \tag{2.630}$$

where $\left[f^{(n-1)};a,b\right]$ is the same as defined in (2.628) *and the remainder $\hat{G}_n(f;a,b)$ satisfies the estimation*

$$\left|\hat{G}_n(f;a,b)\right| \leq \frac{\left[T\left(\hat{\zeta}(t),\hat{\zeta}(t)\right)\right]^{\frac{1}{2}}}{(n-3)!\sqrt{2}} \cdot \frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)\left(f^{(n+1)}(t)\right)^2 dt\right)^{\frac{1}{2}}.$$

Proof. The proof is analogous to the proof of Theorem 2.220. We apply Theorem 1.10 for $f\to\hat{\zeta}$ and $h\to f^{(n)}$ and get the desired results. □

Theorem 2.222 ([94]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n)}$ is absolutely continuous and let $f^{(n+1)}\geq 0$ on $[a,b]$. Let G and ζ be the same as defined in* (1.180) *and* (2.625) *respectively.*

Then we have the representation (2.627) *and the remainder $G_n(f;a,b)$ satisfies the estimation*

$$|G_n(f;a,b)| \leq \frac{\|\zeta'(t)\|_\infty}{(n-3)!}\left(\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2} - \left[f^{(n-2)};a,b\right]\right). \tag{2.631}$$

Proof. The idea of the proof is same as in Theorem 2.9. □

An integral version of Theorem 2.222 states that:

Theorem 2.223 ([94]) *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geq 1$, $f^{(n)}$ is absolutely continuous and let $f^{(n+1)}\geq 0$ on $[a,b]$. Let G and $\hat{\zeta}$ be the same as defined in* (1.180) *and* (2.626) *respectively. Then we have the representation* (2.630) *and the remainder $\hat{G}_n(f;a,b)$ satisfies the estimation*

$$\left|\hat{G}_n(f;a,b)\right| \leq \frac{\|\hat{\zeta}'(t)\|_\infty}{(n-3)!}\left(\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\left[f^{(n-2)};a,b\right]\right).$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.222. We apply Theorem 1.11 for $g\to\hat{\zeta}$ and $h\to f^{(n)}$ and get the desired results. □

An Ostrowski-type inequality related to the generalization of the majorization inequality states that:

Theorem 2.224 ([94]) *Let all the assumptions of Theorem 2.214 be satisfied. Let (p,q) be a pair of conjugate exponents, that is, $p,q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $|f^{(n)}|^p:[a,b]\to\mathbb{R}$ be an $\mathbb{R}$-integrable function for some $n\geq 2$. Then we have*

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{m}p_i f(x_i)-\sum_{i=1}^{m}p_i f(y_i)-\frac{f(b)-f(a)}{b-a}\cdot\sum_{i=1}^{m}p_i(x_i-y_i)\\
&\quad-\sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!(b-a)}\right)\int_a^b\left(\sum_{i=1}^{m}p_iG(x_i,s)-\sum_{i=1}^{m}p_iG(y_i,s)\right)\\
&\quad\cdot\left(f^{(k+1)}(b)(s-b)^k-f^{(k+1)}(a)(s-a)^k\right)ds\Bigg|\\
&\leq\left(\int_a^b\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|\tilde{\zeta}(t)|^q dt\right)^{\frac{1}{q}},
\end{aligned}\tag{2.632}$$

where,

$$\begin{aligned}
\tilde{\zeta}(t):=&\frac{1}{(n-3)!(b-a)}\int_a^b\left(\sum_{i=1}^{m}p_iG(x_i,s)-\sum_{i=1}^{m}p_iG(y_i,s)\right)\\
&\cdot(s-t)^{n-3}k^{[a,b]}(t,s)\,ds.
\end{aligned}$$

The constant $\left(\int_a^b\left|\tilde{\zeta}(t)\right|^q dt\right)^{\frac{1}{q}}$ is sharp for $1<p\leq\infty$ and best possible for $p=1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

The following theorem is the integral version of Theorem 2.224.

Theorem 2.225 ([94]) *Let all the assumptions of Theorem 2.215 be satisfied. Let (p,q) be a pair of conjugate exponents, that is, $p,q \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $|f^{(n)}|^p : [a,b] \to \mathbb{R}$ be an $\mathbb{R}$-integrable function for some $n \geq 2$. Then we have*

$$\left| \begin{array}{c} \int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz - \frac{f(b)-f(a)}{b-a} \\ \cdot \int_c^d p(z)(\varphi(z)-\psi(z))\,dz - \sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!(b-a)}\right) \\ \cdot \int_a^b \left(\int_c^d p(z) G(\varphi(z),s)\,dz - \int_c^d p(z) G(\psi(z),s)\,dz\right) \\ \cdot \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds \end{array} \right|$$
$$\leq \left(\int_a^b \left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\int_a^b \left|\ddot{\zeta}(t)\right|^q dt\right)^{\frac{1}{q}},$$

where,

$$\ddot{\zeta}(t) := \frac{1}{(n-3)!(b-a)} \int_a^b \left(\begin{array}{c} \int_c^d p(z) G(\varphi(z),s)\,dz \\ -\int_c^d p(z) G(\psi(z),s)\,dz \end{array}\right) \cdot (s-t)^{n-3} k^{[a,b]}(t,s)\,ds.$$

The constant $\left(\int_a^b \left|\ddot{\zeta}(t)\right|^q dt\right)^{\frac{1}{q}}$ is sharp for $1 < p \leq \infty$ and best possible for $p=1$.

Proof. The proof is analogous to the proof of Theorem 2.224 but we use identity (2.612) instead of using (2.609). □

Next, we study the n-exponential convexity and log-convexity of the functions associated with the linear functionals Φ_i $(i=1,2)$ as defined in (2.622) and (2.623). In the remaining results of this subsection I denotes an interval in $\mathbb{R}$.

Theorem 2.226 ([94]) *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is n-exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0,\ldots,z_n \in [a,b]$. Let Φ_i $(i=1,2)$ be linear functionals as defined in (2.622) and (2.623). Then the following statements hold:*

(i) The function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^m$ is positive semi-definite for all $m \in \mathbb{N}$, $m \leq n$ and $s_1,\ldots,s_m \in I$. Particularly,

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^m \geq 0, \quad \text{for all } m \in \mathbb{N},\ m \leq n.$$

(ii) If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is n-exponentially convex on I.

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39 but using linear functionals Φ_k $(k=1,2)$ instead of $\digamma_k (k=1,2,..,5)$. □

The following corollary is an immediate consequence of Theorem 2.226.

Corollary 2.45 ([94]) *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0, \ldots, z_n; f_s]$ is exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0, \ldots, z_n \in [a,b]$. Let Φ_i $(i = 1,2)$ be linear functionals as defined in (2.622) and (2.623). Then the following statements hold:*

(i) *The function $s \mapsto \Phi_i(f_s)$ is exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi-definite for all $m \in \mathbb{N}$, $m \le n$ and $s_1, \ldots, s_m \in I$. Particularly,*

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \ge 0, \quad \text{for all } m \in \mathbb{N},\ m \le n.$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is exponentially convex on I.*

Corollary 2.46 ([94]) *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0, \ldots, z_n; f_s]$ is 2-exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0, \ldots, z_n \in [a,b]$. Let Φ_i $(i = 1,2)$ be linear functionals as defined in (2.622) and (2.623). Further, assume that $\Phi_i(f_s)$ $(i = 1,2)$ is strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(i) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is 2-exponentially convex on I and so it is log-convex on I and for $r,s,t \in I$ such that $r < t < s$, we have*

$$[\Phi_i(f_t)]^{s-r} \le [\Phi_i(f_r)]^{s-t}\,[\Phi_i(f_s)]^{t-r}, \qquad i = 1,2. \tag{2.633}$$

If $r < s < t$ or $t < r < s$, then opposite inequalities hold in (2.633).

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is differentiable on I, then for every $s,q,u,v \in I$ such that $s \le u$ and $q \le v$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \le \mu_{u,v}(\Phi_i, \Omega), \qquad i = 1,2, \tag{2.634}$$

where

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\dfrac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \ne q, \\ \exp\left(\dfrac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)}\right), & s = q, \end{cases} \tag{2.635}$$

for $f_s, f_q \in \Omega$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10 but using linear functionals Φ_k $(k = 1,2)$ instead of $\digamma_k (k = 1,2,\ldots,5)$. □

Remark 1.19 is also valid for these functionals.

There are several families of functions which fulfil the conditions of Theorem 2.226, Corollaries 2.45 and 2.46, and so the results of these theorem and corollaries can be applied for them. Here we present an example for such a family of functions but for more examples see [93].

Example 2.6 *Consider the family of functions*

$$\tilde{\Omega} = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)\dots(s-(n-1))}, & s \neq 0,1,\dots,n-1, \\ \frac{x^j \ln x}{(-1)^{n-1-j} j!(n-1-j)!}, & s = j = 0,1,\dots,n-1. \end{cases}$$

Here, $\frac{d^n}{dx^n} f_s(x) = x^{s-n} = e^{(s-n)\ln x} > 0$, *which shows that* f_s *is n-convex for* $x > 0$ *and* $s \mapsto \frac{d^n}{dx^n} f_s(x)$ *is exponentially convex by definition. It is easy to prove that the function* $s \mapsto [z_0,\dots,z_n; f_s]$ *is exponentially convex and by the same arguing as given in [93, Example 5.1], we have* $s \mapsto \Phi_i(f_s)$ $(i = 1,2)$ *is exponentially convex.*

In this case, $\mu_{s,q}(\Phi_i,\Omega)$ $(i = 1,2)$ *defined in* (2.635) *becomes*

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{(-1)^{n-1}(n-1)!\Phi_i(f_0 f_s)}{\Phi_i(f_s)} + \sum_{k=0}^{n-1}\frac{1}{k-s}\right), & s = q \neq 0,1,\dots,n-1, \\ \exp\left(\frac{(-1)^{n-1}(n-1)!\Phi_i(f_0 f_s)}{2\Phi_i(f_s)} + \sum_{\substack{k=0 \\ k\neq s}}^{n-1}\frac{1}{k-s}\right), & s = q = 0,1,\dots,n-1. \end{cases}$$

In particular for $i = 1$, *we have*

$$\begin{aligned} \Phi_1(f_s) &= \sum_{i=1}^m p_i f_s(x_i) - \sum_{i=1}^m p_i f_s(y_i) - \frac{f_s(b) - f_s(a)}{b-a} \cdot \sum_{i=1}^m p_i (x_i - y_i) \\ &- \sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!(b-a)}\right)\int_a^b \left(\sum_{i=1}^m p_i G(x_i,s) - \sum_{i=1}^m p_i G(y_i,s)\right) \\ &\cdot \left(f_s^{(k+1)}(b)(s-b)^k - f_s^{(k+1)}(a)(s-a)^k\right) ds, \end{aligned}$$

and

$$\begin{aligned} \Phi_2(f_0 f_s)A &= \sum_{i=1}^m p_i x_i^s \ln x_i - \sum_{i=1}^m p_i y_i^s \ln y_i - \frac{b^s \ln b - a^s \ln a}{b-a} \cdot \sum_{i=1}^m p_i (x_i - y_i) \\ &- \sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!(b-a)}\right)\int_a^b \left(\sum_{i=1}^m p_i G(x_i,s) - \sum_{i=1}^m p_i G(y_i,s)\right) \\ &\cdot \left(B_{k,s}(b)(s-b)^k - B_{k,s}(a)(s-a)^k\right) ds, \end{aligned}$$

where $A = (-1)^{n-1}(n-1)!\prod_{i=0}^{n-1}(s-i)$ *such that* $s \neq 0,1,\dots,n-1$ *and* $B_{k,s}(x) = x^{s-(k+1)}\left(\prod_{i=0}^{k-1}(s-i)\ln x + \sum_{i=0}^{k-1}\prod_{\substack{j=0 \\ j\neq i}}^{k-1}(s-j)\right)$.

If Φ_i $(i = 1,2)$ *is positive, then Theorem 2.206 applied for* $f = f_s \in \Omega_2$ *and* $k = f_q \in \Omega_2$ *yields that there exist* $\xi_i \in [a,b]$ *such that*

$$\xi_i^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}, \qquad i = 1,2.$$

Since the function $\xi_i \mapsto \xi_i^{s-q}$ *is invertible for* $s \neq q$, *we have*

$$a \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq b, \qquad i = 1,2,$$

which together with the fact that $\mu_{s,q}(\Phi_i, \Omega_2)$ *is continuous, symmetric and monotonous (by* (2.634)*), shows that* $\mu_{s,q}(\Phi_i, \Omega_2)$ *is a mean.*

Remark 2.60 ([94]) *Similar examples can be discussed as given in Section 1.4. We can also give particular cases for* Φ_i $(i = 1,2)$ *as given in example* 2.6.

2.6 Majorization and the Abel-Gontscharoff Interpolating Polynomial

The Abel-Gontscharoff interpolation problem in the real case was introduced in 1935 by Whittaker [170] and subsequently by Gontscharoff [77] and Davis [67]. The Abel-Gontscharoff interpolating polynomial for two points with integral remainder is given in [16] in the form of the following theorem.

Theorem 2.227 *Let* $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$ *and* $\phi \in C^n[a,b]$. *Then we have*

$$\phi(t) = Q_{n-1}(a,b,\phi,t) + R(\phi,t),$$

where Q_{n-1} *is* ***the Abel-Gontscharoff interpolating polynomial*** *for two-points of degree* $n-1$, *i.e.,*

$$Q_{n-1}(a,b,\phi,t) = \sum_{i=0}^{k} \frac{(t-a)^i}{i!}\phi^{(i)}(a)$$
$$+ \sum_{j=0}^{n-k-2}\left[\sum_{i=0}^{j} \frac{(t-a)^{k+1+i}(a-b)^{j-i}}{(k+1+i)!\,(j-i)!}\right]\phi^{(k+1+j)}(b)$$

and the remainder is given by

$$R(\phi,t) = \int_a^b G_n(t,s)\phi^{(n)}(s)ds,$$

where $G_n(t,s)$ *be Green's function [24, p.177]*

$$G_n(t,s) = \frac{1}{(n-1)!}\begin{cases} \sum\limits_{i=0}^{k}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & a \le s \le t; \\ -\sum\limits_{i=k+1}^{n-1}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & t \le s \le b. \end{cases} \tag{2.636}$$

Further, for $a \le s,t \le b$ the following inequalities hold

$$(-1)^{n-k-1}\frac{\partial^i G_n(t,s)}{\partial t^i} \ge 0, \quad 0 \le i \le k, \tag{2.637}$$

$$(-1)^{n-i}\frac{\partial^i G_n(t,s)}{\partial t^i} \ge 0, \quad k+1 \le i \le n-1. \tag{2.638}$$

In this section using interpolation by Abel-Gontscharoff polynomials we give some new identities for the difference of majorization inequalities and present new generalizations of majorization theorems for the class of n-convex functions. We give bounds for identities related to obtained generalized majorization inequalities by using Čebyšev functionals. We also give the Grüss and Ostrowski type inequalities for these functionals. We present the Lagrange and Cauchy type mean value theorems related to the functionals which are the differences of the generalizations of majorization inequality and also give n-exponential convexity which leads to exponential convexity and then log-convexity for these defined functionals. At the end of each subsections, we discuss some families of functions which enable us to construct a large families of functions that are exponentially convex and also give Stolarsky type means with their monotonicity.

2.6.1 Results Obtained by the Abel-Gontscharoff Interpolating Polynomial

We start this subsection with the identities of generalizations of majorization inequality using the Abel-Gontscharoff interpolating polynomial.

Theorem 2.228 ([11]) *Let $n,k \in \mathbb{N}$, $n \ge 2$, $0 \le k \le n-1$, $\mathbf{x} = (x_1,\dots,x_m)$, $\mathbf{y} = (y_1,\dots,y_m)$ and $\mathbf{w} = (w_1,\dots,w_m)$ be m-tuples such that x_r, $y_r \in [a,b]$ and $w_r \in \mathbb{R}$ $(r = 1,\dots,m)$. Let also $\phi \in C^n[a,b]$ and G_n be the Green function defined as in (2.636), then*

$$\begin{aligned} &\sum_{r=1}^{m} w_r\,\phi(x_r) - \sum_{r=1}^{m} w_r\,\phi(y_r) \\ &= \sum_{i=0}^{k}\frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^{m} w_r(x_r-a)^i - \sum_{r=1}^{m} w_r(y_r-a)^i\right] \\ &+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j}\frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\left[\sum_{r=1}^{m} w_r(x_r-a)^{k+1+i} - \sum_{r=1}^{m} w_r(y_r-a)^{k+1+i}\right] \\ &+ \int_a^b\left(\sum_{r=1}^{m} w_r G_n(x_r,s) - \sum_{r=1}^{m} w_r G_n(y_r,s)\right)\phi^{(n)}(s)ds. \end{aligned} \tag{2.639}$$

Proof. Consider the majorization difference

$$\sum_{r=1}^{m} w_r \phi(x_r) - \sum_{r=1}^{m} w_r \phi(y_r). \tag{2.640}$$

By using Theorem 2.227 we have

$$\begin{aligned}\phi(t) &= \sum_{i=0}^{k} \frac{(t-a)^i}{i!}\phi^{(i)}(a)\\ &+ \sum_{j=0}^{n-k-2}\left[\sum_{i=0}^{j} \frac{(t-a)^{k+1+i}(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\right]\phi^{(k+1+j)}(b)\\ &+ \int_a^b G_n(t,s)\,\phi^{(n)}(s)ds. \end{aligned} \tag{2.641}$$

Substituting this value of ϕ in (2.640) and some arrangements, we get (2.639). □

Integral version of the above theorem can be stated as follows.

Theorem 2.229 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and $x,y : [\alpha,\beta] \to [a,b]$, $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions. Let also $\phi \in C^n[a,b]$ and G_n be the Green function defined as in (2.636), then*

$$\begin{aligned} &\int_\alpha^\beta w(t)\,\phi(x(t))\,dt - \int_\alpha^\beta w(t)\,\phi(y(t))\,dt\\ &= \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\int_\alpha^\beta w(t)\,(x(t)-a)^i\,dt - \int_\alpha^\beta w(t)\,(y(t)-a)^i\,dt\right]\\ &+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\\ &\left[\int_\alpha^\beta w(t)\,(x(t)-a)^{k+1+i}\,dt - \int_\alpha^\beta w(t)\,(y(t)-a)^{k+1+i}\,dt\right]\\ &+ \int_a^b \phi^{(n)}(s)\left(\int_\alpha^\beta w(t)G_n(x(t),s)\,dt - \int_\alpha^\beta w(t)G_n(y(t),s)\,dt\right)ds. \end{aligned} \tag{2.642}$$

We give generalizations of majorization inequality for n-convex functions.

Theorem 2.230 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ and $\mathbf{w} = (w_1,\ldots,w_m)$ be m-tuples such that x_r, $y_r \in [a,b]$ and $w_r \in \mathbb{R}$ $(r = 1,\ldots,m)$ and also G_n be the Green function defined as in (2.636).*
If for all $s \in [a,b]$

$$\sum_{r=1}^{m} w_r G_n(y_r,s) \leq \sum_{r=1}^{m} w_r G_n(x_r,s), \tag{2.643}$$

then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\sum_{r=1}^{m} w_r \phi(x_r) - \sum_{r=1}^{m} w_r \phi(y_r) \\
&\geq \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\sum_{r=1}^{m} w_r (x_r - a)^i - \sum_{r=1}^{m} w_r (y_r - a)^i \right] \\
&+ \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \left[\sum_{r=1}^{m} w_r (x_r - a)^{k+1+i} - \sum_{r=1}^{m} w_r (y_r - a)^{k+1+i} \right].
\end{aligned} \tag{2.644}$$

If the reverse inequality in (2.643) holds, then also the reverse inequality in (2.644) holds.

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)}(x) \geq 0$, for all $x \in [a,b]$. Hence we can apply Theorem 2.228 to get (2.644). □

Integral version of the above theorem can be stated as follows.

Theorem 2.231 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and $x,y : [\alpha,\beta] \to [a,b]$, $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and also G_n be the Green function defined as in (2.636).*
If for all $s \in [a,b]$

$$\int_{\alpha}^{\beta} w(t) G_n(y(t),s)\, dt \leq \int_{\alpha}^{\beta} w(t) G_n(x(t),s)\, dt, \tag{2.645}$$

then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\int_{\alpha}^{\beta} w(t) \phi(x(t))\, dt - \int_{\alpha}^{\beta} w(t) \phi(y(t))\, dt \\
&\geq \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\int_{\alpha}^{\beta} w(t) (x(t) - a)^i\, dt - \int_{\alpha}^{\beta} w(t) (y(t) - a)^i\, dt \right] \\
&+ \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\left[\int_{\alpha}^{\beta} w(t) (x(t) - a)^{k+1+i}\, dt - \int_{\alpha}^{\beta} w(t) (y(t) - a)^{k+1+i}\, dt \right].
\end{aligned} \tag{2.646}$$

If the reverse inequality in (2.645) holds, then also the reverse inequality in (2.646) holds.

The following theorem is the generalization of classical majorization theorem:

Theorem 2.232 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\boldsymbol{x} = (x_1, \ldots, x_m)$, $\boldsymbol{y} = (y_1, \ldots, y_m)$ be m-tuples such that x_r, $y_r \in [a,b]$ and $\boldsymbol{x} \succ \boldsymbol{y}$ and also G_n be the Green function defined as in (2.636).*

(i) If k is odd and n is even or k is even and n is odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\sum_{r=1}^{m} \phi(x_r) - \sum_{r=1}^{m} \phi(y_r) \\
&\geq \sum_{i=2}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^{m}(x_r-a)^i - \sum_{r=1}^{m}(y_r-a)^i\right] \\
&+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\left[\sum_{r=1}^{m}(x_r-a)^{k+1+i} - \sum_{r=1}^{m}(y_r-a)^{k+1+i}\right]
\end{aligned} \tag{2.647}$$

(ii) If the inequality (2.647) holds and the function H defined by

$$\begin{aligned}
H(.) &= \sum_{i=2}^{k} \frac{\phi^{(i)}(a)}{i!}\sum_{r=1}^{m}(.-a)^i \\
&+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\sum_{r=1}^{m}(.-a)^{k+1+i}
\end{aligned} \tag{2.648}$$

is convex, then the right hand side of (2.647) will be non negative, that is the following majorization inequality holds:

$$\sum_{r=1}^{m} \phi(y_r) \leq \sum_{r=1}^{m} \phi(x_r) \tag{2.649}$$

(iii) If k and n both are even or odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, the reverse inequality in (2.647) holds.

(iv) If the reverse inequality in (2.647) holds and the function H defined in (2.648) is concave, then the right hand side of the reverse inequality in (2.647) will be non-positive, that is the reverse inequality in (2.649) holds.

Proof. By using (2.637), for $a \leq s,t \leq b$ the following inequalities hold

$$(-1)^{n-k-1}\frac{\partial^2 G_n(t,s)}{\partial t^2} \geq 0,$$

we conclude easily that if k is odd and n is even or k is even and n is odd then $\partial^2 G_n(t,s)/\partial t^2 \geq 0$ and also if k and n both are even or odd then $\partial^2 G_n(t,s)/\partial t^2 \leq 0$.

So k is odd and n is even or k is even and n is odd, G_n is convex with respect to first variable therefore by using Theorem 1.12 we have

$$\sum_{r=1}^{m} G_n(y_r,s) \leq \sum_{r=1}^{m} G_n(x_r,s).$$

Hence by Theorem 2.230 for $w_r = 1$, $(r = 1,\ldots,m)$ we get (2.647).

(ii) The proof is similar to the proof of Theorem 2.112 (ii).

Similarly we can prove other parts. □

The following theorem is the generalization of weighted majorization theorem:

Theorem 2.233 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ be decreasing and $\mathbf{w} = (w_1,\ldots,w_m)$ be any m-tuples with x_r, $y_r \in [a,b]$ and $w_r \in \mathbb{R}$ $(r = 1,\ldots,m)$ such that (1.19) and (1.20) hold. Also let G_n be the Green function defined as in (2.636).*

(i) If k is odd and n is even or k is even and n is odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\sum_{r=1}^{m} w_r\,\phi(x_r) - \sum_{r=1}^{m} w_r\,\phi(y_r) \\
&\geq \sum_{i=2}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^{m} w_r (x_r-a)^i - \sum_{r=1}^{m} w_r (y_r-a)^i\right] \\
&+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\left[\sum_{r=1}^{m} w_r (x_r-a)^{k+1+i} - \sum_{r=1}^{m} w_r (y_r-a)^{k+1+i}\right].
\end{aligned} \tag{2.650}$$

(ii) If the inequality (2.650) holds and the function H defined in (2.648) is convex, then the right hand side of (2.650) will be non-negative, that is the following inequality holds:

$$\sum_{r=1}^{m} w_r\phi(y_r) \leq \sum_{r=1}^{m} w_r\phi(x_r) \tag{2.651}$$

(iii) If k and n both are even or odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, the reverse inequality in (2.650) holds.

(iv) If the reverse inequality in (2.650) holds and the function H defined in (2.648) is concave, then the right hand side of the reverse inequality in (2.650) will be non-positive, that is the reverse inequality in (2.651) holds.

Proof. The proof is similar to the proof of Theorem 2.232 but using Theorem 1.14 instead of Theorem 1.12. □

The following theorem is weighted majorization theorem for n-convex function in integral case:

Theorem 2.234 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $x,y : [\alpha,\beta] \to [a,b]$ be increasing and $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions satisfying (1.27) and (1.28) and also G_n be the Green function defined as in (2.636).*

(i) If k is odd and n is even or k is even and n is odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\int_{\alpha}^{\beta} w(t)\,\phi\,(x(t))\,dt - \int_{\alpha}^{\beta} w(t)\,\phi\,(y(t))\,dt \\
&\geq \sum_{i=2}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\int_{\alpha}^{\beta} w(t)\,(x(t)-a)^{i}\,dt - \int_{\alpha}^{\beta} w(t)\,(y(t)-a)^{i}\,dt\right] \\
&+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\left[\int_{\alpha}^{\beta} w(t)\,(x(t)-a)^{k+1+i}\,dt - \int_{\alpha}^{\beta} w(t)\,(y(t)-a)^{k+1+i}\,dt\right]
\end{aligned} \tag{2.652}$$

(ii) If the inequality (2.652) holds and the function H defined in (2.648) is convex, then the right hand side of (2.652) will be non-negative, that is (1.29) holds.

(iii) If k and n both are even or odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, then the reverse inequality holds in (2.652).

(iv) If the reverse inequality in (2.652) holds and the function H defined in (2.648) is concave, then the right hand side of the reverse inequality in (2.652) will be non-positive that is the reverse inequality in (1.29) holds.

In the sequel we use the above theorems to obtain generalizations of the results proved in the previous.

For m-tuples $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ with $x_r,\ y_r \in [a,b], w_r \in \mathbb{R}$ $(r = 1,\ldots,m)$ and the function G_n as defined above, denote

$$\Upsilon(t) = \sum_{r=1}^{m} w_r G_n\,(x_r,s) - \sum_{r=1}^{m} w_r G_n\,(y_r,s)\,, \quad s \in [a,b], \tag{2.653}$$

similarly for $x,y : [\alpha,\beta] \to [a,b]$ and $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and for all $s \in [a,b]$, denote

$$\widetilde{\Upsilon}(s) = \int_{\alpha}^{\beta} w(t) G_n\,(x(t),s)\,dt - \int_{\alpha}^{\beta} w(t) G_n\,(y(t),s)\,dt. \tag{2.654}$$

Consider the Čebyšev functionals defined as:

$$T(\Upsilon,\Upsilon) = \frac{1}{b-a}\int_a^b \Upsilon^2(s)ds - \left(\frac{1}{b-a}\int_a^b \Upsilon(s)ds\right)^2, \tag{2.655}$$

$$T(\widetilde{\Upsilon},\widetilde{\Upsilon}) = \frac{1}{b-a}\int_a^b \widetilde{\Upsilon}^2(s)ds - \left(\frac{1}{b-a}\int_a^b \widetilde{\Upsilon}(s)ds\right)^2. \tag{2.656}$$

Theorem 2.235 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$ with $(.-a)(b-.)\left[\phi^{(n+1)}\right]^2 \in L[a,b]$, and $\mathbf{x} = (x_1,\ldots,x_m)$, $\mathbf{y} = (y_1,\ldots,y_m)$ and $\mathbf{w} = (w_1,\ldots,w_m)$ be m-tuples such that x_r, $y_r \in [a,b]$ and $w_r \in \mathbb{R}$ $(r = 1,\ldots,m)$. Let the functions G_n, Υ and T be defined in (2.636), (2.653) and (2.655) respectively. Then*

$$\begin{aligned}
&\sum_{r=1}^{m} w_r \phi(x_r) - \sum_{r=1}^{m} w_r \phi(y_r) \\
&= \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\sum_{r=1}^{m} w_r (x_r - a)^i - \sum_{r=1}^{m} w_r (y_r - a)^i\right] \\
&+ \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \left[\sum_{r=1}^{m} w_r (x_r - a)^{k+1+i} - \sum_{r=1}^{m} w_r (y_r - a)^{k+1+i}\right] \\
&+ \frac{\phi^{(n-1)}(b) - \phi^{(n-1)}(a)}{b-a} \int_a^b \Upsilon(t)dt + H_n^1(\phi;a,b),
\end{aligned} \tag{2.657}$$

where the remainder $H_n^1(\phi;a,b)$ satisfies the estimation

$$\left|H_n^1(\phi;a,b)\right| \leq \sqrt{\frac{b-a}{2}} \left[T(\Upsilon,\Upsilon)\right]^{\frac{1}{2}} \left|\int_a^b (t-a)(b-t)\left[\phi^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}. \tag{2.658}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

Integral case of the above theorem can be given as follows.

Theorem 2.236 ([11]) *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$ with $(.-a)(b-.)\left[\phi^{(n+1)}\right]^2 \in L[a,b]$, and $x,y : [\alpha,\beta] \to [a,b]$, $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and also let the functions $G_n, \widetilde{\Upsilon}$ and T be defined in (2.636), (2.654) and (2.656) respectively. Then*

$$\begin{aligned}
&\int_\alpha^\beta w(t)\phi(x(t))dt - \int_\alpha^\beta w(t)\phi(y(t))dt \\
&= \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\int_\alpha^\beta w(t)(x(t)-a)^i dt - \int_\alpha^\beta w(t)(y(t)-a)^i dt\right] \\
&+ \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\left[\int_\alpha^\beta w(t)(x(t)-a)^{k+1+i} dt - \int_\alpha^\beta w(t)(y(t)-a)^{k+1+i} dt\right] \\
&+ \frac{\phi^{(n-1)}(b) - \phi^{(n-1)}(a)}{b-a} \int_a^b \widetilde{\Upsilon}(s)ds + \widetilde{H}_n^1(\phi;a,b),
\end{aligned} \tag{2.659}$$

where the remainder $\widetilde{H}_n^1(\phi;a,b)$ *satisfies the estimation*

$$\left|\widetilde{H}_n^1(\phi;a,b)\right| \leq \sqrt{\frac{b-a}{2}}\left[T(\widetilde{\Upsilon},\widetilde{\Upsilon})\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.$$

Using Theorem 1.11 we obtain the following Grüss type inequalities.

Theorem 2.237 ([11]) *Let* $n,k\in\mathbb{N}$*,* $n\geq 2$*,* $0\leq k\leq n-1$*,* $\phi:[a,b]\to\mathbb{R}$ *be such that* $\phi\in C^n[a,b]$ *and* $\phi^{(n+1)}\geq 0$ *on* $[a,b]$ *and let the function* Υ *and* T *be defined by (2.653) and (2.655) respectively.*
Then we have the representation (2.657) and the remainder $H_n^1(\phi;a,b)$ *satisfies the bound*

$$\left|H_n^1(\phi;a,b)\right| \leq \left\|\Upsilon'\right\|_\infty \left\{\frac{\phi^{(n-1)}(b)+\phi^{(n-1)}(a)}{2} - \frac{\phi^{(n-2)}(b)-\phi^{(n-2)}(a)}{b-a}\right\}. \quad (2.660)$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

Integral version of the above theorem can be given as follows.

Theorem 2.238 ([11]) *Let* $n,k\in\mathbb{N}$*,* $n\geq 2$*,* $0\leq k\leq n-1$*,* $\phi:[a,b]\to\mathbb{R}$ *be such that* $\phi\in C^n[a,b]$ *and* $\phi^{(n+1)}\geq 0$ *on* $[a,b]$ *and also let the functions* $\widetilde{\Upsilon}$ *and* T *be defined by (2.654) and (2.656) respectively.*
Then we have the representation (2.659) and the remainder $\widetilde{H}_n^1(\phi;a,b)$ *satisfies the bound*

$$\left|\widetilde{H}_n^1(\phi;a,b)\right| \leq \left\|\widetilde{\Upsilon}'\right\|_\infty \left\{\frac{\phi^{(n-1)}(b)+\phi^{(n-1)}(a)}{2} - \frac{\phi^{(n-2)}(b)-\phi^{(n-2)}(a)}{b-a}\right\}.$$

We give the Ostrowski type inequalities related to the generalizations of majorization inequality.

Theorem 2.239 ([11]) *Suppose that all the assumptions of Theorem 2.228 hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1\leq p,q\leq\infty$*,* $\frac{1}{p}+\frac{1}{q}=1$*. Let* $\left|\phi^{(n)}\right|^p:[a,b]\to\mathbb{R}$ *be an R-integrable function for some* $n\in\mathbb{N}$*. Then we have*

$$\begin{aligned}
&\Bigg|\sum_{r=1}^m w_r\,\phi(x_r) - \sum_{r=1}^m w_r\,\phi(y_r) \\
&- \sum_{i=0}^{k}\frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^m w_r(x_r-a)^i - \sum_{r=1}^m w_r(y_r-a)^i\right] \\
&- \sum_{j=0}^{n-k-2}\sum_{i=0}^{j}\frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\left[\sum_{r=1}^m w_r(x_r-a)^{k+1+i} - \sum_{r=1}^m w_r(y_r-a)^{k+1+i}\right]\Bigg| \\
&\leq \left\|\phi^{(n)}\right\|_p \left(\int_a^b \left|\sum_{r=1}^m w_r G_n(x_r,s) - \sum_{r=1}^m w_r G_n(y_r,s)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned} \quad (2.661)$$

The constant on the right-hand side of (2.661) is sharp for $1<p\leq\infty$ *and the best possible for* $p=1$*.*

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Integral version of the above theorem can be stated as follows.

Theorem 2.240 ([11]) *Suppose that all the assumptions of Theorem 2.229 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $\left|\phi^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
&\left| \int_\alpha^\beta w(t)\,\phi\left(x(t)\right)dt - \int_\alpha^\beta w(t)\,\phi\left(y(t)\right)dt \right. \\
&- \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^i dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^i dt \right] \\
&- \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\left. \left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^{k+1+i} dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^{k+1+i} dt \right] \right| \\
&\le \left\| \phi^{(n)} \right\|_p \left(\int_a^b \left| \int_\alpha^\beta w(t) G_n\left(x(t),s\right)dt - \int_\alpha^\beta w(t) G_n\left(y(t),s\right)dt \right|^q ds \right)^{\frac{1}{q}}.
\end{aligned} \tag{2.662}$$

The constant on the right-hand side of (2.662) is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Motivated by the inequalities (2.644) and (2.646), we define functional $\Theta_1(\phi)$ and $\Theta_2(\phi)$ by

$$\begin{aligned}
\Theta_1(\phi) = &\sum_{r=1}^{m} w_r\,\phi\left(x_r\right) - \sum_{r=1}^{m} w_r\,\phi\left(y_r\right) \\
&- \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^i - \sum_{r=1}^{m} w_r\left(y_r-a\right)^i \right] \\
&- \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^{k+1+i} - \sum_{r=1}^{m} w_r\left(y_r-a\right)^{k+1+i} \right]
\end{aligned} \tag{2.663}$$

$$\begin{aligned}
\Theta_2(\phi) = &\int_\alpha^\beta w(t)\,\phi\left(x(t)\right)dt - \int_\alpha^\beta w(t)\,\phi\left(y(t)\right)dt \\
&- \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^i dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^i dt \right] \\
&- \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^{k+1+i} dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^{k+1+i} dt \right].
\end{aligned} \tag{2.664}$$

Remark 2.61 ([11]) *Under the assumptions of Theorem 2.230 and Theorem 2.231, it holds $\Theta_i(\phi) \geq 0$, $i = 1,2$ for all n-convex functions ϕ.*

The Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems:

Theorem 2.241 ([11]) *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$. If the inequalities in (2.643), (2.645) hold, then there exist $\xi_i \in [a,b]$ such that*

$$\Theta_i(\phi) = \phi^{(n)}(\xi_i)\Theta_i(\eta), \;\; i = 1,2, \tag{2.665}$$

where $\eta(x) = \frac{x^n}{n!}$ and Θ_1, Θ_2 are defined in (2.663) and (2.664).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (see the proof of Theorem 7 in [30]). □

Theorem 2.242 ([11]) *Let $\phi, \psi : [a,b] \to \mathbb{R}$ be such that $\phi, \psi \in C^n[a,b]$. If the inequalities in (2.643), (2.645) hold, then there exist $\xi_i \in [a,b]$ such that*

$$\frac{\Theta_i(\phi)}{\Theta_i(\varphi)} = \frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)}, \;\; i = 1,2, \tag{2.666}$$

provided that the denominators are non-zero and Θ_1, Θ_2 are defined in (2.663) and (2.664).

Proof. The idea of the proof is the same as that of the proof of Theorem 2.14 (see the proof of Corollary 12 in [30]). □

We use an idea from [84] to give an elegant method of producing n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [142]):

Theorem 2.243 ([11]) *Let $\Phi = \{\phi_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$ such that the function $s \mapsto \phi_s[x_0,\ldots,x_l]$ is an n-exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $x_0,\ldots,x_l \in [a,b]$. Let $\Theta_i(\phi_s)$, $i = 1,2$ be the linear functionals defined as in (2.663) and (2.664). Then the following statements hold:*

(i) The function $s \mapsto \Theta_i(\phi_s)$ is an n-exponentially convex function in the Jensen sense on J and the matrix $\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}$, $m \leq n$, $s_1,\ldots,s_m \in J$. Particularly

$$\det\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^m \geq 0 \;\; for all\ m \in \mathbb{N}, m = 1,\ldots,n.$$

(ii) If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is n-exponentially convex function on J.

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39 but using linear functionals $\Theta_k (k = 1,2)$ instead of $\digamma_k (k = 1,2,..,5)$. □

The following corollaries are immediate consequences of the above theorem.

Corollary 2.47 ([11]) *Let $\Phi = \{\phi_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$ such that the function $s \mapsto \phi_s [x_0,\dots,x_l]$ is an exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $x_0,\dots,x_l \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (2.663) and (2.664). Then the following statements hold:*

(i) The function $s \mapsto \Theta_i(\phi_s)$ is an exponentially convex function in the Jensen sense on J and the matrix $\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}$, $m \le n$, $s_1,\dots,s_m \in J$. Particularly

$$\det\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^{m} \ge 0 \quad for\, all\ m \in \mathbb{N},\, m = 1,\dots,n.$$

(ii) If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is exponentially convex function on J.

Corollary 2.48 ([11]) *Let $\Phi = \{\phi_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s [x_0,\dots,x_l]$ is an 2-exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $x_0,\dots,x_l \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (2.663) and (2.664). Then the following statements hold:*

(i) If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Theta_i(\phi_s)$ is additionally strictly positive, then it is log-convex *on J. Furthermore, the Lypunov's inequality holds true:*

$$[\Theta_i(\phi_s)]^{t-r} \le [\Theta_i(\phi_r)]^{t-s} [\Theta_i(\phi_t)]^{s-r}, \quad i = 1,2, \tag{2.667}$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $s \mapsto \Theta_i(\phi_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v \in J$, such that $s \le u$ and $q \le v$, we have

$$\mu_{s,q}(\Theta_i,\Phi) \le \mu_{u,v}(\Theta_i,\Phi), \tag{2.668}$$

where

$$\mu_{s,q}(\Theta_i,\Phi) = \begin{cases} \left(\dfrac{\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right)^{\frac{1}{s-q}}, & s \ne q, \\ \exp\left(\dfrac{\frac{d}{ds}\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right), & s = q, \end{cases} \tag{2.669}$$

for $\phi_s, \phi_q \in \Phi$.

Proof. The idea of the proof is the same as that of the proof of Corollary 2.11 but using linear functionals $\Theta_k (k = 1,2)$ instead of $\digamma_k (k = 1,2,..,5)$. □

Remark 1.19 is also valid for these functionals.

Remark 2.62 ([11]) *Similar examples can be discussed as given in Section 1.4.*

2.6.2 Results Obtained by Green's Function and the Abel-Gontscharoff Interpolating Polynomial

In this subsection, we use interpolation by the Abel-Gontscharoff polynomials in combination with Green's function to establish new generalizations of majorization theorems for the class of n-convex functions.

Theorem 2.244 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n[\alpha,\beta]$ and $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ be m-tuples such that x_l, $y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l = 1,\ldots,m)$. Also let G and G_n be defined by (1.180) and (2.636) respectively. Then*

$$\begin{aligned}
&\sum_{l=1}^{m} w_l\,\phi\,(x_l) - \sum_{l=1}^{m} w_l\,\phi\,(y_l) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l\,(x_l-y_l) \\
&+\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\right](s-\alpha)^i\,ds \\
&+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}\,(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\right](s-\alpha)^{k+1+i}\,ds \\
&+\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\right]G_{n-2}(s,t)\phi^{(n)}(t)dtds.
\end{aligned} \tag{2.670}$$

Proof. Using (1.181) in $\sum_{l=1}^{m} w_l\,\phi\,(x_l) - \sum_{l=1}^{m} w_l\,\phi\,(y_l)$ we have

$$\begin{aligned}
&\sum_{l=1}^{m} w_l\,\phi\,(x_l) - \sum_{l=1}^{m} w_l\,\phi\,(y_l) \\
&= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l\,(x_l-y_l) + \int_{\alpha}^{\beta}\left[\sum_{l=1}^{m} w_l G(x_l,s) - \sum_{l=1}^{m} w_l G(y_l,s)\right]\phi''(s)ds.
\end{aligned} \tag{2.671}$$

By Theorem 2.227, $\phi''(s)$ can be expressed as

$$\begin{aligned}
\phi''(s) = \sum_{i=0}^{k}\frac{(s-\alpha)^i}{i!}\phi^{(i+2)}(\alpha) + \sum_{j=0}^{n-k-4}\left[\sum_{i=0}^{j}\frac{(s-\alpha)^{k+1+i}\,(\alpha-\beta)^{j-i}}{(k+1+i)!\,(j-i)!}\right]\phi^{(k+3+j)}(\beta) \\
+\int_{\alpha}^{\beta} G_{n-2}(s,t)\phi^{(n)}(t)dt.
\end{aligned} \tag{2.672}$$

Using (2.672) in (2.671) we get (2.670). □

Integral version of the above theorem can be stated as follows.

Theorem 2.245 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n[\alpha,\beta]$, and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and G, G_n be defined by (1.180) and (2.636) respectively. Then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds$$
$$+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!(j-i)!}$$
$$\cdot\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^{k+1+i}ds$$
$$+\int_\alpha^\beta\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)G_{n-2}(s,t)\phi^{(n)}(t)dtds. \tag{2.673}$$

In the following theorem we obtain generalizations of majorization inequality for n-convex functions.

Theorem 2.246 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ be m-tuples such that x_l, $y_l \in [\alpha,\beta], w_l \in \mathbb{R}$ $(l = 1,\ldots,m)$. Also let G and G_n be defined by (1.180) and (2.636) respectively. If $\phi : [\alpha,\beta] \to \mathbb{R}$ is n-convex, and*

$$\int_\alpha^\beta\left(\sum_{l=1}^{m} w_l\left(G(x_l,s)-G(y_l,s)\right)\right)G_{n-2}(s,t)ds \geq 0, \quad t \in [\alpha,\beta]. \tag{2.674}$$

Then

$$\begin{aligned}\sum_{l=1}^{m} w_l\,\phi(x_l) - \sum_{l=1}^{m} w_l\,\phi(y_l) &\geq \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{l=1}^{m} w_l(x_l-y_l)\\ &+\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left[\sum_{l=1}^{m} w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^i ds\\ &+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!(j-i)!}\\ &\int_\alpha^\beta\left[\sum_{l=1}^{m} w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^{k+1+i}ds.\end{aligned} \tag{2.675}$$

If the reverse inequality in (2.674) holds, then also the reverse inequality in (2.675) holds.

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)}(x) \geq 0$ for all $x \in [\alpha, \beta]$ (see [144, p. 16 and p. 293]). Hence, we can apply Theorem 2.244 to obtain (2.675). □

Remark 2.63 ([9]) *As from (2.637) we have $(-1)^{n-k-3}G_{n-2}(s,t) \geq 0$, therefore for the case when n is even and k is odd or n is odd and k is even, it is enough to assume that $\sum_{l=1}^{m} w_l G(x_l, s) - \sum_{l=1}^{m} w_l G(y_l, s) \geq 0, s \in [\alpha, \beta]$, instead of the assumption (2.674) in Theorem 2.246. Similarly we can discuss for the reverse inequality in (2.675).*

Integral version of the above theorem can be stated as follows.

Theorem 2.247 ([9]) *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $x, y : [a,b] \to [\alpha, \beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and G, G_n be defined by (1.180) and (2.636) respectively. If $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex, and*

$$\int_{\alpha}^{\beta} \left(\int_{a}^{b} w(\tau)(G(x(\tau), s) - G(y(\tau), s)) d\tau \right) G_{n-2}(s,t) ds \geq 0. \tag{2.676}$$

Then

$$\begin{aligned}
&\int_{a}^{b} w(\tau)\phi(x(\tau))d\tau - \int_{a}^{b} w(\tau)\phi(y(\tau))d\tau \geq \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \int_{a}^{b} w(\tau)(x(\tau)-y(\tau))d\tau \\
&+\sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left(\int_{a}^{b} w(\tau)(G(x(\tau), s) - G(y(\tau), s))d\tau \right) (s-\alpha)^i ds \\
&+\sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!(j-i)!} \\
&\cdot \int_{\alpha}^{\beta} \left(\int_{a}^{b} w(\tau)(G(x(\tau), s) - G(y(\tau), s))d\tau \right) (s-\alpha)^{k+1+i} ds.
\end{aligned} \tag{2.677}$$

If the reverse inequality in (2.676) holds, then also the reverse inequality in (2.677) holds.

Remark 2.64 ([9]) *As from (2.637) we have $(-1)^{n-k-3}G_{n-2}(s,t) \geq 0$, therefore for the case when n is even and k is odd or n is odd and k is even, it is enough to assume that $\int_a^b w(\tau)(G(x(\tau), s) - G(y(\tau), s))d\tau \geq 0, s \in [\alpha, \beta]$, instead of the assumption (2.676) in Theorem 2.245. Similarly we can discuss for the reverse inequality in (2.677).*

We give a generalization of majorization theorem for majorized m-tuples:

Theorem 2.248 ([9]) *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$ and $\boldsymbol{x} = (x_1, \ldots, x_m)$, $\boldsymbol{y} = (y_1, \ldots, y_m)$ be two m-tuples such that $\boldsymbol{y} \prec \boldsymbol{x}$ with x_l, $y_l \in [\alpha, \beta]$, $(l = 1, \ldots, m)$. Also let G be defined by (1.180). Consider $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex.*

(i) If n is even and k is odd or n is odd and k is even. Then

$$\begin{aligned}
&\sum_{l=1}^{m}\phi\left(x_l\right)-\sum_{l=1}^{m}\phi\left(y_l\right)\\
&\geq \sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_{\alpha}^{\beta}\left[\sum_{l=1}^{m}w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^i\,ds\\
&+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}\\
&\int_{\alpha}^{\beta}\left[\sum_{l=1}^{m}w_l\left(G(x_l,s)-G(y_l,s)\right)\right](s-\alpha)^{k+1+i}\,ds.
\end{aligned}\tag{2.678}$$

(ii) If the inequality (2.678) holds and the function F defined by

$$\begin{aligned}
F(.)=&\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_{\alpha}^{\beta}G(.,s)\,(s-\alpha)^i\,ds\\
&+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}\int_{\alpha}^{\beta}G(.,s)\,(s-\alpha)^{k+1+i}\,ds
\end{aligned}\tag{2.679}$$

is convex, then the right hand side of (2.678) will be non negative, that is (2.153) holds.

(iii) If n and k both are even or both are odd, then reverse inequality holds in (2.678).

(iv) If the reverse inequality in (2.678) holds and the function F defined in (2.679) is concave, then the right hand side of the reverse inequality in (2.678) will be non positive, that is the reverse inequality in (2.153) holds.

Proof. (i) By using (2.637) we have $(-1)^{n-k-3}G_{n-2}(s,t)\geq 0$, $\alpha\leq s,t\leq\beta$, therefore if n is even and k is odd or n is odd and k is even then $G_{n-2}(s,t)\geq 0$. Also as G is convex so by Theorem 1.12 and non negativity of G_{n-2}, the inequality (2.674) holds for $w_l=1$, $l=1,2,..,m$. Hence by Theorem 2.246 for $w_l=1$, $l=1,2,..,m$, the inequality (2.678) holds.
By using the other conditions the non negativity of the right hand side of (2.678) is obvious.

(ii) The proof is similar to the proof of Theorem 2.112(ii).

Similarly we can prove other parts. □

In the following theorem we present generalization of Fuch's majorization theorem.

Theorem 2.249 ([9]) *Let $n,k\in\mathbb{N}$, $n\geq 4$, $0\leq k\leq n-1$, $\boldsymbol{x}=(x_1,\ldots,x_m)$, $\boldsymbol{y}=(y_1,\ldots,y_m)$ be decreasing and $\boldsymbol{w}=(w_1,\ldots,w_m)$ be any m-tuples such that x_l, $y_l\in[\alpha,\beta]$, $w_l\in\mathbb{R}$ $(l=1,\ldots,m)$ which satisfy (1.19) and (1.20). Also let G be defined by (1.180). Consider $\phi:[\alpha,\beta]\to\mathbb{R}$ is n-convex.*

(i) If n is even and k is odd or n is odd and k is even. Then

$$\begin{aligned}
&\sum_{l=1}^{m} w_l\,\phi\,(x_l) - \sum_{l=1}^{m} w_l\,\phi\,(y_l) \\
&\geq \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\right](s-\alpha)^i\,ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\right](s-\alpha)^{k+1+i}\,ds.
\end{aligned} \tag{2.680}$$

(ii) If the inequality (2.680) holds and the function F defined in (2.679) is convex, then the right hand side of (2.680) will be non negative, that is (2.157) holds.

(iii) If n and k both are even or both are odd, then reverse inequality holds in (2.680).

(iv) If the reverse inequality in (2.680) holds and the function F defined in (2.679) is concave, then the right hand side of the reverse inequality in (2.680) will be non positive, that is reverse inequality (2.157) holds.

Proof. The proof is similar to the proof of Theorem 2.248 but using Theorem 1.14 instead of Theorem 1.12. □

The integral version of Theorem 2.249 can be stated as follows.

Theorem 2.250 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $x,y : [a,b] \to [\alpha,\beta]$ be decreasing and $w : [a,b] \to \mathbb{R}$ be any continuous function. Also let G be defined by (1.180). Consider $\phi : [\alpha,\beta] \to \mathbb{R}$ is n-convex and*

$$\int_a^{\nu} w(\tau)y(\tau)\,d\tau \leq \int_a^{\nu} w(\tau)x(\tau)\,d\tau \ \text{for}\ \nu \in [a,b], \tag{2.681}$$

$$\int_a^{b} w(\tau)x(\tau)\,d\tau = \int_a^{b} w(\tau)y(\tau)\,d\tau \tag{2.682}$$

(i) If n is even and k is odd or n is odd and k is even. Then

$$\begin{aligned}
&\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\
&\geq \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\cdot \int_{\alpha}^{\beta} \left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^{k+1+i}\,ds.
\end{aligned} \tag{2.683}$$

(ii) If the inequality (2.683) holds and the function F defined in (2.679) is convex, then the right hand side of (2.683) will be non negative, that is (2.159) holds.

(iii) If n and k both are even or both are odd, then reverse inequality holds in (2.683).

(iv) If the reverse inequality in (2.683) holds and the function F defined in (2.679) is concave, then the right hand side of the reverse inequality in (2.683) will be non positive, that is reverse inequality (2.159) holds.

In the sequel we use the above theorems to obtain generalizations of the previous results in the form of the Grüss and Ostrowski type inequalities.

For m-tuples $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ with $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}\ (l = 1,\ldots,m)$ and the functions G, G_n as defined above, denote

$$\mathfrak{R}(t) = \sum_{l=1}^{m} w_l \int_{\alpha}^{\beta} (G(x_l,s) - G(y_l,s))\, G_{n-2}(s,t)ds, \quad t \in [\alpha,\beta], \tag{2.684}$$

and for continuous functions $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$, denote

$$\tilde{\mathfrak{R}}(t) = \int_{\alpha}^{\beta} \left(\int_{a}^{b} w(\tau)(G(x(\tau),s) - G(y(\tau),s))d\tau \right) G_{n-2}(s,t)ds, \quad t \in [\alpha,\beta], \tag{2.685}$$

Consider the Čebyšev functionals $T(\mathfrak{R},\mathfrak{R})$ and $T(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})$ are given by:

$$T(\mathfrak{R},\mathfrak{R}) = \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathfrak{R}^2(t)dt - \left(\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t)dt \right)^2, \tag{2.686}$$

Theorem 2.251 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n[\alpha,\beta]$ with $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$, $\boldsymbol{w} = (w_1,\ldots,w_m)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\boldsymbol{y} = (y_1,\ldots,y_m)$ be m-tuples such that $x_l,\ y_l \in [\alpha,\beta], w_l \in \mathbb{R}\ (l = 1,\ldots,m)$ and let the functions G, $\mathfrak{R}$ and T be defined by (1.180), (2.684) and (2.686) respectively. Then*

$$\begin{aligned}
\sum_{l=1}^{m} w_l\, \phi(x_l) - \sum_{l=1}^{m} w_l\, \phi(y_l) &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^{m} w_l\,(x_l - y_l) \\
&+ \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G(x_l,s) - G(y_l,s)) \right] (s-\alpha)^i\, ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(\alpha-\beta)^{j-i}\, \phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G(x_l,s) - G(y_l,s)) \right] (s-\alpha)^{k+1+i}\, ds \\
&+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{\beta-\alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t)dt + \kappa_n(\phi;\alpha,\beta).
\end{aligned} \tag{2.687}$$

where the remainder $\kappa_n(\phi;\alpha,\beta)$ satisfies the estimation

$$|\kappa_n(\phi;\alpha,\beta)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}} \left[T(\mathfrak{R},\mathfrak{R})\right]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \tag{2.688}$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.7. □

Integral case of the above theorem can be given as follows.

Theorem 2.252 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n[\alpha,\beta]$ with $(\cdot - \alpha)(\beta - \cdot)[\phi^{(n+1)}]^2 \in L[\alpha,\beta]$ and $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and let the functions G, $\tilde{\mathfrak{R}}$, T be defined by (1.180), (2.685) and (2.686) respectively. Then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{i=0}^{k}\frac{\phi^{(i+2)}(\alpha)}{i!}\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^i ds$$
$$+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(\alpha-\beta)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!}$$
$$\int_\alpha^\beta\left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^{k+1+i}ds$$
$$+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta \tilde{\mathfrak{R}}(t)dt + \tilde{\kappa}_n(\phi;\alpha,\beta). \quad (2.689)$$

where the remainder $\tilde{\kappa}_n(\phi;\alpha,\beta)$ satisfies the estimation

$$|\tilde{\kappa}_n(\phi;\alpha,\beta)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left[T(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})\right]^{\frac{1}{2}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}. \quad (2.690)$$

Using Theorem 1.11 we obtain the following the Grüss type inequalities.

Theorem 2.253 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n[\alpha,\beta]$ such $\phi^{(n)}$ is increasing on $[\alpha,\beta]$ and let the functions G, $\mathfrak{R}$ and T be defined by (1.180), (2.684) and (2.686) respectively. Then the representation* (2.687) *holds and the remainder $\kappa_n(\phi;\alpha,\beta)$ satisfies the bound*

$$|\kappa_n(\phi;\alpha,\beta)| \leq \|\mathfrak{R}'\|_\infty\left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \quad (2.691)$$

Proof. The idea of the proof is the same as that of the proof of Theorem 2.9. □

Integral case of the above theorem can be given as follows.

Theorem 2.254 ([9]) *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\phi \in C^n[\alpha,\beta]$ such that $\phi^{(n)}$ is increasing on $[\alpha,\beta]$ and let the functions G, $\tilde{\mathfrak{R}}$ T be defined by (1.180), (2.685) and (2.686) respectively. Then we have the representation* (2.689) *and the remainder $\tilde{\kappa}_n(\phi;\alpha,\beta)$ satisfies the bound*

$$|\tilde{\kappa}_n(\phi;\alpha,\beta)| \leq \|\tilde{\mathfrak{R}}'\|_\infty\left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \quad (2.692)$$

We present the Ostrowski-type inequalities related to generalizations of majorization inequality.

Theorem 2.255 ([9]) *Suppose that all assumptions of Theorem 2.244 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $\left|\phi^{(n)}\right|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function. Then we have:*

$$\begin{aligned}
&\Bigg| \sum_{l=1}^{m} w_l\, \phi(x_l) - \sum_{l=1}^{m} w_l\, \phi(y_l) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^{m} w_l\,(x_l-y_l) \\
&- \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left(\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s)) \right) (s-\alpha)^i ds \\
&- \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\int_{\alpha}^{\beta} \left[\sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s)) \right] (s-\alpha)^{k+1+i} ds \Bigg| \\
&\leq \left\|\phi^{(n)}\right\|_p \|\mathfrak{R}\|_q,
\end{aligned} \tag{2.693}$$

$$\textit{where } \mathfrak{R}(t) = \int_{\alpha}^{\beta} \sum_{l=1}^{m} w_l\,(G(x_l,s)-G(y_l,s))\, G_{n-2}(s,t)ds, \quad t \in [\alpha,\beta].$$

The constant on the right-hand side of (2.693) is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.11. □

Integral case can be given as:

Theorem 2.256 ([9]) *Suppose that all assumptions of Theorem 2.245 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $\left|\phi^{(n)}\right|^p : [\alpha,\beta] \to \mathbb{R}$ be an R-integrable function. Then we have:*

$$\begin{aligned}
&\Bigg| \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \\
&- \sum_{i=0}^{k} \frac{\phi^{(i+2)}(\alpha)}{i!} \int_{\alpha}^{\beta} \left(\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau \right) (s-\alpha)^i ds \\
&- \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\,\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\
&\int_{\alpha}^{\beta} \left[\int_a^b w(\tau)(G(x(\tau),s)-G(y(\tau),s))d\tau \right] (s-\alpha)^{k+1+i} ds \Bigg| \\
&\leq \left\|\phi^{(n)}\right\|_p \|\tilde{\mathfrak{R}}\|_q,
\end{aligned} \tag{2.694}$$

where $\tilde{\mathfrak{R}}(t) = \int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau),s) - G(y(\tau),s))d\tau \right) G_{n-2}(s,t)ds, \quad t \in [\alpha,\beta].$

The constant on the right-hand side of (2.694) is sharp for $1 < p \leq \infty$ *and the best possible for* $p = 1$.

We use an idea from [84] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals to a given family with the same property (see [142]).

Motivated by inequalities (2.675) and (2.677), under the assumptions of Theorems 2.246 and 2.247 we define the following linear functionals:

$$\begin{aligned} \digamma_1(\phi) &= \sum_{l=1}^m w_l\, \phi(x_l) - \sum_{l=1}^m w_l\, \phi(y_l) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{l=1}^m w_l\,(x_l - y_l) \\ &- \sum_{i=0}^k \frac{\phi^{(i+2)}(\alpha)}{i!} \int_\alpha^\beta \left[\sum_{l=1}^m w_l\,(G(x_l,s) - G(y_l,s)) \right] (s-\alpha)^i\, ds \\ &- \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\ &\int_\alpha^\beta \left[\sum_{l=1}^m w_l\,(G(x_l,s) - G(y_l,s)) \right] (s-\alpha)^{k+1+i}\, ds \end{aligned} \tag{2.695}$$

$$\begin{aligned} \digamma_2(\phi) &= \int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\ &- \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \\ &- \sum_{i=0}^k \frac{\phi^{(i+2)}(\alpha)}{i!} \int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-\alpha)^i ds \\ &- \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(\beta-\alpha)^{j-i}\phi^{(k+3+j)}(\beta)}{(k+1+i)!\,(j-i)!} \\ &\cdot \int_\alpha^\beta \left(\int_a^b w(\tau)(G(x(\tau),s) - G(y(\tau),s))d\tau \right) (s-\alpha)^{k+1+i} ds. \end{aligned} \tag{2.696}$$

Remark 2.65 Under the assumptions of Theorems 2.246 and 2.245, it holds $\digamma_i(\phi) \geq 0$, $i = 1,2$ for all n-convex functions ϕ.

The Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

Theorem 2.257 ([9]) *Let* $\phi : [\alpha,\beta] \to \mathbb{R}$ *be such that* $\phi \in C^n[\alpha,\beta]$. *If the inequalities in (2.675), (2.677) hold, then there exist* $\xi_i \in [\alpha,\beta]$ *such that*

$$\digamma_i(\phi) = \phi^{(n)}(\xi_i)\digamma_i(\varphi), \quad i = 1,2, \tag{2.697}$$

where $\varphi(x) = \frac{x^n}{n!}$ *and* $\digamma_1$, $\digamma_2$ *are defined by (2.695) and (2.696) respectively.*

Proof. The idea of the proof is the same as that of the proof of Theorem 2.13 (see the proof of Theorem 4.1 in [86]). □

Theorem 2.258 ([9]) *Let $\phi,\psi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi,\psi \in C^n[\alpha,\beta]$. If the inequalities in (2.675) and (2.677) hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\frac{\digamma_i(\phi)}{\digamma_i(\psi)} = \frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)}, \quad i = 1,2. \tag{2.698}$$

provided that the denominators are non-zero and $\digamma_1$, $\digamma_2$ are defined by (2.695) and (2.696) respectively.

Proof. The idea of the proof is the same as that of the proof of Theorem 2.14 (see the proof of Corollary 4.2 in [86]). □

We use the idea from [84] to give an elegant method of producing n-exponentially convex functions and exponentially convex functions associated with the above functional (see [142]):

Theorem 2.259 ([9]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0,\dots,x_k;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every $(k+1)$ mutually different points $x_0,\dots,x_k \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1,2)$ as defined by (2.695) and (2.696), the following statements hold:*

(i) The function $t \to \digamma_i(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \textit{ for all } m \in \mathbb{N},\ m = 1,2,\dots,n.$$

(ii) If the function $t \to \digamma_i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. The idea of the proof is the same as that of the proof of Theorem 1.39. □

The following corollaries is an immediate consequence of the above theorem

Corollary 2.49 ([9]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\dots,x_k;\phi_t]$ is exponentially convex in the Jensen sense on J for every $(k+1)$ mutually different points $x_0,\dots,x_k \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1,2)$ as defined by (2.695) and (2.696), the following statements hold:*

(i) The function $t \to \digamma_i(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \le n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geq 0 \textit{ for all } m \in \mathbb{N},\ m = 1,2,\dots,n.$$

(ii) If the function $t \to \digamma_i(\phi_t)$ is continuous on J, then it is exponentially convex on J.

Corollary 2.50 ([9]) *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_k; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(k+1)$ mutually different points $x_0, \ldots, x_k \in I$. Let $\digamma_i$, $i = 1,2$ be linear functionals defined by* (2.695) *and* (2.696). *Then the following statements hold:*

(i) If the function $t \mapsto \digamma_i(\phi_t)$ is continuous on J, then it is 2-exponentially convex function on J. If $t \mapsto \digamma_i(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\digamma_i(\phi_s)]^{t-r} \leq [\digamma_i(\phi_r)]^{t-s} [\digamma_i(\phi_t)]^{s-r}, \quad i = 1,2.$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \digamma_i(\phi_t)$ is strictly positive and differentiable on J, then for every $p,q,u,v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\digamma_i, \Omega) \leq \mu_{u,v}(\digamma_i, \Omega), \tag{2.699}$$

where

$$\mu_{p,q}(\digamma_i, \Omega) = \begin{cases} \left(\dfrac{\digamma_i(\phi_p)}{\digamma_i(\phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\dfrac{\frac{d}{dp}\digamma_i(\phi_p)}{\digamma_i(\phi_p)} \right), & p = q, \end{cases} \tag{2.700}$$

for $\phi_p, \phi_q \in \Omega$.

Proof. The idea of the proof is the same as that of the proof of Corollary 1.10. □

Remark 2.66 *Remark 1.19 is also valid for these functionals. Similar examples can be discussed as given in Section 1.4.*

Chapter 3

Majorization in Information Theory

One of the most important issues in many applications of Probability Theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. Distance or divergence measures are of key importance in different fields like theoretical and applied statistical inference and data processing problems, such as estimation, detection, classification, compression, recognition, indexation, diagnosis and model selection etc. A number of divergence measures for this purpose have been proposed and extensively studied by Csiszár [64], Kullback and Leibler [101], Rényi [151], Rao [149] and Lin [113] and others. These measures have been applied in a variety of fields such as: anthropology [149], genetics [126], finance, economics, and political science [158, 165, 166], biology [147], the analysis of contingency tables [75], approximations of probability distributions [59, 92], signal processing [88, 89] and pattern recognition [37, 58]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied. The literature on such types of issues is wide and has considerably expanded in the recent years. In particular, following the set of some books published during the second half of the eighties [22, 46, 62, 78] and the number of some books have been published during the last decade or so [23, 34, 40, 150]. In a report on divergence measures and their tight connections with the notion of entropy, information and mean values, an attempt has been made to describe various procedures for building divergences measures from entropy functions or from generalized mean values and conversely for defining entropies from divergence measures [38], [39].

Well over a century ago measures were derived for assessing the distance between two models of probability distributions. Most relevant is Boltzmann's [48] concept of generalized entropy in physics and thermodynamics (see Akaike [19] for a brief review). Shannon [159] employed entropy in his famous treatise on communication theory. Kullback-Leibler [101] derived an information measure that happened to be the negative of Boltzmann's entropy, now referred as the Kullback-Leibler (K-L) distance. The motivation of the Kullback-Leibler work was to provide a rigorous definition of information in relation to Fisher's sufficient statistics. The K-L distance has also been called the K-L discrepancy, divergence, information and number – these terms are synonyms, we tend to use distance or information in the material to follow.

A fundamental result related to the notion of the Shannon entropy is the following **Shannon's inequality** (see [120])

$$\sum_{i=1}^{n} p_i \log \frac{1}{p_i} \leq \sum_{i=1}^{n} p_i \log \frac{1}{q_i}, \tag{3.1}$$

for all positive real numbers p_i and q_i with

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i. \tag{3.2}$$

Here, 'log' denotes the logarithmic function taken to a fixed base $b > 1$. Equality holds in (3.1) iff $q_i = p_i$ for all i. For details see [128, p.635-650]. This result, sometimes called the fundamental lemma of information theory, has extensive applications (see for example [124]).

Matić et al. [118, 119, 120, 122] continuously worked on Shannon's inequality and related inequalities in the probability distribution and information science. They studied and discussed in [120, 122] several aspects of Shannon's inequality in discrete as well as in integral forms, by presenting upper estimates of the difference between its two sides. In [120, 122], they considered a discrete-valued random variable X with finite range $\{x_i\}_{i=1}^{r}$. Assume $p_i = P\{X = x_i\}$. The **b-entropy** of X is defined by

$$H_b(X) := \sum_{i=1}^{r} p_i \log(1/p_i). \tag{3.3}$$

In [120], they proved that

$$H_b(X) \leq \log r, \tag{3.4}$$

which shows that the entropy function $H_b(X)$ achieves its maximum value on the discrete uniform probability distribution.

They introduced the idea by giving the general setting of the above inequality by using Majorization theorem for the function $f(x) = x \log x$ which is convex and continuous on $\mathbb{R}_+$. Suppose X and Y are discrete random variables with finite ranges and probability distributions $\mathbf{p} = \{p_i\}_{i=1}^{r}$ and $\mathbf{q} = \{q_i\}_{i=1}^{r}$ $(\sum_{i=1}^{r} p_i = \sum_{i=1}^{r} q_i = 1)$ such that $\mathbf{p} \succ \mathbf{q}$. Then by the majorization theorem

$$H_b(X) \leq H_b(Y). \tag{3.5}$$

By substituting $\mathbf{p} = (1/r, \ldots, 1/r)$ we get (3.4).

It is generally common to take log with base of 2 in the introduced notions, but in our investigations this is not essential.

3.1 Majorization, Csiszár Divergence and Zipf-Mandelbrot Law in Discrete Case

In this section, we give some generalized results for majorization inequality by using the following Csiszár f-divergence functional. Upplying f-divergence to some special convex functions it reduces to the results for majorization inequality in the form of Shannon entropy and the Kullback-Leibler divergence. We give several applications by using the Zipf-Mandelbrot law which we introduce in the sequel.

Csiszár introduced in [64] and then discussed in [63] the following notion.

Definition 3.1 *(**Csiszár f-divergence functional**) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function, and let $\mathbf{p} := (p_1, \ldots, p_n)$ and $\mathbf{q} := (q_1, \ldots, q_n)$ be positive probability distributions. The f-divergence functional is*

$$I_f(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right).$$

It is possible to use non-negative probability distributions in the f-divergence functional, by defining

$$f(0) := \lim_{t \to 0^+} f(t); \quad 0f\left(\frac{0}{0}\right) := 0; \quad 0f\left(\frac{a}{0}\right) := \lim_{t \to 0^+} tf\left(\frac{a}{t}\right), \quad a > 0.$$

Horváth et al. [83, p.3] considered functional based on the previous definition.

Definition 3.2 *Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function. Let $\mathbf{p} := (p_1, \ldots, p_n) \in \mathbb{R}^n$, and $\mathbf{q} := (q_1, \ldots, q_n) \in \,]0, \infty[^n$ be such that*

$$\frac{p_i}{q_i} \in J, \quad i = 1, \ldots, n. \tag{3.6}$$

Then we denote

$$\hat{I}_f(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right).$$

Motivated by the ideas in [120] and [122], in the following part we study and discuss the majorization results in the form of divergences and entropies. The following theorem is a generalization of the result given in [120] i.e., (3.5).
Assume $\mathbf{p}$ and $\mathbf{q}$ be n-tuples, then we define

$$\frac{\mathbf{p}}{\mathbf{q}} := \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}\right).$$

The following theorem is the connection between Csiszár f-divergence and weighted majorization inequality when one of sequence is monotonic.

Theorem 3.1 *Assume $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous convex function, p_i, r_i $(i = 1,\ldots,n)$ be real numbers and q_i $(i = 1,\ldots,n)$ be positive real numbers such that*

$$\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} p_i \quad \text{for } k = 1,\ldots,n-1, \tag{3.7}$$

and

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} p_i, \tag{3.8}$$

with $\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J$ $(i = 1,\ldots,n)$.

(a) If $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing, then

$$\hat{I}_f(\mathbf{r},\mathbf{q}) \leq \hat{I}_f(\mathbf{p},\mathbf{q}). \tag{3.9}$$

(b) If $\frac{\mathbf{p}}{\mathbf{q}}$ is increasing, then

$$\hat{I}_f(\mathbf{r},\mathbf{q}) \geq \hat{I}_f(\mathbf{p},\mathbf{q}). \tag{3.10}$$

If f is a continuous concave function, then the reverse inequalities hold in (3.9) and (3.10).

Proof. (a): We use Theorem 1.15 (a) with substitutions $x_i := \frac{p_i}{q_i}$, $y_i := \frac{r_i}{q_i}$, $w_i := q_i$ as $q_i > 0$, $(i = 1,\ldots,n)$ then we get (3.9).
We can prove part (b) with the similar substitutions in Theorem 1.15 (b). □

Theorem 3.2 *Assume $J \subset \mathbb{R}$ be an interval, $g : J \to \mathbb{R}$ be a function such that $x \to xg(x)$ $(x \in J)$ be a continuous convex function, p_i, r_i $(i = 1,\ldots,n)$ be real numbers and q_i $(i = 1,\ldots,n)$ be positive real numbers satisfying (3.7) and (3.8) with*

$$\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J \quad (i = 1,\ldots,n).$$

(a) If $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing, then

$$\hat{I}_g(\mathbf{r},\mathbf{q}) := \sum_{i=1}^{n} r_i g\left(\frac{r_i}{q_i}\right) \leq \hat{I}_g(\mathbf{p},\mathbf{q}). \tag{3.11}$$

(b) If $\frac{\mathbf{p}}{\mathbf{q}}$ is increasing, then

$$\hat{I}_g(\mathbf{r},\mathbf{q}) \geq \hat{I}_g(\mathbf{p},\mathbf{q}). \tag{3.12}$$

If $xg(x)$ is a continuous concave function, then the reverse inequalities hold in (3.11) and (3.12).

Proof. (a): We use Theorem 1.15 (a) with substitutions $x_i = \frac{p_i}{q_i}$, $y_i = \frac{r_i}{q_i}$, $w_i = q_i$ as $q_i > 0$, $(i = 1,\ldots,n)$ and $f(x) := xg(x)$ then we get (3.11).
We can prove part (b) with the similar substitutions in Theorem 1.15 (b) for $f(x) := xg(x)$. □

The theory of majorization and the notion of entropic measure of disorder are closely related. Based on this fact, the aim of the following results is to look for majorization relations with the connection to entropic inequalities. This was interesting to do for two main reasons. The first one is the fact that the majorization relations are usually stronger than the entropic inequalities, in the sense that they imply these entropic inequalities, but that the converse is not true. The second reason is the fact that when we dispose of majorization relations between two different quantum states, we know that we can transform one of the states into the other using some unitary transformation. The concept of entropy alone would not allow us to prove such a property.
The Shannon entropy was introduced by Shannon himself in the field of classical information. There are two ways of viewing the Shannon entropy. Suppose we have a random variable X, and we learn its value. In one point of view, the Shannon entropy quantifies the amount of information about the value of X (after measurement). In another point of view, the Shannon entropy tells us the amount of uncertainty about the variable of X before we learn its value (before measurement).
We mention two special cases of the previous result.
The first case corresponds to the entropy of a discrete probability distribution.

Definition 3.3 ***(Shannon's entropy)*** *of a positive probability distribution* $\mathbf{p} := (p_1, \dots, p_n)$ *is defined by*

$$H(p) := -\sum_{i=1}^{n} p_i \log p_i. \tag{3.13}$$

Note that there is no problem with the definition in the case of a zero probability, since

$$\lim_{x \to 0} x \log x = 0. \tag{3.14}$$

Corollary 3.1 *Assume p_i, r_i and q_i $(i = 1, \dots, n)$ be positive real numbers satisfying (3.7) and (3.8) with*

$$\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J \;\; (i = 1, \dots, n).$$

(a) If $\frac{\mathbf{r}}{\mathbf{q}}$ is a decreasing n-tuple and the base of log *is greater than 1, then the following estimates for the Shannon entropy of* $\mathbf{q}$ *hold*

$$\sum_{i=1}^{n} q_i \log(\frac{r_i}{q_i}) \geq H(\mathbf{q}). \tag{3.15}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.15).*

(b) If $\frac{\mathbf{p}}{\mathbf{q}}$ is an increasing n-tuple and the base of log *is greater than 1, then the following estimates for the Shannon entropy of* $\mathbf{q}$ *hold*

$$H(\mathbf{q}) \leq \sum_{i=1}^{n} q_i \log(\frac{p_i}{q_i}). \tag{3.16}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.16).*

Proof. (a): Substitute $f(x) := \log x$ and $p_i = 1$, $(i = 1,\ldots,n)$ in Theorem 3.1 (a) then we get (3.15).
We can prove the part (b) with the similar substitutions for $r_i = 1$, $(i = 1,\ldots,n)$. □

Corollary 3.2 *Assume p_i and r_i $(i = 1,\ldots,n)$ be positive real numbers satisfying (3.7) and (3.8).*

(a) If $\mathbf{r}$ is a decreasing n-tuple and the base of log *is greater than 1, then the connection between Shannon entropies of $\mathbf{p}$ and $\mathbf{r}$*

$$H(\mathbf{r}) \geq H(\mathbf{p}). \tag{3.17}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.17).*

(b) If $\mathbf{p}$ is an increasing n-tuple and the base of log *is greater than 1, then the connection between Shannon entropies of $\mathbf{p}$ and $\mathbf{r}$*

$$H(\mathbf{r}) \leq H(\mathbf{p}). \tag{3.18}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.18).*

Proof. (a): Substitute $g(x) := \log x$ and $q_i = 1$, $(i = 1,\ldots,n)$ in Theorem 3.2 (a) then we get (3.17).
We can prove part (b) with the similar substitutions. □

The second case corresponds to the relative entropy or the Kullback-Leibler divergence between two probability distributions:

Definition 3.4 ***(Kullback-Leibler divergence)*** *(K-L divergence) divergence between the positive probability distributions $\mathbf{p} := (p_1,\ldots,p_n)$ and $\mathbf{q} := (q_1,\ldots,q_n)$ is defined by*

$$L(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

Corollary 3.3 *Assume $J \subset \mathbb{R}$ be an interval, p_i, r_i and $q_i\,(i = 1,\ldots,n)$ be positive real numbers satisfying (3.7) and (3.8) with*

$$\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J \;\; (i = 1,\ldots,n).$$

(a) If $\frac{\mathbf{r}}{\mathbf{q}}$ is a decreasing n-tuple and the base of log *is greater than 1, then*

$$\sum_{i=1}^{n} q_i \log\left(\frac{r_i}{q_i}\right) \geq \sum_{i=1}^{n} q_i \log\left(\frac{p_i}{q_i}\right). \tag{3.19}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.19).*

(b) If $\frac{\mathbf{p}}{\mathbf{q}}$ is an increasing n-tuple and the base of log *is greater than 1, then*

$$\sum_{i=1}^{n} q_i \log\left(\frac{r_i}{q_i}\right) \le \sum_{i=1}^{n} q_i \log\left(\frac{p_i}{q_i}\right). \tag{3.20}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.20).*

Proof. (a): Substitute $f(x) := \log x$ in Theorem 3.1 (a) then we get (3.19).
We can prove part (b) with substitution $f(x) := \log x$ in Theorem 3.1 (b). □

Corollary 3.4 *Let $J \subset \mathbb{R}$ be an interval and assume p_i, r_i and q_i $(i = 1,\ldots,n)$ be positive real numbers satisfying (3.7) and (3.8) with*

$$\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J \ \ (i = 1,\ldots,n).$$

(a) If $\frac{\mathbf{r}}{\mathbf{q}}$ is a decreasing n-tuple and the base of log *is greater than 1, then the following comparison inequality between K-L divergence of $(\mathbf{r},\mathbf{q})$ and $(\mathbf{p},\mathbf{q})$ holds*

$$L(\mathbf{r},\mathbf{q}) := \sum_{i=1}^{n} r_i \log\left(\frac{r_i}{q_i}\right) \le L(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right). \tag{3.21}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.21).*

(b) If $\frac{\mathbf{p}}{\mathbf{q}}$ is an increasing n-tuple and the base of log *is greater than 1, then the following comparison inequality between K-L divergence of $(\mathbf{r},\mathbf{q})$ and $(\mathbf{p},\mathbf{q})$ holds*

$$\sum_{i=1}^{n} r_i \log\left(\frac{r_i}{q_i}\right) \ge \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right). \tag{3.22}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.22).*

Proof. (a): Substitute $g(x) := \log x$ in Theorem 3.2 (a) then we get (3.21).
We can prove part (b) with substitution $g(x) := \log x$ in Theorem 3.2 (b). □

Remark 3.1 *We give the above results when one sequence is monotone by using Theorem 1.15, but we can give all the above results when both sequences are monotone via using the Weighted Majorization Theorem 1.14 for $w_i > 0$ $(i = 1,\ldots,n)$.*

Next we introduce the concept of the Zipf-Mandelbrot law and give several applications by using it.
The term Zipfian distribution refers to a distribution of probabilities of occurrence that follows the Zipf's Law. Zipf's law is an experimential law, not a theoretical one; i.e. it describes an occurrence rather than predicting it from some kind of theory. The observation that, in many natural and man-made phenomena, the probability of occurrence of many random items starts high and tapers off. Thus, a few occur very often while many others occur rarely. The formal definition of this law is: $\mathbf{P_n} = 1/\mathbf{n^a}$, where $\mathbf{P_n}$ is the frequency of occurrence of the **n**th ranked item and **a** is closed to 1.

Applied to language, this means that the rank of a word (in terms of its frequency) is approximately inversely proportional to its actual frequency, and so produces a hyperbolic distribution. To put the George Zipf's Law in another way (see [1, 160]): $fr = C$, where $r =$ the rank of a word, $f =$ the frequency of occurrence of that word, and $C =$ a constant (the value of which depends on the subject under consideration). Essentially this shows an inverse proportional relationship between a word's frequency and its frequency rank. Zipf calls this curve the 'standard curve'. Texts from natural languages do not, of course, behave with such absolute mathematical precision. They can not, because, for one thing, any curve representing empirical data from large texts will be a stepped graph, since many non-high-frequency words will share the same frequency. But the overall consensus is that texts match the standard curve significantly well. Li [112] writes "this distribution, also called the Zipf's law, has been checked for accuracy for the standard corpus of the present-day English [Kučera and Francis] with very good results." See Miller [127] for a concise summary of the match between actual data and the standard curve.
Zipf also studied the relationship between the frequency of occurrence of a word and its length. In The Psycho-Biology of Language, he stated that "it seems reasonably clear that shorter words are distinctly more favoured in language other than words."
Apart from the use of this law in information science and linguistics, Zipf's law is used in economics. This distribution in economics is known as Pareto's law which analyze the distribution of the wealthiest members of the community [74, p.125] . These two laws are the same in the mathematical sense, but they are applied in a different context [71, p.294]. The same type of distribution that we have in the Zipf's and Pareto's law, also known as the Power law, can be also found in other scientific disciplines, such as: physics, biology, earth and planetary sciences, computer science, demography and the social sciences [132]. Benoit Mandelbrot in [116] gave generalization of Zipf's law, now known as **the Zipf-Mandelbrot law** which gave improvement in account for the low-rank words in corpus where $k < 100$ [130]:

$$f(k) = \frac{C}{(k+q)^s},$$

when $q = 0$, we get **Zipf's law**.
For $n \in \mathbb{N}$, $q \geq 0$, $s > 0$, $k \in \{1,2,\ldots,n\}$, in a more clear form, the Zipf-Mandelbrot law (probability mass function) is defined with

$$f(k,n,q,s) := \frac{1/(k+q)^s}{H_{n,q,s}}, \tag{3.23}$$

where,

$$H_{n,q,s} := \sum_{i=1}^{n} \frac{1}{(i+q)^s}, \tag{3.24}$$

$n \in \mathbb{N}$, $q \geq 0$, $s > 0$, $k \in \{1,2,\ldots,n\}$.
Application of the Zipf-Mandelbrot law can also be found in linguistics [130], information sciences [71, 160] and ecological field studies [131].
In probability theory and statistics, **the cumulative distribution function** of a real-valued

random variable X, or just distribution function of X, evaluated at x, is the probability that X will take a value less than or equal to x and we often denote **CDF** as the following ratio:

$$CDF := \frac{H_{k,t,s}}{H_{n,t,s}}. \tag{3.25}$$

The cumulative distribution function is an important application of majorization.
In the case of a continuous distribution, it gives the area under the probability distribution functions are also used to specify the distribution of multivariable random variables.
There are various applications of CDF, for example, in learning to rank, the cumulative distribution function (CDF) arises naturally as a probability measure over inequality events of the type $\{X \leq x\}$. The joint CDF lends itself to problems that are easily described in terms of inequality events in which statistical dependence relationships also among events. Examples of this type of problem include web search and document retrieval [51, 56, 87, 173], predicting rating of movies [150] or predicting multiplayer game outcomes with a team structure [81]. In contrast to the canonical problems of classification or regression, in learning to rank we are required to learn some mapping from inputs to inter-dependent output variables so that we may wish to model both stochastic orderings of variable states that statistical dependence relationships between variables.
In the following application, we use two the Zipf-Mandelbrot laws for different parameters.

Corollary 3.5 *Assume* $\mathbf{p}$ *and* $\mathbf{r}$ *be the Zipf-Mandelbrot laws with parameters* $n \in \{1,2,\ldots\}$, $t_1, t_2 \geq 0$ *and* $s_1, s_2 > 0$ *respectively satisfying*

$$\frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \leq \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1,\ldots,n-1, \tag{3.26}$$

and also let $q_i > 0$ $(i = 1,2,\ldots,n)$.

(a) *If* $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \log\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \leq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \log\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \tag{3.27}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.27).*

(b) *If* $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \log\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \log\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \tag{3.28}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.28).*

Proof. (a) Assume $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $r_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, then

$$\sum_{i=1}^{k} p_i := \sum_{i=1}^{k} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} = \frac{1}{H_{n,t_1,s_1}} \sum_{i=1}^{k} \frac{1}{(i+t_1)^{s_1}} = \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}},\ k = 1,\dots,n-1,$$

similarly $\sum_{i=1}^{k} r_i := \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}},\ k = 1,\dots,n-1$.
This implies that

$$\sum_{i=1}^{k} r_i \le \sum_{i=1}^{k} p_i \quad \Leftrightarrow \quad \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \le \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}},\quad k = 1,\dots,n-1.$$

We can easily check that $\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ is decreasing over $i = 1,\dots,n$ and similarly r_i too. Now, we investigate the behaviour of $\frac{\mathbf{r}}{\mathbf{q}}$ for $q_i > 0$ $(i = 1,2,\dots,n)$, take

$$\frac{r_i}{q_i} = \frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \quad \textit{and} \quad \frac{r_{i+1}}{q_{i+1}} = \frac{1}{q_{i+1}(i+1+t_2)^{s_2} H_{n,t_2,s_2}},$$

$$\frac{r_{i+1}}{q_{i+1}} - \frac{r_i}{q_i} = \frac{1}{H_{n,t_2,s_2}} \left[\frac{1}{q_{i+1}(i+1+t_2)^{s_2}} - \frac{1}{q_i (i+t_2)^{s_2}} \right] \le 0,$$

$$\Leftrightarrow \frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \le \frac{q_{i+1}}{q_i},$$

which shows that $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing. So, all the assumptions of Corollary 3.4 (a) are true, then by using (3.21) we get (3.27).
(b) If we switch the role of r_i into p_i, then by using (3.22) in Corollary 3.4 (b) we get (3.28). □

The following application is a special case of the above result.

Corollary 3.6 *Assume* $\mathbf{p}$ *and* $\mathbf{r}$ *be the Zipf-Mandelbrot laws with parameters* $n \in \{1,2,\dots\}$, $t_1, t_2 \ge 0$ *and* $s_1, s_2 > 0$ *respectively satisfying (3.26).*
If the base of log *is greater than 1, then*

$$\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \log\left(\frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \right) \le \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \log\left(\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \right). \tag{3.29}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.29).*

Proof. Substitute $q_i := 1$ $(i = 1,2,\dots,n)$ in (3.27) we get (3.29). □

Corollary 3.7 *Assume* $\mathbf{p}$ *and* $\mathbf{r}$ *be the Zipf-Mandelbrot laws with parameters* $n \in \{1,2,\ldots\}$, $t_1,t_2 \geq 0$ *and* $s_1,s_2 > 0$ *respectively satisfying (3.26) and also let* $q_i > 0$ $(i = 1,2,\ldots,n)$.

(a) *If* $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \geq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i(i+t_1)^{s_1} H_{1,t_1,s_1}}\right). \tag{3.30}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.30).*

(b) *If* $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i(i+t_1)^{s_1} H_{1,t_1,s_1}}\right). \tag{3.31}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.31).*

Proof. We can prove by the similar method as given in Corollary 3.5 with substitutions $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $r_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$ in Corollary 3.3 instead of Corollary 3.4 we get the required results. □

The following result is a special case of the previous corollary.

Corollary 3.8 *Assume* $\mathbf{p}$ *and* $\mathbf{r}$ *be the Zipf-Mandelbrot laws with parameters* $n \in \{1,2,\ldots\}$, $t_1,t_2 \geq 0$ *and* $s_1,s_2 > 0$ *respectively satisfying (3.26).*
If the base of log *is greater than 1, then*

$$\sum_{i=1}^{n} \log\left(\frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \geq \sum_{i=1}^{n} \log\left(\frac{1}{(i+t_1)^{s_1} H_{1,t_1,s_1}}\right). \tag{3.32}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.32).*

Proof. Substitute $q_i := 1$ $(i = 1,2,\ldots,n)$ in (3.30) we get (3.32). □

Corollary 3.9 *Assume* $\mathbf{p}$ *and* $\mathbf{r}$ *be the Zipf-Mandelbrot laws with parameters* $n \in \{1,2,\ldots\}$, $t_1,t_2 \geq 0$ *and* $s_1,s_2 > 0$ *respectively satisfying (3.26) and also let* $q_i > 0$ $(i = 1,2,\ldots,n)$.

(a) *If* $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \geq H(\mathbf{q}). \tag{3.33}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.33).*

(b) If $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, \ldots, n)$ *and the base of* log *is greater than 1, then*

$$H(\mathbf{q}) \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \tag{3.34}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.34).*

Proof. (a) We can prove by the similar method as given in Corollary 3.5 with substitutions $p_i := 1$ and $r_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$ in Corollary 3.1 (a) instead of Corollary 3.4 (a) we get (3.33). (b) For this part switch the role of $\mathbf{p}$ and $\mathbf{r}$ in part (a) like $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $r_i := 1$ $(i = 1, 2, \ldots, n)$ and applying Corollary 3.1 (b) instead of Corollary 3.4 (b) we get (3.34). □

At the end, in the following application, we use three the Zipf-Mandelbrot laws for different parameters:

Corollary 3.10 *Assume* $\mathbf{p}$*,* $\mathbf{q}$ *and* $\mathbf{r}$ *be the Zipf-Mandelbrot laws with parameters* $n \in \{1, 2, \ldots\}$*,* $t_1, t_2, t_3 \geq 0$ *and* $s_1, s_2, s_3 > 0$ *respectively satisfying (3.26).*

(a) If $\frac{(i+1+t_2)^{s_2}}{(i+1+t_3)^{s_3}} \leq \frac{(i+t_2)^{s_2}}{(i+t_3)^{s_3}}$ $(i = 1, \ldots, n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \log\left(\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_3)^{s_3} H_{n,t_3,s_3}}\right) \leq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \log\left(\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \tag{3.35}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.35).*

(b) If $\frac{(i+1+t_2)^{s_2}}{(i+1+t_3)^{s_3}} \geq \frac{(i+t_2)^{s_2}}{(i+t_3)^{s_3}}$ $(i = 1, \ldots, n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \log\left(\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_3)^{s_3} H_{n,t_3,s_3}}\right) \geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \log\left(\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \tag{3.36}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.36).*

Proof. (a) Let $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$, $q_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$ and $r_i := \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$, here p_i, q_i and r_i are decreasing over $i = 1, \ldots, n$. Now, we investigate the behaviour of $\frac{\mathbf{r}}{\mathbf{q}}$. Take

$$\frac{r_i}{q_i} = \frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \quad \textit{and} \quad \frac{r_{i+1}}{q_{i+1}} = \frac{(i+1+t_2)^{s_2} H_{n,t_2,s_2}}{(i+1+t_3)^{s_3} H_{n,t_3,s_3}},$$

$$\frac{r_{i+1}}{q_{i+1}} - \frac{r_i}{q_i} = \frac{(i+1+t_2)^{s_2} H_{n,t_2,s_2}}{(i+1+t_3)^{s_3} H_{n,t_3,s_3}} - \frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_3)^{s_3} H_{n,t_3,s_3}},$$

$$\frac{r_{i+1}}{q_{i+1}} - \frac{r_i}{q_i} = \frac{H_{n,t_2,s_2}}{H_{n,t_3,s_3}} \left[\frac{(i+1+t_2)^{s_2}}{(i+1+t_3)^{s_3}} - \frac{(i+t_2)^{s_2}}{(i+t_3)^{s_3}} \right],$$

the right hand side is non-positive by using the assumption, which shows that $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing, therefore using Corollary 3.4 (a) we get (3.35).
(b) If we replace $\frac{\mathbf{r}}{\mathbf{q}}$ with $\frac{\mathbf{p}}{\mathbf{q}}$ in the part (a) and using Corollary 3.4 (b), we get (3.36). □

3.2 Majorization and Csiszár Divergence in Integral Case

In this section, we give the generalized results for majorization inequality in integral form by using integral Csiszár f-divergence which we introduce in the sequel. We also give Shannon entropy and the Kullback-Leibler divergence for obtained results. As applications, we present the majorization inequality for various distances like variational distance, Hellinger distance, χ^2-divergence, Bhattacharyya distance, Harmonic distance, Jeffreys distance and triangular discrimination which obtain by applying some special type of convex functions. For more information about different types of divergences see monograph [69].

The following counterpart of the integral Shannon inequality (see also Definition 3.7) was proved in [120, p.505-509]).

Theorem 3.3 *Let I be a measurable subset of the real line and $p(x)$ and $q(x)$ positive integrable functions on I such that $\int_I p(x)dx = 1$ and $\alpha := \int_I q(x)dx < \infty$. Suppose that for $b > 1$ at least one of the integrals*

$$J_p := \int_I p(x) \log \frac{1}{p(x)} dx \quad \text{and} \quad J_q := \int_I p(x) \log \frac{1}{q(x)} dx$$

is finite. If $\int_I \left(p^2(x)/q(x)\right) dx < \infty$, then both J_p and J_q are finite and

$$0 \leq J_q - J_p + \log_b \alpha \leq \log \left[\alpha \int_I \frac{p^2(x)}{q(x)} dx \right] \leq \frac{1}{lnb} \alpha \left[\int_I \frac{p^2(x)}{q(x)} dx - 1 \right],$$

with equality throughout if and only if $q(x) = \alpha p(x)$ a.e. on I.

The notion of entropy $H_b(X) := \sum_{i=1}^{\infty} p_i \log(1/p_i)$ can be extended to the case of a general random variable X, by approximating X by discrete random variables. In the case

when X is non-discrete, $H_b(X)$ is usually infinite. For example, this always happens when X is continuous (see [124, p.38]).
In the case when X is continuous random variable with density $p(x)$ (a nonnegative measurable function on $\mathbb{R}$ such that $\int_{\mathbb{R}} p(x)dx = 1$), we may define the so-called **differential entropy** of X by

$$h_b(X) := \int_{\mathbb{R}} p(x) \log \frac{1}{p(x)} dx \quad (b > 1),$$

whenever the integral exists.
They showed that $h_b(X) \approx \log\left(s\sqrt{2\pi e}\right)$ when the distribution of X is 'close' to the Gaussian distribution with variance s^2. Also, $h_b(x) \approx \log(\mu e)$ if the distribution of X is 'close' to the exponential distribution with mean μ. Finally,

$$h_b(x) \approx \log(l) \tag{3.37}$$

whenever the distribution of X is 'close' to the uniform distribution over an interval of length l.

Csiszár [64, 63] introduced the notion of integral f-divergence as follows.

Definition 3.5 ***(Integral f-divergence)*** *Let $f : (0,\infty) \to (0,\infty)$ be a convex function. Let $p,q : [a,b] \to (0,\infty)$ be positive probability densities. The f-divergence functional is*

$$D_f(p,q) := \int_a^b q(t) f\left(\frac{p(t)}{q(t)}\right) dt.$$

Based on the previous definition, we introduce a new integral functional.

Definition 3.6 *Let $J := [0,\infty) \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function. Let $p,q : [a,b] \to (0,\infty)$ such that*

$$\frac{p(x)}{q(x)} \in J, \quad \forall x \in [a,b].$$

We define

$$\hat{D}_f(p,q) := \int_a^b q(t) f\left(\frac{p(t)}{q(t)}\right) dt.$$

The special case of the above functional, we define

$$\hat{D}_{id_J f}(r,q) := \int_a^b r(t) f\left(\frac{r(t)}{q(t)}\right) dt.$$

Motivated by the ideas in [120] (2000) and [122] (2002), we discuss the behaviour of the results in the form of divergences and entropies. We present the following theorem is the connection between Csiszár f-divergence and weighted majorization inequality in integral form as one function is monotonic. It is generalization of the result (3.37) in integral case given in [120].

Theorem 3.4 *Let $J := [0,\infty) \subset \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ be a convex function. Let also $p,q,r : [a,b] \to (0,\infty)$ such that*

$$\int_a^{\upsilon} r(t)dt \leq \int_a^{\upsilon} p(t)dt, \quad \upsilon \in [a,b] \tag{3.38}$$

and

$$\int_a^b r(t)dt = \int_a^b p(t)dt, \tag{3.39}$$

with

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J, \quad \forall t \in [a,b]. \tag{3.40}$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function on $[a,b]$, then

$$\hat{D}_f(r,q) \leq \hat{D}_f(p,q). \tag{3.41}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function on $[a,b]$, then the inequality is reversed, i.e.

$$\hat{D}_f(r,q) \geq \hat{D}_f(p,q). \tag{3.42}$$

If f is strictly convex function and $p(t) \neq r(t)$ $(a.e.)$, then strict inequality holds in (3.41) *and* (3.42)*.*
If f is concave function then the reverse inequalities hold in (3.41) and (3.42). If f is strictly concave and $p(t) \neq r(t)$ $(a.e.)$, then the strict reverse inequalities hold in (3.41) and (3.42).

Proof. (i): We use Theorem 1.20 (i) with substitutions $x(t) := \frac{p(t)}{q(t)}$, $y(t) := \frac{r(t)}{q(t)}$, $w(t) := q(t) > 0$ $\forall t \in [a,b]$ and $f := f$ and also using the fact that $\frac{r(t)}{q(t)}$ is a decreasing function then we get (3.41).
(ii) We can prove with the similar substitutions as in the first part by using Theorem 1.20 (ii) that is the fact that $\frac{p(t)}{q(t)}$ is an increasing function. □

Theorem 3.5 *Let $J := [0,\infty) \subset \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ be a function such that $x \to xf(x)$, $x \in J$ is a convex function. Let also $p,q,r : [a,b] \to (0,\infty)$ such that satisfying (3.38) and (3.39) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J, \quad \forall t \in [a,b]. \tag{3.43}$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function on $[a,b]$, then

$$\hat{D}_{id_J f}(r,q) \leq \hat{D}_{id_J f}(p,q). \tag{3.44}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function on $[a,b]$, then the inequality is reversed, i.e.

$$\hat{D}_{id_J f}(r,q) \geq \hat{D}_{id_J f}(p,q). \tag{3.45}$$

If $xf(x)$ is strictly convex function and $p(t) \neq r(t)$ $(a.e.)$, then (3.44) *and* (3.45) *are strict.*
If $xf(x)$ is concave function then the reverse inequalities hold in (3.44) and (3.45). If $xf(x)$ is strictly concave and $p(t) \neq r(t)$ $(a.e.)$ then the strict reverse inequalities hold in (3.44) and (3.45).

Proof. (i): We use Theorem 1.20 (i) with substitutions $x(t) := \frac{p(t)}{q(t)}$, $y(t) := \frac{r(t)}{q(t)}$, $w(t) = q(t) > 0, \forall t \in [a,b]$ and $f(x) := xf(x)$ and also using the fact that $\frac{r(t)}{q(t)}$ is decreasing function then we get (3.44).
(ii) We can prove with the similar substitutions as Part (i) in Theorem 1.20 (ii) for $f(x) := xf(x)$ and $\frac{p(t)}{q(t)}$ is an increasing function. □

We mention several special cases of the previous results.
The first case corresponds to the entropy of a continuous probability density (see [120, p.506]):

Definition 3.7 ***(Integral Shannon's entropy)*** *Let $p : [a,b] \to (0,\infty)$ be a positive probability density. The Shannon entropy of $p(x)$ is defined by*

$$H(p(x)) := -\int_a^b p(x) \log p(x) dx \quad (b > 1), \tag{3.46}$$

whenever the integral exists.

Note that there is no problem with the definition in the case of a zero probability, since

$$\lim_{x \to 0} x \log x = 0. \tag{3.47}$$

Corollary 3.11 *Let $p,q,r : [a,b] \to (0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \quad \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function and the base of log *is greater than 1, then we have estimates for the Shannon entropy of $q(t)$*

$$\int_a^b q(t) \log\left(\frac{r(t)}{q(t)}\right) \geq H(q(t)). \tag{3.48}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.48).*

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function and the base of log *is greater than 1, then we have estimates for the Shannon entropy of $q(t)$*

$$H(q(t)) \leq \int_a^b q(t) \log\left(\frac{p(t)}{q(t)}\right). \tag{3.49}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.49).*

Proof. (i): Substitute $f(x) := -\log x$ and $p(t) := 1, \forall t \in [a,b]$ in Theorem 3.4 (i) then we get (3.48).
(ii) We can prove by switching the role of $p(t)$ with $r(t)$ i.e., $r(t) := 1 \forall t \in [a,b]$ and $f(x) := -\log x$ in Theorem 3.4 (ii) then we get (3.49). □

Corollary 3.12 *Let $p, r : [a,b] \to (0,\infty)$ be functions such that satisfying (3.38).*

(i) If $r(t)$ is a decreasing function and the base of log *is greater than 1, then the following comparison inequality between Shannon entropies of $p(t)$ and $r(t)$*

$$H(r(t)) \geq H(p(t)). \tag{3.50}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.50).*

(ii) If $p(t)$ is an increasing function and the base of log *is greater than 1, then the following comparison inequality between Shannon entropies of $p(t)$ and $r(t)$*

$$H(r(t)) \leq H(p(t)). \tag{3.51}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.51).*

Proof. (i): Consider the function $f(x) := \log x$. Then the function $xf(x) := x \log x$ is a convex function. Substitute $f(x) := \log x$ and $q(t) := 1, \forall t \in [a,b]$ in Theorem 3.5 (i) then we get (3.50).
(ii) We can prove with the similar substitutions as Part (i) in Theorem 3.5 (ii) then we get (3.51). □

The second case corresponds to the relative entropy or Kullback-Leibler divergence between two probability densities:

Definition 3.8 ***(Integral Kullback-Leibler divergence)*** *Let $p, q : [a,b] \to (0,\infty)$ be a positive probability densities. The Kullback-Leibler (K-L) divergence between $p(t)$ and $q(t)$ is defined by*

$$L(p(t), q(t)) := \int_a^b p(t) \log\left(\frac{p(t)}{q(t)}\right) dt.$$

Corollary 3.13 *Let $p, q, r : [a,b] \to (0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function and the base of log *is greater than 1, then*

$$\hat{D}_{\log x}(r,q) \geq \hat{D}_{\log x}(p,q). \tag{3.52}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.52).*

(ii) If $\frac{p(t)}{q(t)}$ *is an increasing function and the base of* log *is greater than 1, then*

$$\hat{D}_{\log x}(r,q) \leq \hat{D}_{\log x}(p,q). \tag{3.53}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.53).*

Proof. (i): Substitute $f(x) := -\log x$ in Theorem 3.4 (i) then we get (3.52).
(ii) We can prove with substitution $f(x) := -\log x$ in Theorem 3.4 (ii). □

Corollary 3.14 *Let* $p,q,r : [a,b] \to (0,\infty)$ *be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ *is a decreasing function and the base of* log *is greater than 1, then the connection between K-L divergence of* $(r(t),q(t))$ *and* $(p(t),q(t))$

$$L(r(t),q(t)) \leq L(p(t),q(t)). \tag{3.54}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.54).*

(ii) If $\frac{p(t)}{q(t)}$ *is an increasing function and the base of* log *is greater than 1, then the connection between K-L divergence of* $(r(t),q(t))$ *and* $(p(t),q(t))$

$$L(r(t),q(t)) \geq L(p(t),q(t)). \tag{3.55}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.55).*

Proof. (i): Substitute $f(x) := \log x$ in Theorem 3.5 (i) then we get (3.54).
(ii) We can prove with substitution $f(x) := \log x$ in Theorem 3.5 (ii) we get (3.55). □

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence.

Definition 3.9 ***(Variational distance)*** *Let* $p,q : [a,b] \to (0,\infty)$ *be a positive probability densities. The variation distance between* $p(t)$ *and* $q(t)$ *is defined by*

$$\hat{D}_v(p(t),q(t)) := \int_a^b |p(t)-q(t)|dt.$$

Corollary 3.15 *Let* $p,q,r : [a,b] \to (0,\infty)$ *be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ *is a decreasing function, then*

$$\hat{D}_v(r(t),q(t)) \leq \hat{D}_v(p(t),q(t)). \tag{3.56}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{D}_v(r(t),q(t)) \geq \hat{D}_v(p(t),q(t)). \tag{3.57}$$

Proof. (i): Since $f(x) := |x-1|$ be a convex function for $x \in \mathbb{R}^+$, therefore substitute $f(x) := |x-1|$ in Theorem 3.4 (i) then

$$\int_a^b q(t)\left|\frac{r(t)}{q(t)}-1\right| dt \leq \int_a^b q(t)\left|\frac{p(t)}{q(t)}-1\right| dt,$$

$$\int_a^b q(t)\frac{|r(t)-q(t)|}{|q(t)|}dt \leq \int_a^b q(t)\frac{|p(t)-q(t)|}{|q(t)|}dt,$$

since $q(t) > 0$ then we get (3.56).
(ii) We can prove with substitution $f(x) := |x-1|$ in Theorem 3.4 (ii). □

Definition 3.10 ***(Hellinger distance)*** *Let $p,q : [a,b] \to (0,\infty)$ be a positive probability densities. The Hellinger distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{D}_H(p(t),q(t)) := \int_a^b \left[\sqrt{p(t)}-\sqrt{q(t)}\right]^2 dt.$$

Corollary 3.16 *Let $p,q,r : [a,b] \to (0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{D}_H(r(t),q(t)) \leq \hat{D}_H(r(t),q(t)). \tag{3.58}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{D}_H(r(t),q(t)) \geq \hat{D}_H(r(t),q(t)). \tag{3.59}$$

Proof. (i): Since $f(x) := (\sqrt{x}-1)^2$ be a convex function for $x \in \mathbb{R}^+$, therefore substitute $f(x) := (\sqrt{x}-1)^2$ in Theorem 3.4 (i) then

$$\int_a^b q(t)\left[\sqrt{\frac{r(t)}{q(t)}}-1\right]^2 dt \leq \int_a^b q(t)\left[\sqrt{\frac{p(t)}{q(t)}}-1\right]^2 dt,$$

since $q(t) > 0$ then we get (3.58).
(ii) We can prove with substitution $f(x) := (\sqrt{x}-1)^2$ in Theorem 3.4 (ii). □

Definition 3.11 ***(χ^2-Divergence)*** *Let $p,q : [a,b] \to (0,\infty)$ be a positive probability densities. The χ^2-divergence between $p(t)$ and $q(t)$ is defined by*

$$\hat{D}_{id_J\,\chi^2}(p(t),q(t)) := \int_a^b p(t)\left[\left(\frac{q(t)}{p(t)}\right)^2-1\right] dt.$$

Corollary 3.17 *Let $p,q,r:[a,b]\to(0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)},\frac{r(t)}{q(t)}\in J:=(0,\infty),\ \forall t\in[a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{D}_{id_J\chi^2}\left(r(t),q(t)\right)\le\hat{D}_{id_J\chi^2}\left(p(t),q(t)\right). \tag{3.60}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{D}_{id_J\chi^2}\left(r(t),q(t)\right)\ge\hat{D}_{id_J\chi^2}\left(p(t),q(t)\right). \tag{3.61}$$

Proof. (i): Since $f(x):=x\left(\frac{1}{x^2}-1\right)$ be a convex function for $x\in\mathbb{R}^+$, therefore substitute $f(x):=x\left(\frac{1}{x^2}-1\right)$ in Theorem 3.4 (i) then

$$\int_a^b q(t)\frac{r(t)}{q(t)}\left[\left(\frac{q(t)}{r(t)}\right)^2-1\right]dt\le\int_a^b q(t)\frac{p(t)}{q(t)}\left[\left(\frac{q(t)}{p(t)}\right)^2-1\right]dt,$$

we get (3.60). We can also prove by using Theorem 5 (i) for function $f(x):=\frac{1}{x^2}-1$ such that $xf(x):=x\left(\frac{1}{x^2}-1\right)$ be convex function for $x\in\mathbb{R}^+$, we get (3.60).
(ii) We can prove with substitution $f(x):=x\left(\frac{1}{x^2}-1\right)$ in Theorem 3.4 (ii). □

Definition 3.12 ***(Bhattacharyya distance)*** *Let $p,q:[a,b]\to(0,\infty)$ be a positive probability densities. The Bhattacharyya distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{D}_B\left(p(t),q(t)\right):=\int_a^b\sqrt{p(t)q(t)}dt.$$

Corollary 3.18 *Let $p,q,r:[a,b]\to(0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)},\frac{r(t)}{q(t)}\in J:=(0,\infty),\ \forall t\in[a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{D}_B\left(p(t),q(t)\right)\le\hat{D}_B\left(r(t),q(t)\right). \tag{3.62}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{D}_B\left(p(t),q(t)\right)\ge\hat{D}_B\left(r(t),q(t)\right). \tag{3.63}$$

Proof. (i): Since $f(x) := -\sqrt{x}$ be a convex function for $x \in \mathbb{R}^+$, therefore substitute $f(x) := -\sqrt{x}$ in Theorem 3.4 (i) then

$$\int_a^b q(t)\left(-\sqrt{\frac{r(t)}{q(t)}}\right) dt \leq \int_a^b q(t)\left(-\sqrt{\frac{p(t)}{q(t)}}\right) dt,$$

we get (3.62).
(ii) We can prove with substitution $f(x) := -\sqrt{x}$ in Theorem 3.4 (ii). □

Definition 3.13 ***(Harmonic distance)*** *Let $p,q : [a,b] \to (0,\infty)$ be a positive probability densities. The Harmonic distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{D}_{id_J\,Ha}(p(t),q(t)) := \int_a^b \frac{2p(t)q(t)}{p(t)+q(t)} dt.$$

Corollary 3.19 *Let $p,q,r : [a,b] \to (0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{D}_{id_J\,Ha}(p(t),q(t)) \leq \hat{D}_{id_J\,Ha}(r(t),q(t)). \tag{3.64}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{D}_{id_J\,Ha}(p(t),q(t)) \geq \hat{D}_{id_J\,Ha}(r(t),q(t)). \tag{3.65}$$

Proof. (i): Since $f(x) := \frac{2}{x+1}$, then $xf(x) := \frac{2x}{x+1}$ be a concave function for $x \geq 0$, therefore substitute $f(x) := \frac{2}{x+1}$ in Theorem 3.4 (i) then

$$\int_a^b p(t)\frac{2}{p(t)/q(t)+1} dt \leq \int_a^b r(t)\frac{2}{r(t)/q(t)+1} dt,$$

we get (3.64).
(ii) We can prove with substitution $f(x) := \frac{2}{x+1}$ in Theorem 3.4 (ii). □

Definition 3.14 ***(Jeffreys distance)*** *Let $p,q : [a,b] \to (0,\infty)$ be a positive probability densities. The Jeffreys distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{D}_J(p(t),q(t)) := \int_a^b [p(t)-q(t)] \ln\left[\frac{p(t)}{q(t)}\right] dt.$$

Corollary 3.20 *Let $p,q,r : [a,b] \to (0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{D}_J(r(t),q(t)) \leq \hat{D}_J(p(t),q(t)). \tag{3.66}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{D}_J(r(t),q(t)) \geq \hat{D}_J(p(t),q(t)). \tag{3.67}$$

Proof. (i): Since $f(x) := (x-1)\ln x$ be a convex function for $x \in \mathbb{R}^+$, therefore substitute $f(x) := (x-1)\ln x$ in Theorem 3.4 (i) then

$$\int_a^b q(t)\left(\frac{r(t)}{q(t)}-1\right)\ln\left(\frac{r(t)}{q(t)}\right)dt \leq \int_a^b q(t)\left(\frac{p(t)}{q(t)}-1\right)\ln\left(\frac{p(t)}{q(t)}\right)dt,$$

we get (3.66).
(ii) We can prove with substitution $f(x) := (x-1)\ln x$ in Theorem 3.4 (ii). □

Definition 3.15 ***(Triangular discrimination)*** *Let $p,q:[a,b]\to(0,\infty)$ be a positive probability densities. The triangular discrimination between $p(t)$ and $q(t)$ is defined by*

$$\hat{D}_\Delta(p(t),q(t)) := \int_a^b \frac{[p(t)-q(t)]^2}{p(t)+q(t)}\,dt.$$

Corollary 3.21 *Let $p,q,r:[a,b]\to(0,\infty)$ be functions such that satisfying (3.38) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{D}_\Delta(r(t),q(t)) \leq \hat{D}_\Delta(p(t),q(t)). \tag{3.68}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{D}_\Delta(r(t),q(t)) \geq \hat{D}_\Delta(p(t),q(t)). \tag{3.69}$$

Proof. (i): Since $f(x) := \frac{(x-1)^2}{x+1}$ be a convex function for $x \geq 0$, therefore substitute $f(x) := \frac{(x-1)^2}{x+1}$ in Theorem 3.4 (i) then

$$\int_a^b q(t)\frac{(r(t)/q(t)-1)^2}{r(t)/q(t)+1}\,dt \leq \int_a^b q(t)\frac{(p(t)/q(t)-1)^2}{p(t)/q(t)+1}\,dt,$$

$$\int_a^b q(t)\frac{((r(t)-q(t))/q(t))^2}{(r(t)+q(t))/q(t)}\,dt \leq \int_a^b q(t)\frac{((p(t)-q(t))/q(t))^2}{(p(t)+q(t))/q(t)}\,dt,$$

we get (3.68).
(ii) We can prove with substitution $f(x) := \frac{(x-1)^2}{x+1}$ in Theorem 3.4 (ii). □

3.3 Further Results on Majorization and Zipf-Mandelbrot Law

In this section, motivated by an idea in [120] and [122], we discuss the behavior of the results in the form of divergences, majorization and Zipf-Mandelbrot law. We consider the Csiszár f-divergence for Zipf-Mandelbrot law and to develop several important majorization inequalities via **CDF** as the condition of majorization. We discuss some special cases of our generalized results. We also present several applications of our results by constructing distances in the Zipf-Mandelbrot law i.e., the R*é*nyi α-order entropy for Z-M law, variational distance for Z-M law, the Hellinger discrimination for Z-M law, triangular discrimination for Z-M law and χ^2-distance for Z-M law. At the end, we give important applications of the Zipf's law in linguistics and obtain the bounds for the Kullback-Leibler divergence of the distributions associated to the English and Russian languages.

We introduce the following two definitions of Csiszár divergence [64, 63] for Zipf-Mandelbrot law.

Definition 3.16 ***(Csiszár Divergence for Z-M law)*** *Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function. Let $n \in \{1,2,3,\ldots\}$, $t_1 \geq 0$, $s_1 > 0$ and also let $q_i > 0$ for $(i = 1,\ldots,n)$ such that*

$$\frac{1}{q_i(i+t_1)^{s_1}H_{n,t_1,s_1}} \in J, \quad i = 1,\ldots,n, \tag{3.70}$$

then let

$$\hat{I}_f(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i(i+t_1)^{s_1}H_{n,t_1,s_1}}\right).$$

Definition 3.17 *Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function. Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that*

$$\frac{(i+t_2)^{s_2}H_{n,t_2,s_2}}{(i+t_1)^{s_1}H_{n,t_1,s_1}} \in J, \quad i = 1,\ldots,n, \tag{3.71}$$

then let

$$\widetilde{I}_f(i,n,t_1,t_2,s_1,s_2) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}} f\left(\frac{(i+t_2)^{s_2}H_{n,t_2,s_2}}{(i+t_1)^{s_1}H_{n,t_1,s_1}}\right).$$

Remark 3.2 *It is obvious that the second Csiszár divergence for Zipf-Mandelbrot law is a special case of the first one.*

We present the following theorem is the connection between Csiszár f-divergence, Zipf-Mandelbrot law and weighted majorization inequality.

Theorem 3.6 *Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1,2,3,\ldots\}$, $t_1,t_2,t_3 \geq 0$ and $s_1,s_2,s_3 > 0$ such that satisfying*

$$\frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \leq \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1,\ldots,n-1, \tag{3.72}$$

with

$$\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J, \ \ (i = 1,\ldots,n),$$

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned}\widetilde{I}_f(i,n,t_2,t_3,s_2,s_3) &:= \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq \widetilde{I}_f(i,n,t_1,t_3,s_1,s_3) &:= \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right).\end{aligned} \tag{3.73}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned}&\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\geq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right),\end{aligned} \tag{3.74}$$

If f is continuous concave function, then the reverse inequalities hold in (3.73) and (3.74).

Proof. Let us consider $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $w_i := \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$ for $(i = 1,\ldots,n)$, then

$$\begin{aligned}\sum_{i=1}^{k} w_i x_i :&= \sum_{i=1}^{k} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}} \\ &= \frac{1}{H_{n,t_1,s_1}} \sum_{i=1}^{k} \frac{1}{(i+t_1)^{s_1}} \\ &= \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1,\ldots,n-1,\end{aligned}$$

similarly

$$\sum_{i=1}^{k} w_i y_i := \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}}, \ k = 1,\ldots,n-1.$$

This implies that

$$\sum_{i=1}^{k} w_i y_i \le \sum_{i=1}^{k} w_i x_i \quad \Leftrightarrow \quad \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \le \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \ \ k = 1,\dots,n-1.$$

We can easily check that $\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ is decreasing over $i = 1,\dots,n$ and similarly the others too. Now, we investigate the behaviour of y_i for $(i = 1,2,\dots,n)$, take

$$y_i = \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \quad \textit{and} \quad y_{i+1} = \frac{(i+1+t_3)^{s_3} H_{n,t_3,s_3}}{(i+1+t_2)^{s_2} H_{n,t_2,s_2}},$$

$$y_{i+1} - y_i = \frac{H_{n,t_3,s_3}}{H_{n,t_2,s_2}} \left[\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} - \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}} \right] \le 0,$$

$$\Leftrightarrow \frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \le \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}, \ (i = 1,\dots,n),$$

which shows that y_i is decreasing.
Therefore, substitute $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $w_i := \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$ for $(i = 1,\dots,n)$ and $f := f$ in Theorem 1.15 (a), then we get (3.73).
(b) We can prove part (b) with the similar substitutions as in Part (a) but switch the role of y_i with x_i that is increasing sequence, in Theorem 1.15 (b). □

Theorem 3.7 *Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1,2,3,\dots\}$, $t_1, t_2 \ge 0$ and $s_1, s_2 > 0$ such that satisfying (3.72) and also let $q_i > 0$, $(i = 1,\dots,n)$ with*

$$\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i = 1,\dots,n),$$

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \le \frac{q_{i+1}}{q_i}$ $(i = 1,\dots,n)$, then

$$\begin{aligned} \hat{I}_f(i,n,t_2,s_2,\mathbf{q}) &:= \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \right) \\ &\le \hat{I}_f(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}} \right). \end{aligned} \tag{3.75}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \ge \frac{q_{i+1}}{q_i}$ $(i = 1,\dots,n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \right) \\ &\ge \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}} \right). \end{aligned} \tag{3.76}$$

If f is continuous concave function, then the reverse inequalities hold in (3.75) and (3.76).

Proof. Let us consider $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{q_i}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{q_i}$, and $w_i = q_i > 0$, $(i = 1,\ldots,n)$ then we can get as in the previous proof

$$\sum_{i=1}^{k} w_i y_i \leq \sum_{i=1}^{k} w_i x_i \quad \Leftrightarrow \quad \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \leq \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1,\ldots,n-1.$$

Now, we investigate the behaviour of y_i for $(i = 1,2,\ldots,n)$, take

$$y_i = \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \quad \textit{and} \quad y_{i+1} = \frac{1}{q_{i+1}(i+1+t_2)^{s_2} H_{n,t_2,s_2}},$$

$$y_{i+1} - y_i = \frac{1}{H_{n,t_2,s_2}} \left[\frac{1}{q_{i+1}(i+1+t_2)^{s_2}} - \frac{1}{q_i(i+t_2)^{s_2}} \right] \leq 0,$$

$$\Leftrightarrow \frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}, \ (i = 1,\ldots,n),$$

which shows that y_i is decreasing. Therefore, substitute $x_i := \frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}$, $y_i := \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}$, $w_i = q_i > 0$, $(i = 1,\ldots,n)$ and also $f := f$ in Theorem 1.15, we get (3.75).
(b) If we switch the role of y_i into x_i as increasing sequence in the similar fashion as the proof of Part (a), then by using Theorem 1.15 (b) we get (3.76). □

Corollary 3.22 *Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72) with*

$$\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ (i = 1,\ldots,n),$$

then the following inequality holds

$$\begin{aligned} \hat{I}_f(i,n,t_2,s_2,\mathbf{1}) &:= \sum_{i=1}^{n} f\left(\frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq \hat{I}_f(i,n,t_1,s_1,\mathbf{1}) &:= \sum_{i=1}^{n} f\left(\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.77}$$

If f is continuous concave function, then the reverse inequality hold in (3.77).

Proof. Let us consider $x_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $y_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, we can easily check that $y_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$ is decreasing over $i = 1,\ldots,n$. Therefore, substitute $x_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$, $y_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, $q_i = 1$, $(i = 1,\ldots,n)$ and also $f := f$ in (3.75), we get (3.77). □

Theorem 3.8 *Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ be a function such that $x \to xf(x)$ $(x \in J)$ is a continuous convex function.*

Let $n \in \{1,2,3,\dots\}$, $t_1, t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (3.72) with

$$\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J, \;\; (i = 1,\dots,n),$$

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,\dots,n)$, then

$$\begin{aligned} \widetilde{I}_{id_J f}(i,n,t_2,t_3,s_2,s_3) &:= \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq \widetilde{I}_{id_J f}(i,n,t_1,t_3,s_1,s_3) &:= \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.78}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,\dots,n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\quad \geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.79}$$

If $xf(x)$ is continuous concave function, then the reverse inequalities hold in (3.78) and (3.79).

Proof. Let us substitute $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $w_i := \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$ for $(i = 1,\dots,n)$ in Theorem 3.6 (a) and follow the proof of Theorem 3.7 for function $f(x) := xf(x)$, then we get (3.78).
(b) We can prove part (b) with the similar substitutions as in Part (a) but switch the role of y_i with x_i that is an increasing sequence, in Theorem 1.15 (b) for function $f(x) := xf(x)$. □

Theorem 3.9 *Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ be a function such that $x \to xf(x)$ $(x \in J)$ is a continuous convex function. Let $n \in \{1,2,3,\dots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72) and also let $q_i > 0$ with*

$$\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \;\; (i = 1,\dots,n),$$

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\dots,n)$, then

$$\begin{aligned} \hat{I}_{id_J f}(i,n,t_2,s_2,\mathbf{q}) &:= \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq \hat{I}_{id_J f}(i,n,t_1,s_1,\mathbf{q}) &:= \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.80}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, \ldots, n)$*, then*

$$\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \quad (3.81)$$

If $xf(x)$ is continuous concave function, then the reverse inequalities hold in (3.80) and (3.81).

Proof. Let us consider $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{q_i}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{q_i}$, and $w_i = q_i > 0$, $(i = 1, \ldots, n)$ and also $f(x) := xf(x)$ in Theorem 1.15 (a) by follow the proof of Theorem 3.6 (a), we get (3.80).
(b) If we switch the role of y_i into x_i as an increasing sequence with the similar substitutions as in Part (a), then by using Theorem 1.15 (b) we get (3.81). □

Corollary 3.23 *Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ be a function such that $x \to xf(x)$ $(x \in J)$ is a continuous convex function. Let $n \in \{1, 2, 3, \ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72) with*

$$\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i = 1, \ldots, n),$$

then the following inequality holds

$$\hat{I}_{id_J f}(i, n, t_2, s_2, \mathbf{1}) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \leq \hat{I}_{id_J f}(i, n, t_1, s_1, \mathbf{1}) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \quad (3.82)$$

If $xf(x)$ is continuous concave function, then the reverse inequality hold in (3.82).

Proof. Since $y_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$ is decreasing over $i = 1, \ldots, n$. Therefore, substitute $x_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $y_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, and $q_i = 1$, $(i = 1, \ldots, n)$ and also $f(x) := xf(x)$ in (3.80), we get (3.82).
□

Previous results can be easily applied on the functions $-\log x$ and $\log x$ to get various results for the Kullback-Leibler divergence for the Zipf-Mandelbrot law as given in [106]. Next we give applications to some other known divergences divergences for the Zipf-Mandelbrot law. In the purpose of that, we introduce the following definitions are the R*é*nyi α-order entropy for the Zipf-Madelbrot law.

Definition 3.18 ***(Rényi α-order entropy for Z-M law)*** *If $n \in \{1,2,3,\ldots\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$, $(i = 1,\ldots,n)$, then the Rényi α-order entropy ($\alpha > 1$) for Zipf-Madelbrot law is defined by*

$$\hat{R_\alpha}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} q_i^{\alpha-1}.$$

Definition 3.19 *If $n \in \{1,2,3,\ldots\}$, $t_1,t_2 \geq 0$, $s_1,s_2 > 0$, then for ($\alpha > 1$)*

$$\widetilde{R_\alpha}(i,n,t_1,t_2,s_1,s_2) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{\alpha-1}.$$

Corollary 3.24 *Let $n \in \{1,2,3,\ldots\}$, $t_1,t_2,t_3 \geq 0$ and $s_1,s_2,s_3 > 0$ such that satisfying (3.72),*

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,\ldots,n)$, then the following inequality holds for ($\alpha > 1$)

$$\begin{aligned}\widetilde{R_\alpha}(i,n,t_2,t_3,s_2,s_3) &:= \sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1} \\ \leq \widetilde{R_\alpha}(i,n,t_1,t_3,s_1,s_3) &:= \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1}.\end{aligned} \tag{3.83}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,\ldots,n)$, then the following inequality holds for ($\alpha > 1$)

$$\begin{aligned}&\sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1} \\ &\quad \geq \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1}.\end{aligned} \tag{3.84}$$

Proof. If we choose $f(t) := t^\alpha$, $t \in \mathbb{R}^+$ ($\alpha > 1$), then by using (3.73) we get

$$\begin{aligned}&\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right)^{\alpha} \\ &\quad \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right)^{\alpha},\end{aligned}$$

we get (3.83).
(b) Similarly as Part (a), we can prove (3.84) by using (3.74) and $f(t) := t^\alpha$, $t \in \mathbb{R}^+$ ($\alpha > 1$).
□

Corollary 3.25 *If $n \in \{1,2,3,\ldots\}$, t_1, $t_2 \geq 0$ and $s_1,s_2 > 0$ such that satisfying (3.72) and also $q_i > 0$, $(i = 1,\ldots,n)$,* *(a) if* $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, *then*

$$\begin{aligned}\hat{R_\alpha}(i,n,t_2,s_2,\mathbf{q}) &:= \sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} q_i^{1-\alpha} \\ &\leq \hat{R_\alpha}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} q_i^{1-\alpha}. \end{aligned} \tag{3.85}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, *then*

$$\sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} q_i^{1-\alpha} \geq \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} q_i^{1-\alpha}. \tag{3.86}$$

Proof. If we choose $f(t) := t^\alpha$, $t \in \mathbb{R}^+$ $(\alpha > 1)$, then by using (3.75) we get

$$\sum_{i=1}^{n} q_i \left[\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right]^\alpha \leq \sum_{i=1}^{n} q_i \left[\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right]^\alpha,$$

we get (3.85).
(b) Similarly as Part (a), we can prove (3.86) by using (3.76) and $f(t) := t^\alpha$, $t \in \mathbb{R}^+$ $(\alpha > 1)$.
□

Corollary 3.26 *If* $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ *and* $s_1, s_2 > 0$ *such that satisfying (3.72), then the following inequality holds*

$$\begin{aligned}\hat{R_\alpha}(i,n,t_2,s_2,\mathbf{1}) &:= \sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} \\ &\leq \hat{R_\alpha}(i,n,t_1,s_1,\mathbf{1}) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha}. \end{aligned} \tag{3.87}$$

Proof. If we choose $f(t) := t^\alpha$, $t \in \mathbb{R}^+$ $(\alpha > 1)$, and $q_i := 1$, $(i = 1,\ldots,n)$ in (3.85), then we get (3.87). □

The following definitions are the variational distance for Zipf-Madelbrot law.

Definition 3.20 ***(Variational Distance for Z-M law)*** *If* $n \in \{1,2,3,\ldots\}$, $t_1 \geq 0$, $s_1 > 0$ *and also* $q_i > 0$, $(i = 1,\ldots,n)$, *then the variational distance for Zipf-Mandelbrot law is defined by*

$$\hat{V}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - q_i \right|.$$

Definition 3.21 *If* $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$, *then*

$$\widetilde{V}(i,n,t_1,t_2,s_1,s_2) := \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \right|.$$

Corollary 3.27 *Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (3.72),*

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned}\widetilde{V}(i,n,t_2,t_3,s_2,s_3) &:= \sum_{i=1}^{n}\left|\frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}} - \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\right| \\ &\leq \widetilde{V}(i,n,t_1,t_3,s_1,s_3) := \sum_{i=1}^{n}\left|\frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}} - \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\right|.\end{aligned} \tag{3.88}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,\ldots,n)$, then

$$\sum_{i=1}^{n}\left|\frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}} - \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\right| \geq \sum_{i=1}^{n}\left|\frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}} - \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\right|. \tag{3.89}$$

Proof. If we choose $f(t) := |t-1|$, $t \in \mathbb{R}^+$, then by using (3.73) we get

$$\sum_{i=1}^{n}\frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\left|\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_2)^{s_2}H_{n,t_2,s_2}} - 1\right| \leq \sum_{i=1}^{n}\frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\left|\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_1)^{s_1}H_{n,t_1,s_1}} - 1\right|,$$

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\left|\frac{(i+t_3)^{s_3}H_{n,t_3,s_3} - (i+t_2)^{s_2}H_{n,t_2,s_2}}{(i+t_2)^{s_2}H_{n,t_2,s_2}}\right| \\ &\leq \sum_{i=1}^{n}\frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\left|\frac{(i+t_3)^{s_3}H_{n,t_3,s_3} - (i+t_1)^{s_1}H_{n,t_1,s_1}}{(i+t_1)^{s_1}H_{n,t_1,s_1}}\right|,\end{aligned}$$

since $(i+t_3)^{s_3}H_{n,t_3,s_3} > 0$, we get (3.88).
(b) Similarly as Part (a), we can prove (3.89) by using (3.74) and $f(t) := |t-1|$, $t \in \mathbb{R}^+$. □

Corollary 3.28 *Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72) and also let $q_i > 0$, $(i = 1,\ldots,n)$,*

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned}\hat{V}(i,n,t_2,s_2,\mathbf{q}) &:= \sum_{i=1}^{n}\left|\frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}} - q_i\right| \\ &\leq \hat{V}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n}\left|\frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}} - q_i\right|.\end{aligned} \tag{3.90}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\sum_{i=1}^{n}\left|\frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}} - q_i\right| \geq \sum_{i=1}^{n}\left|\frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}} - q_i\right|. \tag{3.91}$$

Proof. If we choose $f(t) := |t-1|$, $t \in \mathbb{R}^+$, then by using (3.75) we get

$$\sum_{i=1}^{n} q_i \left| \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} - 1 \right| \leq \sum_{i=1}^{n} q_i \left| \frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}} - 1 \right|,$$

$$\sum_{i=1}^{n} q_i \left| \frac{1 - q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \right| \leq \sum_{i=1}^{n} q_i \left| \frac{1 - q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}} \right|,$$

since $q_i > 0$ $(i = 1, \ldots, n)$, we get (3.90).
(b) Similarly as Part (a), we can prove (3.91) by using (3.76) and $f(t) := |t-1|$, $t \in \mathbb{R}^+$. □

Corollary 3.29 *If $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72), then the following inequality hold*

$$\begin{aligned} \hat{V}(i,n,t_2,s_2,\mathbf{1}) &:= \sum_{i=1}^{n} \left| \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - 1 \right| \\ \leq \hat{V}(i,n,t_1,s_1,\mathbf{1}) &:= \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - 1 \right|. \end{aligned} \tag{3.92}$$

Proof. If we choose $f(t) := |t-1|$, $t \in \mathbb{R}^+$ and $q_i := 1$, $(i = 1, \ldots, n)$ in (3.90), then we get (3.92). □

The following definitions are the Hellinger discrimination for Zipf-Madelbrot law.

Definition 3.22 ***(Hellinger Discrimination for Z-M law)*** *If $n \in \{1,2,3,\ldots\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1, \ldots, n)$, then the Hellinger discrimination for Zipf-Mandelbrot law is defined by*

$$\hat{h}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \sqrt{q_i} \right)^2.$$

Definition 3.23 *If $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$, then*

$$\widetilde{h}(i,n,t_1,t_2,s_1,s_2) := \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} \right)^2.$$

Corollary 3.30 *Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (3.72),*

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1, \ldots, n)$, then

$$\begin{aligned} \widetilde{h}(i,n,t_2,t_3,s_2,s_3) &:= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} - \frac{1}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}} \right)^2 \\ \leq \widetilde{h}(i,n,t_1,t_3,s_1,s_3) &:= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \frac{1}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}} \right)^2. \end{aligned} \tag{3.93}$$

(b) *if* $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i=1,\ldots,n)$, *then*

$$\sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_2)^{s_2}H_{n,t_2,s_2}}}-\frac{1}{\sqrt{(i+t_3)^{s_3}H_{n,t_3,s_3}}}\right)^2 \geq \sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_1)^{s_1}H_{n,t_1,s_1}}}-\frac{1}{\sqrt{(i+t_3)^{s_3}H_{n,t_3,s_3}}}\right)^2. \tag{3.94}$$

Proof. If we choose $f(t) := \frac{1}{2}\left(\sqrt{t}-1\right)^2$, $t \in \mathbb{R}^+$, then by using (3.73) we get

$$\sum_{i=1}^{n}\frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\left(\sqrt{\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_2)^{s_2}H_{n,t_2,s_2}}}-1\right)^2 \leq \sum_{i=1}^{n}\frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}\left(\sqrt{\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_1)^{s_1}H_{n,t_1,s_1}}}-1\right)^2,$$

$$\sum_{i=1}^{n}\left(\frac{\sqrt{(i+t_3)^{s_3}H_{n,t_3,s_3}}-\sqrt{(i+t_2)^{s_2}H_{n,t_2,s_2}}}{\sqrt{(i+t_3)^{s_3}H_{n,t_3,s_3}}\sqrt{(i+t_2)^{s_2}H_{n,t_2,s_2}}}\right)^2 \leq \sum_{i=1}^{n}\left(\frac{\sqrt{(i+t_3)^{s_3}H_{n,t_3,s_3}}-\sqrt{(i+t_1)^{s_1}H_{n,t_1,s_1}}}{\sqrt{(i+t_3)^{s_3}H_{n,t_3,s_3}}\sqrt{(i+t_1)^{s_1}H_{n,t_1,s_1}}}\right)^2,$$

we get (3.93).
(b) Similarly as Part (a), we can prove (3.94) by using (3.74) and $f(t) := \frac{1}{2}\left(\sqrt{t}-1\right)^2$, $t \in \mathbb{R}^+$. □

Corollary 3.31 *Let $n \in \{1,2,3,\ldots\}$, $t_1,t_2 \geq 0$ and $s_1,s_2 > 0$ such that satisfying (3.72) and also let $q_i > 0$, $(i=1,\ldots,n)$,*

(a) *if* $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i=1,\ldots,n)$, *then*

$$\hat{h}(i,n,t_2,s_2,\mathbf{q}) := \sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_2)^{s_2}H_{n,t_2,s_2}}}-\sqrt{q_i}\right)^2 \leq h(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_1)^{s_1}H_{n,t_1,s_1}}}-\sqrt{q_i}\right)^2. \tag{3.95}$$

(b) *if* $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i=1,\ldots,n)$, *then*

$$\sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_2)^{s_2}H_{n,t_2,s_2}}}-\sqrt{q_i}\right)^2 \geq \sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_1)^{s_1}H_{n,t_1,s_1}}}-\sqrt{q_i}\right)^2. \tag{3.96}$$

Proof. If we choose $f(t) := \frac{1}{2}\left(\sqrt{t}-1\right)^2$, $t \in \mathbb{R}^+$, then by using (3.75) we get

$$\sum_{i=1}^{n} \frac{q_i}{2}\left(\frac{1}{\sqrt{q_i(i+t_2)^{s_2}H_{n,t_2,s_2}}}-1\right)^2 \leq \sum_{i=1}^{n} \frac{q_i}{2}\left(\frac{1}{\sqrt{q_i(i+t_1)^{s_1}H_{n,t_1,s_1}}}-1\right)^2,$$

$$\sum_{i=1}^{n} q_i \frac{\left(1-\sqrt{q_i(i+t_2)^{s_2}H_{n,t_2,s_2}}\right)^2}{q_i(i+t_2)^{s_2}H_{n,t_2,s_2}} \leq \sum_{i=1}^{n} q_i \frac{\left(1-\sqrt{q_i(i+t_1)^{s_1}H_{n,t_1,s_1}}\right)^2}{q_i(i+t_1)^{s_1}H_{n,t_1,s_1}},$$

we get (3.95).
(b) Similarly as Part (a), we can prove (3.96) by using (3.76) and $f(t) := \frac{1}{2}\left(\sqrt{t}-1\right)^2$, $t \in \mathbb{R}^+$. □

Corollary 3.32 *If $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72), then the following inequality holds*

$$\begin{aligned}\hat{h}(i,n,t_2,s_2,\mathbf{1}) &:= \sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_2)^{s_2}H_{n,t_2,s_2}}}-1\right)^2 \\ \leq \hat{h}(i,n,t_1,s_1,\mathbf{1}) &:= \sum_{i=1}^{n}\left(\frac{1}{\sqrt{(i+t_1)^{s_1}H_{n,t_1,s_1}}}-1\right)^2. \end{aligned} \tag{3.97}$$

Proof. Substitute $q_i = 1$, $(i = 1,\ldots,n)$ in (3.95), we get (3.97). □

The following definitions are the Triangular discrimination for Zipf-Madelbrot law.

Definition 3.24 ***(Triangular Descrimination in Z-M Law)*** *If $n \in \{1,2,3,\ldots\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1,\ldots,n)$, then the Triangular discrimination for Zipf-Mandelbrot law is defined by*

$$\hat{\Delta}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}}\left[\frac{(1-q_i(i+t_1)^{s_1}H_{n,t_1,s_1})^2}{1+q_i(i+t_1)^{s_1}H_{n,t_1,s_1}}\right].$$

Definition 3.25 *If $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$, $s_1\, s_2 > 0$, then the Triangular discrimination for Zipf-Mandelbrot law is defined by*

$$\begin{aligned}\widetilde{\Delta}(i,n,t_1,t_2,s_1,s_2) := \sum_{i=1}^{n} &\frac{1}{[(i+t_1)^{s_1}H_{n,t_1,s_1}][(i+t_2)^{s_2}H_{n,t_2,s_2}]} \\ &\left[\frac{[(i+t_2)^{s_2}H_{n,t_2,s_2}-(i+t_1)^{s_1}H_{n,t_1,s_1}]^2}{(i+t_2)^{s_2}H_{n,t_2,s_2}+(i+t_1)^{s_1}H_{n,t_1,s_1}}\right].\end{aligned}$$

Corollary 3.33 *Let $n \in \{1,2,3,\ldots\}$, $t_1,t_2,t_3 \geq 0$ and $s_1,s_2,s_3 > 0$ such that satisfying (3.72),*

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i=1,\ldots,n)$, then

$$\begin{aligned}
&\widetilde{\Delta}(i,n,t_2,t_3,s_2,s_3)\\
&:= \sum_{i=1}^{n} \frac{1}{\left[(i+t_2)^{s_2}H_{n,t_2,s_2}\right]\left[(i+t_3)^{s_3}H_{n,t_3,s_3}\right]}\left[\frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}-(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}+(i+t_2)^{s_2}H_{n,t_2,s_2}}\right]\\
&\leq \widetilde{\Delta}(i,n,t_1,t_3,s_1,s_3)\\
&:= \frac{1}{\left[(i+t_1)^{s_1}H_{n,t_1,s_1}\right]\left[(i+t_3)^{s_3}H_{n,t_3,s_3}\right]}\left[\frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}-(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}+(i+t_1)^{s_1}H_{n,t_1,s_1}}\right].
\end{aligned} \tag{3.98}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i=1,\ldots,n)$, then

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{\left[(i+t_2)^{s_2}H_{n,t_2,s_2}\right]\left[(i+t_3)^{s_3}H_{n,t_3,s_3}\right]}\left[\frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}-(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}+(i+t_2)^{s_2}H_{n,t_2,s_2}}\right]\\
&\geq \frac{1}{\left[(i+t_1)^{s_1}H_{n,t_1,s_1}\right]\left[(i+t_3)^{s_3}H_{n,t_3,s_3}\right]}\left[\frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}-(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}+(i+t_1)^{s_1}H_{n,t_1,s_1}}\right].
\end{aligned} \tag{3.99}$$

Proof. If we choose $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$, then by using (3.73) we get

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}/(i+t_2)^{s_2}H_{n,t_2,s_2}-1\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}/(i+t_2)^{s_2}H_{n,t_2,s_2}+1}\\
&\leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}/(i+t_1)^{s_1}H_{n,t_1,s_1}-1\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}/(i+t_1)^{s_1}H_{n,t_1,s_1}+1},
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}-(i+t_2)^{s_2}H_{n,t_2,s_2}/(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}+(i+t_2)^{s_2}H_{n,t_2,s_2}/(i+t_2)^{s_2}H_{n,t_2,s_2}}\\
&\leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3}H_{n,t_3,s_3}-(i+t_1)^{s_1}H_{n,t_1,s_1}/(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}{(i+t_3)^{s_3}H_{n,t_3,s_3}+(i+t_1)^{s_1}H_{n,t_1,s_1}/(i+t_1)^{s_1}H_{n,t_1,s_1}},
\end{aligned}$$

we get (3.98).

(b) Similarly as Part (a), we can prove (3.99) by using (3.74) and $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$. □

Corollary 3.34 *Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72) and also let $q_i > 0$, $(i = 1,\ldots,n)$,*

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\hat{\Delta}(i,n,t_2,s_2,\mathbf{q}) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \left[\frac{\left(1 - q_i(i+t_2)^{s_2} H_{n,t_2,s_2}\right)^2}{1 + q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \right]$$
$$\leq \hat{\Delta}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \left[\frac{\left(1 - q_i(i+t_1)^{s_1} H_{n,t_1,s_1}\right)^2}{1 + q_i(i+t_1)^{s_1} H_{n,t_1,s_1}} \right]. \quad (3.100)$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \left[\frac{\left(1 - q_i(i+t_2)^{s_2} H_{n,t_2,s_2}\right)^2}{1 + q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \right]$$
$$\geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \left[\frac{\left(1 - q_i(i+t_1)^{s_1} H_{n,t_1,s_1}\right)^2}{1 + q_i(i+t_1)^{s_1} H_{n,t_1,s_1}} \right]. \quad (3.101)$$

Proof. If we choose $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$, then by using (3.75) we get

$$\sum_{i=1}^{n} q_i \frac{\left(1/q_i(i+t_2)^{s_2} H_{n,t_2,s_2} - 1\right)^2}{1/q_i(i+t_2)^{s_2} H_{n,t_2,s_2} + 1} \leq \sum_{i=1}^{n} q_i \frac{\left(1/q_i(i+t_1)^{s_1} H_{n,t_1,s_1} - 1\right)^2}{1/q_i(i+t_1)^{s_1} H_{n,t_1,s_1} + 1},$$

$$\sum_{i=1}^{n} q_i \frac{\left[1 - q_i(i+t_2)^{s_2} H_{n,t_2,s_2} / q_i(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{1 + q_i(i+t_2)^{s_2} H_{n,t_2,s_2} / q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}$$
$$\leq \sum_{i=1}^{n} q_i \frac{\left[1 - q_i(i+t_1)^{s_1} H_{n,t_1,s_1} / q_i(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{1 + q_i(i+t_1)^{s_1} H_{n,t_1,s_1} / q_i(i+t_1)^{s_1} H_{n,t_1,s_1}},$$

we get (3.100).

(b) Similar way as Part (a), we can prove (3.101) by using (3.76) and $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$. □

Corollary 3.35 *If $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.72), then the following inequality holds*

$$\hat{\Delta}(i,n,t_2,s_2,\mathbf{1}) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \left[\frac{\left(1 - (i+t_2)^{s_2} H_{n,t_2,s_2}\right)^2}{1 + (i+t_2)^{s_2} H_{n,t_2,s_2}} \right]$$
$$\leq \hat{\Delta}(i,n,t_1,s_1,\mathbf{1}) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \left[\frac{\left(1 - (i+t_1)^{s_1} H_{n,t_1,s_1}\right)^2}{1 + (i+t_1)^{s_1} H_{n,t_1,s_1}} \right]. \quad (3.102)$$

Proof. Substitute $q_i = 1$, $(i = 1,\ldots,n)$ in (3.100), we get (3.102). □

The following definitions are the χ^2-distance (chi-square distance) for Zipf-Madelbrot law.

Definition 3.26 *(χ^2-distance for Z-M law) If $n \in \{1,2,3,\ldots\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1,\ldots,n)$, then the χ^2-distance for Zipf-Mandelbrot law is defined by*

$$\hat{\chi^2}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \frac{[1 - q_i(i+t_1)^{s_1} H_{n,t_1,s_1}]^2}{q_i\,[(i+t_1)^{s_1} H_{n,t_1,s_1}]^2}.$$

Definition 3.27 *If $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$, then*

$$\widetilde{\chi^2}(i,n,t_1,t_2,s_1,s_2) := \sum_{i=1}^{n} \frac{[(i+t_2)^{s_2} H_{n,t_2,s_2} - (i+t_1)^{s_1} H_{n,t_1,s_1}]^2}{[(i+t_2)^{s_2} H_{n,t_2,s_2}]\,[(i+t_1)^{s_1} H_{n,t_1,s_1}]^2}.$$

Corollary 3.36 *Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (3.72),*

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned}\widetilde{\chi^2}(i,n,t_2,t_3,s_2,s_3) &:= \sum_{i=1}^{n} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{\left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2 \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]} \\ \leq \widetilde{\chi^2}(i,n,t_1,t_3,s_1,s_3) &:= \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2 \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]}.\end{aligned} \tag{3.103}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,\ldots,n)$, then

$$\sum_{i=1}^{n} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{\left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2 \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]} \geq \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2 \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]}. \tag{3.104}$$

Proof. If we choose $f(t) := (t-1)^2$, $t \in [0,\infty)$, then by using (3.73) we get

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left[\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - 1\right]^2 \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left[\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - 1\right]^2,$$

$$\begin{aligned}&\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{[(i+t_2)^{s_2} H_{n,t_2,s_2}]^2} \\ &\quad \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{[(i+t_1)^{s_1} H_{n,t_1,s_1}]^2},\end{aligned}$$

we get (3.103).
(b) Similarly as Part (a), we can prove (3.104) by using (3.74) and $f(t) := (t-1)^2$, $t \in [0,\infty)$. □

Corollary 3.37 *Let $n \in \{1,2,3,\ldots\}$, $t_1,t_2 \geq 0$ and $s_1,s_2 > 0$ such that satisfying (3.72) and also let $q_i > 0$, $(i = 1,\ldots,n)$,*

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\hat{\chi}^2(i,n,t_2,s_2,\mathbf{q}) := \sum_{i=1}^{n} \frac{\left[1-q_i(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}{q_i\left[(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}$$
$$\leq \hat{\chi}^2(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \frac{\left[1-q_i(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}{q_i\left[(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}. \tag{3.105}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\sum_{i=1}^{n} \frac{\left[1-q_i(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}{q_i\left[(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2} \geq \sum_{i=1}^{n} \frac{\left[1-q_i(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}{q_i\left[(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}. \tag{3.106}$$

Proof. If we choose $f(t) := (t-1)^2$, $t \in [0,\infty)$, then by using (3.75) we get

$$\sum_{i=1}^{n} q_i \left(\frac{1}{q_i(i+t_2)^{s_2}H_{n,t_2,s_2}} - 1\right)^2 \leq \sum_{i=1}^{n} q_i \left(\frac{1}{q_i(i+t_1)^{s_1}H_{n,t_1,s_1}} - 1\right)^2,$$

$$\sum_{i=1}^{n} q_i \left(\frac{1-q_i(i+t_2)^{s_2}H_{n,t_2,s_2}}{q_i(i+t_2)^{s_2}H_{n,t_2,s_2}}\right)^2 \leq \sum_{i=1}^{n} q_i \left(\frac{1-q_i(i+t_1)^{s_1}H_{n,t_1,s_1}}{q_i(i+t_1)^{s_1}H_{n,t_1,s_1}}\right)^2,$$

we get (3.105).
(b) Similar way as Part (a), we can prove (3.106) by using (3.76) and $f(t) := (t-1)^2$, $t \in [0,\infty)$. □

Corollary 3.38 *If $n \in \{1,2,3,\ldots\}$, $t_1,t_2 \geq 0$ and $s_1,s_2 > 0$ such that satisfying (3.72), then the following inequality holds*

$$\hat{\chi}^2(i,n,t_2,s_2,\mathbf{1}) := \sum_{i=1}^{n} \frac{\left[1-(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}{\left[(i+t_2)^{s_2}H_{n,t_2,s_2}\right]^2}$$
$$\leq \hat{\chi}^2(i,n,t_1,s_1,\mathbf{1}) := \sum_{i=1}^{n} \frac{\left[1-(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}{\left[(i+t_1)^{s_1}H_{n,t_1,s_1}\right]^2}. \tag{3.107}$$

Proof. Substitute $q_i = 1$, $(i = 1,\ldots,n)$ in (3.105), we get (3.107). □

For finite n and $t = 0$ the Zipf-Mandelbrot law becomes Zipf's law. Therefore (3.23) and (3.24) becomes

$$f(k,n,s) := \frac{1/k^s}{H_{n,s}}, \qquad \text{where } H_{n,s} := \sum_{i=1}^{n} \frac{1}{i^s}. \tag{3.108}$$

Gelbukh and Sidorov in [76] observed the difference between the coefficients s_1 and s_2 in Zipf's law for the English and Russian languages. They processed 39 literature texts for each language, chosen randomly from different genres, with the requirement that the size be greater than 10,000 running words each. They calculated coefficients for each of the mentioned texts and as the result they obtained the average of $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language.

The following definitions are the Kullback-Leibler divergence for Zipf's law.

Definition 3.28 ***(Kullback-Leibler divergence for Zipf Law)*** *If $n \in \{1,2,3,\ldots\}$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1,\ldots,n)$, then the Kullback-Leibler divergence for Zipf's law is defined by*

$$\hat{KL}(i,n,s_1,\mathbf{q}) := \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right).$$

Definition 3.29 *If $n \in \{1,2,3,\ldots\}$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1,\ldots,n)$, then the Kullback-Leibler divergence for Zipf's law is defined by*

$$K\hat{L}_{id}(i,n,s_1,\mathbf{q}) := \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right).$$

Remark 3.3 *The majorization conditions (3.72) for $t_1 = t_2 = 0$ becomes*

$$\frac{H_{k,s_2}}{H_{n,s_2}} \leq \frac{H_{k,s_1}}{H_{n,s_1}}, \quad for \;\; k = 2,\ldots,n-1, \tag{3.109}$$

and for $k = 1$, the above inequality becomes

$$H_{n,s_1} \leq H_{n,s_2} \qquad \Leftrightarrow \qquad s_2 \leq s_1, \tag{3.110}$$

which means that the generalized harmonic number of order n of s_1 is less or equal to the generalized harmonic number of order n of s_2.

Corollary 3.39 *Let $n \in \{1,2,3,\ldots\}$ and $s_1, s_2 > 0$ such that $s_2 \leq s_1$ satisfying (3.109) and also let $q_i > 0$, $(i = 1,\ldots,n)$,*

(a) if $\frac{i^{s_2}}{(i+1)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ and the base of log *is greater than 1, then*

$$\begin{aligned} \hat{KL}(i,n,s_2,\mathbf{q}) &:= \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{s_2} H_{n,s_2}}\right) \\ &\geq \hat{KL}(i,n,s_1,\mathbf{q}) := \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right). \end{aligned} \tag{3.111}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.111).*

(b) if $\frac{i^{s_1}}{(i+1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{s_2}\, H_{n,s_2}}\right) \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{s_1}\, H_{n,s_1}}\right). \tag{3.112}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.112).*

Proof. If we choose the function $f(x) := \log x$ and $t_1 = t_2 = 0$ in Theorem 3.8, we get the required results. □

Corollary 3.40 *If* $n \in \{1,2,3,\ldots\}$ *and* $s_1, s_2 > 0$ *such that* $s_2 \leq s_1$ *satisfying (3.109) and also the base of* log *is greater than 1, then*

$$\begin{aligned} \hat{KL}(i,n,s_2,\mathbf{1}) &:= \sum_{i=1}^{n} \log\left(\frac{1}{i^{s_2}\, H_{n,s_2}}\right) \\ \geq \hat{KL}(i,n,s_1,\mathbf{1}) &:= \sum_{i=1}^{n} \log\left(\frac{1}{i^{s_1} H_{n,s_1}}\right). \end{aligned} \tag{3.113}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.113).*

Proof. If we choose $q_i := 1$, $(i = 1,\ldots,n)$ in (3.111), then we get (3.113). □

Corollary 3.41 *Let* $n \in \{1,2,3,\ldots\}$, $s_1 = 0,973863$ *for the English language and* $s_2 = 0,892869$ *for the Russian language such that satisfying (3.109) and also let* $q_i > 0$, $(i = 1,\ldots,n)$*,*

(a) if $\frac{i^{0,892869}}{(i+1)^{0,892869}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then the following bound for the Kullback-Leibler divergence of the distributions associated to the English and Russian languages depending only on the parameter n hold*

$$\begin{aligned} 0 \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) \\ \leq \log\left(n^{0,080994}\, \frac{H_{n,0,973863}}{H_{n,0,892869}}\right) \sum_{i=1}^{n} q_i. \end{aligned} \tag{3.114}$$

If the base of log *is in between 0 and 1, then*

$$\begin{aligned} 0 \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) \\ \leq \log\left(\frac{H_{n,0,892869}}{H_{n,0,973863}}\right) \sum_{i=1}^{n} q_i \end{aligned} \tag{3.115}$$

(b) if $\frac{i^{0,973863}}{(i+1)^{0,973863}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then (3.115) holds. If the base of* log *is in between 0 and 1, then (3.114) holds.*

Proof. (a) Take the difference of Left Hand and the Right Hand sides of (3.111) and then putting the experimental values of s_1 and s_2, we have

$$0 \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right)$$

$$= \sum_{i=1}^{n} q_i \log\left(i^{0,080994}\, \frac{H_{n,0,973863}}{H_{n,0,892869}}\right) \leq \log\left(n^{0,080994}\, \frac{H_{n,0,973863}}{H_{n,0,892869}}\right) \sum_{i=1}^{n} q_i.$$

In the similar fashion, we can prove the other bounds. □

Corollary 3.42 *If* $n \in \{1,2,3,\ldots\}$, $s_1 = 0,973863$ *for the English language and* $s_2 = 0,892869$ *for the Russian language such that satisfying (3.109) and the base of* log *is greater than 1, then we give the following bound associated to the English and Russian languages:*

$$0 \leq \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) - \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) \leq \log\left(n^{0,080994}\, \frac{H_{n,0,973863}}{H_{n,0,892869}}\right)^{n}. \tag{3.116}$$

If the base of log *is in between 0 and 1, then*

$$0 \leq \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) - \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) \leq \log\left(\frac{H_{n,0,892869}}{H_{n,0,973863}}\right)^{n}. \tag{3.117}$$

Corollary 3.43 *Let* $n \in \{1,2,3,\ldots\}$ *and* $s_1, s_2 > 0$ *such that* $s_2 \leq s_1$ *satisfying (3.109) and also let* $q_i > 0$, $(i = 1,\ldots,n)$,

(a) if $\frac{i^{s_2}}{(i+1)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\begin{aligned} \hat{KL}_{id}(i,n,s_2,\mathbf{q}) &:= \sum_{i=1}^{n} \frac{1}{i^{s_2} H_{n,s_2}} \log\left(\frac{1}{q_i\, i^{s_2} H_{n,s_2}}\right) \\ &\leq \hat{KL}_{id}(i,n,s_1,\mathbf{q}) := \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right). \end{aligned} \tag{3.118}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.118).*

(b) if $\frac{i^{s_1}}{(i+1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$ *and the base of* log *is greater than 1, then*

$$\sum_{i=1}^{n} \frac{1}{i^{s_2} H_{n,s_2}} \log\left(\frac{1}{q_i\, i^{s_2} H_{n,s_2}}\right) \geq \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right). \tag{3.119}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.119).*

Proof. If we choose the function $xf(x) := x\log x$ and $t_1 = t_2 = 0$ in Theorem 3.9, we get the required results. □

Corollary 3.44 *If $n \in \{1,2,3,\ldots\}$ and $s_1, s_2 > 0$ such that $s_2 \le s_1$ satisfying (3.109) and the base of* log *is greater than 1, then*

$$\begin{aligned}\hat{KL}_{id}(i,n,s_2,\mathbf{1}) &:= \sum_{i=1}^{n} \frac{1}{i^{s_2} H_{n,s_2}} \log\left(\frac{1}{i^{s_2} H_{n,s_2}}\right) \\ \le \hat{KL}_{id}(i,n,s_1,\mathbf{1}) &:= \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \log\left(\frac{1}{i^{s_1} H_{n,s_1}}\right).\end{aligned} \tag{3.120}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.120).*

Proof. If we choose $q_i := 1$, $(i = 1, \ldots, n)$ in (3.118), then we get (3.120). □

Corollary 3.45 *Let $n \in \{1,2,3,\ldots\}$, $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language such that satisfying (3.109) and also let $q_i > 0$, $(i = 1, \ldots, n)$,*

(a) if $\frac{i^{0,892869}}{(i+1)^{0,892869}} \le \frac{q_{i+1}}{q_i}$ $(i = 1, \ldots, n)$ and the base of log *is greater than 1, then*

$$\begin{aligned}0 \le\ & \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) \\ & - \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) \\ \le\ & \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i H_{n,0,973863}}\right) \\ & - \frac{1}{n^{0,892869} H_{n,0,892869}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i\, n^{0,892869} H_{n,0,892869}}\right).\end{aligned} \tag{3.121}$$

If the base of log *is in between 0 and 1, then*

$$\begin{aligned}0 \le\ & \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) \\ & - \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) \\ \le\ & \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i H_{n,0,892869}}\right) \\ & - \frac{1}{n^{0,973863} H_{n,0,973863}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i\, n^{0,973863} H_{n,0,973863}}\right).\end{aligned} \tag{3.122}$$

(b) if $\frac{i^{0,973863}}{(i+1)^{0,973863}} \ge \frac{q_{i+1}}{q_i}$ $(i = 1, \ldots, n)$ and the base of log *is greater than 1, then (3.122) holds. If the base of* log *is in between 0 and 1, then (3.121) holds.*

Corollary 3.46 *If $n \in \{1,2,3,\ldots\}$, $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language such that satisfying (3.109) and the base of* log *is greater than 1, then*

$$\begin{aligned}
0 \leq & \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) \\
& - \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) \\
\leq & \frac{1}{H_{n,0,973863}} \log\left(\frac{1}{H_{n,0,973863}}\right)^{n} - \frac{n^{0,107131}}{H_{n,0,892869}} \log\left(\frac{1}{n^{0,892869} H_{n,0,892869}}\right).
\end{aligned} \tag{3.123}$$

If the base of log *is in between 0 and 1, then*

$$\begin{aligned}
0 \leq & \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) \\
& - \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) \\
\leq & \frac{1}{H_{n,0,892869}} \log\left(\frac{1}{H_{n,0,892869}}\right)^{n} - \frac{n^{0,026137}}{H_{n,0,973863}} \log\left(\frac{1}{n^{0,973863} H_{n,0,973863}}\right).
\end{aligned} \tag{3.124}$$

3.4 Majorization via Hybrid Zipf-Mandelbrot Law in Information Theory

Latif et al. (2018) [107] considered the following two definitions of Csiszár divergence [64, 63] for Zipf-Mandelbrot law.

Definition 3.30 ***(Csiszár Divergence for Z-M law)*** *Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function.*

Let $n \in \{1,2,3,\ldots\}$, $t_1 \geq 0$, $s_1 > 0$ and also let $q_i > 0$ for $(i = 1,\ldots,n)$ such that

$$\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}} \in J, \quad i = 1,\ldots,n,$$

then we denote

$$\hat{I}_f(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right).$$

Definition 3.31 *Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function. Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that*

$$\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \in J, \quad i = 1,\ldots,n,$$

then we denote

$$\widetilde{I}_f\,(i,n,t_1,t_2,s_1,s_2) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right).$$

Remark 3.4 *It is obvious that the second Csiszár divergence for Zipf-Mandelbrot law is a special case of the first one.*

Jakšetić et al. [85] gave the following theorem which includes the hybrid Zipf-Mandelbrot law.

Theorem 3.10 *If $I = \{1,\ldots,N\}$ or $I = N$, then probability distribution that maximizes Shannon entropy under constraints*

$$\sum_{i\in I} p_i = 1, \ \sum_{i\in I} p_i \ln(i+t) = \chi, \ \sum_{i\in I} i\,p_i = \mu \tag{3.125}$$

is textbfthe hybrid Zipf-Mandelbrot law:

$$p_i = \frac{w^i}{(i+t)^s \Phi^*(s,t,w)}, \ i \in I,$$

where

$$\Phi_I^*(s,t,w) = \sum_{i\in I} \frac{w^i}{(i+t)^s}.$$

They denoted

$$f_h\,(w,N,i,t,s) = \frac{w^i}{(i+t)^s\,\Phi_I^*(s,t,w)}, \ i = 1,\ldots,N \tag{3.126}$$

and

$$f_h\,(w,i,t,s) = \frac{w^i}{(i+t)^s\,\Phi^*(s,t,w)}, \tag{3.127}$$

hybrid Zipf-Mandelbrot law on finite and infinite state space, respectively.
They also observed, further, that for hybrid Zipf-Mandelbrot law (3.126) Shannon entropy can be bounded from above:

$$S = -\sum_{k=1}^{\infty} f_h\,(i,t,s) \ln f_h\,(i,t,s) \leq -\sum_{i=1}^{\infty} f_h\,(i,t,s) \ln q_i, \tag{3.128}$$

where $\{t_i : i \in \mathbb{N}\}$ is any sequence of positive numbers such that $\sum_{i=1}^{\infty} t_i = 1$.

Motivated the idea in [85] (2018), we discuss the behaviour of the results in the form of divergences, majorization and hybrid Zipf-Mandelbrot law.

We can consider the following two definitions of Csiszár divergence [64, 63] for hybrid Zipf-Mandelbrot law:

Definition 3.32 ***(Csiszár Divergence for Hybrid Z-M law)*** *Let $J \subset \mathbb{R}$ be an interval, and let $f_h : J \to \mathbb{R}$ be a function. Let $n \in \mathbb{N}$, $t_1 \geq 0$, $s_1 > 0$, $w \geq 0$ and also let $q_i > 0$ for $(i \in \mathbb{N})$ such that*

$$\frac{w^i}{q_i(i+t_1)^{s_1}\phi^*(s_1,t_1,w)} \in J, \tag{3.129}$$

then we define

$$\hat{\mathbb{I}}_{f_h}(i,n,t_1,s_1,w,\mathbf{q}) := \sum_{i=1}^{n} q_i f_h\left(\frac{w^i}{q_i(i+t_1)^{s_1}\phi^*(s_1,t_1,w)}\right).$$

Definition 3.33 *Let $J \subset \mathbb{R}$ be an interval, and let $f_h : J \to \mathbb{R}$ be a function. Let $n \in \mathbb{N}$, $t_1,t_2 \geq 0$, $s_1,s_2 > 0$ and $w \geq 0$ such that*

$$\frac{(i+t_2)^{s_2}\phi^*(s_2,t_2,w)}{(i+t_1)^{s_1}\phi^*(s_1,t_1,w)} \in J, \quad i \in \mathbb{N}, \tag{3.130}$$

then we define

$$\widetilde{\mathbb{I}}_{f_h}(i,n,t_1,t_2,s_1,s_2,w) := \sum_{i=1}^{n} \frac{w^i}{(i+t_2)^{s_2}\phi^*(s_2,t_2,w)} f\left(\frac{(i+t_2)^{s_2}\phi^*(s_2,t_2,w)}{(i+t_1)^{s_1}\phi^*(s_1,t_1,w)}\right).$$

Remark 3.5 *It is obvious that the second Csiszár divergence for Hybrid Zipf-Mandelbrot law is a special case of the first one.*

We present the following theorem is the connection between Csiszár f-divergence, hybrid Zipf-Mandelbrot law and weighted majorization inequality:

Theorem 3.11 *Let $J \subset \mathbb{R}$ is an interval and $f_h : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1,2,3,\ldots\}$, $t_1,t_2,t_3 \geq 0$, $s_1,s_2,s_3 > 0$ and $w \geq 0$ such that satisfying*

$$\frac{\phi_k^*(t_2,s_2,w)}{\phi_n^*(t_2,s_2,w)} \leq \frac{\phi_k^*(t_1,s_1,w)}{\phi_n^*(t_1,s_1,w)}, \quad k = 1,\ldots,n-1, \tag{3.131}$$

with

$$\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}, \frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \in J, \ (i = 1,\ldots,n),$$

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned}\tilde{\mathbb{I}}_{f_h}(i,n,t_2,t_3,s_2,s_3,w) &:= \sum_{i=1}^{n} \frac{w^i}{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)} f_h\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right) \\ \leq \tilde{\mathbb{I}}_{f_h}(i,n,t_1,t_3,s_1,s_3,w) &:= \sum_{i=1}^{n} \frac{w^i}{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)} f_h\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right).\end{aligned} \tag{3.132}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,\ldots,n)$*, then*

$$\sum_{i=1}^{n} \frac{w^i}{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)} f_h\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right) \geq \sum_{i=1}^{n} \frac{w^i}{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)} f_h\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \tag{3.133}$$

If f_h is continuous concave function, then the reverse inequalities hold in (3.132) and (3.133).

Proof. Let us consider $x_i := \frac{w^i/(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}{w^i/(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}$, $y_i := \frac{w^i/(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}{w^i/(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}$, $r_i := \frac{w^i}{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}$ for $(i = 1,\ldots,n)$, then

$$\begin{aligned}\sum_{i=1}^{k} r_i x_i : &= \sum_{i=1}^{k} \frac{w^i}{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)} \frac{w^i/(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}{w^i/(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)} \\ &= \frac{1}{\phi_n^*(t_1,s_1,w)} \sum_{i=1}^{k} \frac{w^i}{(i+t_1)^{s_1}} \\ &= \frac{\phi_k^*(t_1,s_1,w)}{\phi_n^*(t_1,s_1,w)}, \quad k = 1,\ldots,n-1,\end{aligned}$$

similarly

$$\sum_{i=1}^{k} r_i y_i := \frac{\phi_k^*(t_2,s_2,w)}{\phi_n^*(t_2,s_2,w)}, \; k = 1,\ldots,n-1.$$

This implies that

$$\sum_{i=1}^{k} r_i y_i \leq \sum_{i=1}^{k} r_i x_i \quad \Leftrightarrow \quad \frac{\phi_k^*(t_2,s_2,w)}{\phi_n^*(t_2,s_2,w)} \leq \frac{\phi_k^*(t_1,s_1,w)}{\phi_n^*(t_1,s_1,w)}, \;\; k = 1,\ldots,n-1.$$

We can easily check that $\frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)} = \frac{w_i/(i+t_1)^{s_1}}{\phi_n^*(t_1,s_1,w)} = \frac{w_i/(i+t_1)^{s_1}}{\sum_{i=1}^{n} w^i/(i+t_1)^{s_1}}$ is decreasing over $i = 1,\ldots,n$ and similarly the other too. Now, we investigate the behaviour of y_i for $(i = 1,2,\ldots,n)$, take

$$y_i = \frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \quad \textit{and} \quad y_{i+1} = \frac{(i+1+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+1+t_2)^{s_2}\phi_n^*(t_2,s_2,w)},$$

$$y_{i+1} - y_i = \frac{\phi_n^*(t_3,s_3,w)}{\phi_n^*(t_2,s_2,w)} \left[\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} - \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}\right] \leq 0,$$

$$\Leftrightarrow \frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}, \; (i = 1,\ldots,n),$$

which shows that y_i is decreasing.

Therefore, substitute $x_i := \frac{w^i/(i+t_1)^{s_1}\phi^*(t_1,s_1,w)}{w^i/(i+t_3)^{s_3}\phi^*(t_3,s_3,w)}$, $y_i := \frac{w^i/(i+t_2)^{s_2}\phi^*(t_2,s_2,w)}{w^i/(i+t_3)^{s_3}\phi^*(t_3,s_3,w)}$, $r_i := \frac{w^i}{(i+t_3)^{s_3}\phi^*(t_3,s_3,w)}$ for $(i=1,\ldots,n)$ and $f := f_h$ in Theorem 1.15 (a), then we get (3.132).

(b) We can prove part (b) with the similar substitutions as in Part (a) but switch the role of y_i with x_i that is increasing sequence, in Theorem 1.15 (b). □

Theorem 3.12 *Let $J \subset \mathbb{R}$ is an interval and $f_h : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$ and $w \geq 0$ such that satisfying (3.131) and also let $q_i > 0$, $(i=1,\ldots,n)$ with*

$$\frac{1}{q_i(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}, \frac{1}{q_i(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \in J \ \ (i=1,\ldots,n),$$

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i=1,\ldots,n)$, then

$$\begin{aligned} \hat{\mathbb{I}}_{f_h}(i,n,t_2,s_2,w,\mathbf{q}) &:= \sum_{i=1}^{n} q_i f_h\left(\frac{1}{q_i(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right) \\ \leq \hat{\mathbb{I}}_{f_h}(i,n,t_1,s_1,w,\mathbf{q}) &:= \sum_{i=1}^{n} q_i f_h\left(\frac{1}{q_i(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \end{aligned} \tag{3.134}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i=1,\ldots,n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} q_i f_h\left(\frac{1}{q_i(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right) \\ &\quad \geq \sum_{i=1}^{n} q_i f_h\left(\frac{1}{q_i(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \end{aligned} \tag{3.135}$$

If f_h is continuous concave function, then the reverse inequalities hold in (3.134) and (3.135).

Proof. Let us consider $x_i := \frac{w^i/(i+t_1)^{s_1}\phi^*(t_1,s_1,w)}{q_i}$, $y_i := \frac{w^i/(i+t_2)^{s_2}\phi^*(t_2,s_2,w)}{q_i}$, and $w_i = q_i > 0$, $(i=1,\ldots,n)$ then we can get as in the previous proof

$$\sum_{i=1}^{k} r_i y_i \leq \sum_{i=1}^{k} r_i x_i \quad \Leftrightarrow \quad \frac{\phi_k^*(t_2,s_2,w)}{\phi_n^*(t_2,s_2,w)} \leq \frac{\phi_k^*(t_1,s_1,w)}{\phi_n^*(t_1,s_1,w)}, \ \ k=1,\ldots,n-1.$$

Now, we investigate the behaviour of y_i for $(i=1,2,\ldots,n)$, take

$$y_i = \frac{1}{q_i(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \quad \textit{and} \quad y_{i+1} = \frac{1}{q_{i+1}(i+1+t_2)^{s_2}\phi_n^*(t_2,s_2,w)},$$

$$y_{i+1} - y_i = \frac{1}{\phi_n^*(t_2,s_2,w)}\left[\frac{1}{q_{i+1}(i+1+t_2)^{s_2}} - \frac{1}{q_i(i+t_2)^{s_2}}\right] \leq 0,$$

$$\Leftrightarrow \frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}, \ \ (i=1,\ldots,n),$$

which shows that y_i is decreasing. Therefore, substitute $x_i := \frac{w^i}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}$, $y_i := \frac{w^i}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}$, $r_i = q_i > 0$, $(i = 1, \ldots, n)$ and also $f := f_h$ in Theorem 1.15, we get (3.134).
(b) If we switch the role of y_i into x_i as increasing sequence in the similar fashion as the proof of Part (a), then by using Theorem 1.15 (b) we get (3.135). □

Corollary 3.47 *Let $J \subset \mathbb{R}$ is an interval and $f_h : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$ and $w \geq 0$ such that satisfying (3.131) with*

$$\frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}, \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \in J \ \ (i = 1,\ldots,n),$$

then the following inequality holds

$$\begin{aligned} \hat{\mathbb{I}}_f(i,n,t_2,s_2,w,\mathbf{1}) &:= \sum_{i=1}^{n} f_h\left(\frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right) \\ &\leq \hat{\mathbb{I}}_f(i,n,t_1,s_1,w,\mathbf{1}) := \sum_{i=1}^{n} f_h\left(\frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \end{aligned} \tag{3.136}$$

If f_h is continuous concave function, then the reverse inequality hold in (3.136).

Proof. Let us consider $x_i := \frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}$ and $y_i := \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}$, we can easily check that $y_i := \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}$ is decreasing over $i = 1,\ldots,n$. Therefore, substitute $x_i := \frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}$, $y_i := \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}$, $q_i = 1$, $(i = 1,\ldots,n)$ and also $f := f_h$ in (3.134), we get (3.136). □

Theorem 3.13 *Let $J \subset \mathbb{R}$ is an interval and $f_h : J \to \mathbb{R}$ be a function such that $x \to x f_h(x)$ $(x \in J)$ is a continuous convex function.*

Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2, t_3 \geq 0$, $s_1, s_2, s_3 > 0$ and $w \geq 0$ such that satisfying (3.131) with

$$\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}, \frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \in J, \ \ (i = 1,\ldots,n),$$

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned} &\widetilde{\mathbb{I}}_{id_J f}(i,n,t_2,t_3,s_2,s_3,w) \\ &:= \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} f_h\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right) \\ &\leq \widetilde{\mathbb{I}}_{id_J f}(i,n,t_1,t_3,s_1,s_3,w) \\ &:= \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)} f_h\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \end{aligned} \tag{3.137}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i=1,\ldots,n)$*, then*

$$\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} f\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right)$$
$$\geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)} f\left(\frac{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \tag{3.138}$$

If $xf_h(x)$ is continuous concave function, then the reverse inequalities hold in (3.137) and (3.138).

Proof. Let us substitute $x_i := \frac{w^i/(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}{w^i/(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}$, $y_i := \frac{w^i/(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}{w^i/(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}$, $r_i := \frac{w^i}{(i+t_3)^{s_3}\phi_n^*(t_3,s_3,w)}$ for $(i=1,\ldots,n)$ in Theorem 3.11 (a) and follow the proof of Theorem 3.12 for function $f(x) := xf_h(x)$, then we get (3.137).
(b) We can prove part (b) with the similar substitutions as in Part (a) but switch the role of y_i with x_i that is an increasing sequence, in Theorem 1.15 (b) for function $f(x){:=}xf_h(x)$. □

Theorem 3.14 *Let $J \subset \mathbb{R}$ is an interval and $f_h : J \to \mathbb{R}$ be a function such that $x \to xf_h(x)$ $(x \in J)$ is a continuous convex function. Let $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$ and $w \geq 0$ such that satisfying (3.131) and also let $q_i > 0$ with*

$$\frac{w^i}{q_i(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}, \frac{w^i}{q_i(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \in J \ \ (i=1,\ldots,n),$$

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i=1,\ldots,n)$*, then*

$$\hat{\mathbb{I}}_{id_J\, f_h}(i,n,t_2,s_2,w,\mathbf{q}) := \sum_{i=1}^{n} \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} f_h\left(\frac{w^i}{q_i(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right)$$
$$\leq \hat{\mathbb{I}}_{id_J\, f_h}(i,n,t_1,s_1,w,\mathbf{q}) := \sum_{i=1}^{n} \frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)} f_h\left(\frac{w^i}{q_i(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \tag{3.139}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i=1,\ldots,n)$*, then*

$$\sum_{i=1}^{n} \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} f_h\left(\frac{w^i}{q_i(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right)$$
$$\geq \sum_{i=1}^{n} \frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)} f_h\left(\frac{w^i}{q_i(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right). \tag{3.140}$$

If $xf_h(x)$ is continuous concave function, then the reverse inequalities hold in (3.139) and (3.140).

Proof. Let us consider $x_i := \frac{w^i/(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}{q_i}$, $y_i := \frac{w^i/(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}{q_i}$, and $r_i = q_i > 0$, $(i = 1,\dots,n)$ and also $f(x) := xf_h(x)$ in Theorem 1.15 (a) by follow the proof of Theorem 3.13(a), we get (3.139).
(b) If we switch the role of y_i into x_i as an increasing sequence with the similar substitutions as in Part (a), then by using Theorem 1.15 (b) we get (3.140). □

Corollary 3.48 *Let $J \subset \mathbb{R}$ is an interval and $f_h : J \to \mathbb{R}$ be a function such that $x \to xf_h(x)$ $(x \in J)$ is a continuous convex function. Let $n \in \{1,2,3,\dots\}$, t_1, $t_2 \geq 0$, $s_1, s_2 > 0$ and $w \geq 0$ such that satisfying (3.131) with*

$$\frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}, \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} \in J \ \ (i = 1,\dots,n),$$

then the following inequality holds

$$\begin{aligned}\hat{\mathbb{I}}_{id_J\, f_h}(i,n,t_2,s_2,w,\mathbf{1}) &:= \sum_{i=1}^{n} \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)} f_h\left(\frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}\right) \\ \leq \hat{\mathbb{I}}_{id_J\, f_h}(i,n,t_1,s_1,w,\mathbf{1}) &:= \sum_{i=1}^{n} \frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)} f_h\left(\frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}\right).\end{aligned} \tag{3.141}$$

If $xf_h(x)$ is continuous concave function, then the reverse inequality hold in (3.141).

Proof. Since $y_i := \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}$ is decreasing over $i = 1,\dots,n$. Therefore, substitute $x_i := \frac{w^i}{(i+t_1)^{s_1}\phi_n^*(t_1,s_1,w)}$ and $y_i := \frac{w^i}{(i+t_2)^{s_2}\phi_n^*(t_2,s_2,w)}$, and $q_i = 1$, $(i = 1,\dots,n)$ and also $f(x) := xf_h(x)$ in (3.139), we get (3.141). □

3.5 Majoriztion,“useful” Csiszár Divergence and “useful” Zipf-Mandelbrot Law

In this section, we consider the definition of “useful” Csiszár divergence and “useful” Zipf-Mandelbrot law associated with the real utility distribution to give the results for majorization inequalities by using monotonic sequences. We obtain the equivalent statements between continuous convex functions and Green functions via majorization inequalities, “useful” Csiszár functional and “useful” Zipf-Mandelbrot law. By considering “useful” Csisźar divergence in integral case, we give the results for integral majorization inequality. At the end, some applications are given.
Zipf’s law ([155], [175], [176]) and power laws in general ([41], [61], [132]) have and continue to attract considerable attention in a wide variety of disciplines from astronomy to demographics to software structure to economics to zoology, and even to warfare [152]. Typically one is dealing with integer-valued observables (numbers of objects, people, cities,

words, animals, corpses), with $n \in \{1,2,3,\dots\}$. As given in [169], sometimes the range of values is allowed to be infinite (at least in principle), sometimes a hard upper bound N is fixed (e.g., total population if one is interested in subdividing a fixed population into sub-classes). Particularly interesting probability distributions are probability laws of the form:

- Zipf’s law: $p_n \propto 1/n$;
- power laws: $p_n \propto 1/n^z$;
- hybrid geometric/power laws: $p_n \propto w^n/n^z$.

Distance or divergence measures are of key importance in different fields like theoretical and applied statistical inference and data processing problems, such as estimation, detection, classification, compression, recognition, indexation, diagnosis and model selection etc. Traditionally, the information conveyed by observing X is measured by the entropy (see [121, p.111])

$$H(\mathbf{p}) := \sum_{i=1}^{n} p_i \log_2 1/p_i,$$

associated with the distribution $\mathbf{p}$, $p_i > 0$ $(1 \le i \le n)$, where $\sum_{i=1}^n p_i = 1$. A generalization of this is to attach a utility $q_i > 0$ to the outcome x_i $(1 \le i \le n)$ and speak of **the ”useful” information measure**

$$H(\mathbf{p};\mathbf{q}) := \sum_{i=1}^{n} q_i p_i \log_2 1/p_i,$$

associated with the utility distribution $\mathbf{q} = (q_1,\dots,q_n)$.
Bhaker and Hooda [44] (see also [121, p.112]) introduced the measures

$$E(\mathbf{p};\mathbf{q}) := \frac{\sum_{k=1}^n q_k p_k \log_2 1/p_k}{\sum_{k=1}^n q_k p_k} \tag{3.142}$$

and

$$E_\alpha(\mathbf{p};\mathbf{q}) := \frac{1}{1-\alpha} \log_2 \frac{\sum_{k=1}^n q_k p_k^\alpha}{\sum_{k=1}^n q_k p_k}, \quad 0 < \alpha \neq 1, \tag{3.143}$$

which have a number of useful properties. It is readily verified that these alternations leave intact the property that (3.143) reduces to (3.142) when $\alpha \to 1$. Also, if $u \equiv 1$ so that there are effectively no utilities, (3.142) and (3.143) reduce to Renyi’s entropies of order 1 and α, respectively.
Csiszár introduced functional in [64] and then discussed in [63], we consider “useful” Csiszár divergence (see [83, p.3], [121, 106, 105]):

Definition 3.34 ***("Useful" Csiszár divergence)*** *Assume $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function with distribution $\mathbf{p} := (p_1, \ldots, p_n)$, associated with the utility distribution $\mathbf{u} := (u_1, \ldots, u_n)$, where $p_i, u_i \in \mathbb{R}$ for $1 \leq i \leq n$, and $\mathbf{q} := (q_1, \ldots, q_n) \in]0, \infty[^n$ be such that*

$$\frac{p_i}{q_i} \in J, \quad i = 1, \ldots, n, \tag{3.144}$$

then we denote the "useful" Csiszár divergence

$$I_f(\mathbf{p}, \mathbf{q}, \mathbf{u}) := \sum_{i=1}^{n} u_i q_i f\left(\frac{p_i}{q_i}\right). \tag{3.145}$$

Remark 3.6 *One can easily seen that if we substitute $\mathbf{u} = \mathbf{1}$, then (3.145) becomes*

$$I_f(\mathbf{p}, \mathbf{q}, \mathbf{1}) := I_f(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right).$$

One can see the various results in information theory in [176, 118, 119].

We consider the following definition of "useful" Zipf-Mandelbrot law (see [121, 64, 63, 106, 105]):

Definition 3.35 ***("Useful" Zipf-Mandelbrot law)*** *Assume $J \subset \mathbb{R}$ be an interval, and $f : J \to \mathbb{R}$ be a function with $n \in \{1, 2, 3, \ldots\}$, $t_1 \geq 0$. Let also distribution $q_i > 0$ and associated with the utility distribution $u_i \in \mathbb{R}$ for $(i = 1, \ldots, n)$ such that*

$$\frac{1}{q_i (i + t_1)^{s_1} H_{n,t_1,s_1}} \in J, \quad i = 1, \ldots, n, \tag{3.146}$$

then we denote "useful" Zipf-Mandelbrot law as

$$I_f(i, n, t_1, s_1, \mathbf{q}, \mathbf{u}) := \sum_{i=1}^{n} u_i q_i f\left(\frac{1}{q_i (i + t_1)^{s_1} H_{n,t_1,s_1}}\right).$$

Remark 3.7 *One can easily seen that for $\mathbf{u} = \mathbf{1}$, then*

$$I_f(i, n, t_1, s_1, \mathbf{q}, \mathbf{1}) = I_f(i, n, t_1, s_1, \mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i + t_1)^{s_1} H_{n,t_1,s_1}}\right).$$

If we substitute $q_i = \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$, then

$$I_f(i, n, t_1, t_3, s_1, s_3) := \sum_{i=1}^{n} \frac{1}{(i + t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i + t_3)^{s_3} H_{n,t_3,s_3}}{(i + t_1)^{s_1} H_{n,t_1,s_1}}\right).$$

This section is oragnised in this manner, firstly we give the results as the connection between useful Csisár divergence, useful Zipf-Mandelbrot law and majorization inequality for one monotonic sequence or both of them. We obtain some corollaries for our obtained results. After that, we present the equivalent statements between continuous convex functions and defined Green functions. We give the results for integral majorization inequality

for considering the integral form of useful Csisár divergence. At the end, we give some applications for obtained results.
Assume $\mathbf{p}$ and $\mathbf{q}$ be n-tuples such that $q_i > 0$ $(i = 1, \ldots, n)$, then define

$$\frac{\mathbf{p}}{\mathbf{q}} := \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n} \right).$$

We start the following theorem is the connection between “useful” Csiszár divergence and weighted majorization as one sequence is monotonic:

Theorem 3.15 *Assume $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous convex function, p_i, r_i $(i = 1, \ldots, n)$ be real numbers and q_i, u_i $(i = 1, \ldots, n)$ be positive real numbers such that*

$$\sum_{i=1}^{k} u_i r_i \leq \sum_{i=1}^{k} u_i p_i \quad \textit{for } k = 1, \ldots, n-1, \tag{3.147}$$

and

$$\sum_{i=1}^{n} u_i r_i = \sum_{i=1}^{n} u_i p_i, \tag{3.148}$$

with $\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J$ $(i = 1, \ldots, n)$.

(a) If $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing, then

$$I_f(\mathbf{r}, \mathbf{q}, \mathbf{u}) \leq I_f(\mathbf{p}, \mathbf{q}, \mathbf{u}). \tag{3.149}$$

(b) If $\frac{\mathbf{p}}{\mathbf{q}}$ is increasing, then

$$I_f(\mathbf{r}, \mathbf{q}, \mathbf{u}) \geq I_f(\mathbf{p}, \mathbf{q}, \mathbf{u}). \tag{3.150}$$

If f is a continuous concave function, then the reverse inequalities hold in (3.149) and (3.150).

Proof. (a): We use Theorem 1.15 (a) with substitutions $x_i := \frac{p_i}{q_i}$, $y_i := \frac{r_i}{q_i}$, $w_i = u_i q_i$ as $q_i > 0$, $(i = 1, \ldots, n)$ then we get (3.149).
We can prove part (b) with the similar substitutions in Theorem 1.15 (b). □

We present the following theorem as the connection between “useful” Csiszár divergence and weighted majorization theorem as both sequences are decreasing:

Theorem 3.16 *Assume $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous convex function, p_i, r_i, u_i $(i = 1, \ldots, n)$ be real numbers and q_i $(i = 1, \ldots, n)$ be positive real numbers such that $\frac{\mathbf{p}}{\mathbf{q}}$ and $\frac{\mathbf{r}}{\mathbf{q}}$ be decreasing satisfying (3.147) and (3.148) with $\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J$ $(i = 1, \ldots, n)$, then*

$$I_f(\mathbf{r}, \mathbf{q}, \mathbf{u}) \leq I_f(\mathbf{p}, \mathbf{q}, \mathbf{u}). \tag{3.151}$$

Proof. We use Theorem 1.15 with substitutions $x_i := \frac{p_i}{q_i}$, $y_i := \frac{r_i}{q_i}$ and $w_i = u_i q_i$ as $q_i > 0$ $(i = 1, \ldots, n)$ then we get (3.151). □

The following two theorem gives the connection between “useful” Zipf-Mandelbrot law and weighted majorization inequality:

Theorem 3.17 *Assume $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous convex function with $u_i > 0$, $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying*

$$\sum_{i=1}^{k} \frac{u_i}{(i+t_2)^{s_2}} \leq \frac{H_{n,t_2,s_2}}{H_{n,t_1,s_1}} \sum_{i=1}^{k} \frac{u_i}{(i+t_1)^{s_1}}, \quad k = 1,\ldots,n-1, \tag{3.152}$$

and

$$\sum_{i=1}^{n} \frac{u_i}{(i+t_2)^{s_2}} = \frac{H_{n,t_2,s_2}}{H_{n,t_1,s_1}} \sum_{i=1}^{n} \frac{u_i}{(i+t_1)^{s_1}}, \tag{3.153}$$

and also let $q_i > 0$, $(i = 1,\ldots,n)$ with

$$\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i = 1,\ldots,n).$$

(a) If $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned} I_f(i,n,t_2,s_2,\mathbf{q},\mathbf{u}) &:= \sum_{i=1}^{n} u_i q_i f\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\leq I_f(i,n,t_1,s_1,\mathbf{q},\mathbf{u}) := \sum_{i=1}^{n} u_i q_i f\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.154}$$

(b) If $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$, then

$$\sum_{i=1}^{n} u_i q_i f\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \geq \sum_{i=1}^{n} u_i q_i f\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \tag{3.155}$$

If f is continuous concave function, then the reverse inequalities hold in (3.154) and (3.155).

Proof. (a) Let us consider that $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $r_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, then

$$\sum_{i=1}^{k} u_i p_i := \sum_{i=1}^{k} \frac{u_i}{(i+t_1)^{s_1} H_{n,t_1,s_1}} = \frac{1}{H_{n,t_1,s_1}} \sum_{i=1}^{k} \frac{u_i}{(i+t_1)^{s_1}}, \ k = 1,\ldots,n-1,$$

similarly

$$\sum_{i=1}^{k} u_i r_i := \frac{1}{H_{n,t_2,s_2}} \sum_{i=1}^{k} \frac{u_i}{(i+t_2)^{s_2}}, \ k = 1,\ldots,n-1,$$

then we get

$$\sum_{i=1}^{k} u_i r_i \leq \sum_{i=1}^{k} u_i p_i \quad \Leftrightarrow \quad \sum_{i=1}^{k} \frac{u_i}{(i+t_2)^{s_2}} \leq \frac{H_{n,t_2,s_2}}{H_{n,t_1,s_1}} \sum_{i=1}^{k} \frac{u_i}{(i+t_1)^{s_1}}, \quad k = 1,\ldots,n-1.$$

One can see easily that $\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ is decreasing over $i = 1,\ldots,n$ and similarly r_i too. Now, we find the behaviour of $\frac{\mathbf{r}}{\mathbf{q}}$ for $q_i > 0$ $(i = 1,2,\ldots,n)$, take

$$\frac{r_i}{q_i} = \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \quad \textit{and} \quad \frac{r_{i+1}}{q_{i+1}} = \frac{1}{q_{i+1}(i+1+t_2)^{s_2} H_{n,t_2,s_2}},$$

$$\frac{r_{i+1}}{q_{i+1}} - \frac{r_i}{q_i} = \frac{1}{H_{n,t_2,s_2}}\left[\frac{1}{q_{i+1}(i+1+t_2)^{s_2}} - \frac{1}{q_i(i+t_2)^{s_2}}\right] \leq 0,$$

$$\Leftrightarrow \frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i},$$

which shows that $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing. So, all the assumptions of Theorem 3.15 (a) are true, then by using (3.149) we get (3.154).
(b) If we switch the role of r_i to p_i in the first part (a), then by using (3.150) in Theorem 3.15 (b) we get (3.155). □

Theorem 3.18 *Assume $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous convex function with $u_i \in \mathbb{R}$, $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$, such that satisfying (3.152), (3.153) and*

- $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$,
- $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,\ldots,n)$,

hold and also let $q_i > 0$, $(i = 1,\ldots,n)$ with

$$\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i = 1,\ldots,n),$$

then the following inequality holds

$$\begin{aligned} I_f(i,n,t_2,s_2,\mathbf{q},\mathbf{u}) &:= \sum_{i=1}^{n} u_i q_i f\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\leq I_f(i,n,t_1,s_1,\mathbf{q},\mathbf{u}) := \sum_{i=1}^{n} u_i q_i f\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.156}$$

Proof. Let us consider that $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $r_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, so as given in the proof of Theorem 3.17, we get $\mathbf{y} = \mathbf{r}/\mathbf{q}$ is decreasing $\Leftrightarrow \frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$, for $(i = 1,\ldots,n)$, similarly we can prove that $\mathbf{x} = \mathbf{p}/\mathbf{q}$ is also decreasing $\Leftrightarrow \frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \leq \frac{q_{i+1}}{q_i}$ for $(i = 1,\ldots,n)$. So, all the assumptions of Theorem 3.16 are true, then by using (3.151) we get (3.156). □

The following two corollaries obtain form Theorem 3.17 and Theorem 3.18 respectively but we use three the Zipf-Mandelbrot laws for different parameters:

Corollary 3.49 *Assume $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous convex function with $u_i > 0$, $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (3.152) and (3.153) and*

$$\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ (i = 1,\ldots,n).$$

(a) If $\frac{(i+1+t_2)^{s_2}}{(i+1+t_3)^{s_3}} \leq \frac{(i+t_2)^{s_2}}{(i+t_3)^{s_3}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned} I_f(i,n,t_2,s_2,t_3,s_3,\mathbf{u}) &:= \sum_{i=1}^{n} \frac{u_i}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq I_f(i,n,t_1,s_1,t_3,s_3,\mathbf{u}) &:= \sum_{i=1}^{n} \frac{u_i}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.157}$$

(b) If $\frac{(i+1+t_2)^{s_2}}{(i+1+t_3)^{s_3}} \geq \frac{(i+t_2)^{s_2}}{(i+t_3)^{s_3}}$ $(i = 1,\ldots,n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} \frac{u_i}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\geq \sum_{i=1}^{n} \frac{u_i}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.158}$$

If f is continuous concave function, then the reverse inequalities hold in (3.157) and (3.158).

Proof. (a) Let $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$, $q_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$ and $r_i := \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$, here p_i, q_i and r_i are decreasing over $i = 1,\ldots,n$. Now, we investigate the behaviour of $\frac{\mathbf{r}}{\mathbf{q}}$, take

$$\frac{r_i}{q_i} = \frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \quad \text{and} \quad \frac{r_{i+1}}{q_{i+1}} = \frac{(i+1+t_2)^{s_2} H_{n,t_2,s_2}}{(i+1+t_3)^{s_3} H_{n,t_3,s_3}},$$

$$\frac{r_{i+1}}{q_{i+1}} - \frac{r_i}{q_i} = \frac{(i+1+t_2)^{s_2} H_{n,t_2,s_2}}{(i+1+t_3)^{s_3} H_{n,t_3,s_3}} - \frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_3)^{s_3} H_{n,t_3,s_3}},$$

$$\frac{r_{i+1}}{q_{i+1}} - \frac{r_i}{q_i} = \frac{H_{n,t_2,s_2}}{H_{n,t_3,s_3}} \left[\frac{(i+1+t_2)^{s_2}}{(i+1+t_3)^{s_3}} - \frac{(i+t_2)^{s_2}}{(i+t_3)^{s_3}}\right],$$

the right hand side is non-positive by using the assumption, which shows that $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing, therefore using Theorem 3.17 (a) we get (3.157).
(b) If we switch the role of $\frac{\mathbf{r}}{\mathbf{q}}$ with $\frac{\mathbf{p}}{\mathbf{q}}$ in the part (a) and using Theorem 3.17(b), we get (3.158). □

Corollary 3.50 *Assume $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a continuous convex function with $u_i \in \mathbb{R}$, $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ and $s_1, s_2 > 0$, such that satisfying (3.152) and (3.153) and*

- $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \leq \frac{(i+t_3)^{s_3}}{(i+1+t_3)^{s_3}}$ $(i = 1,\ldots,n)$,
- $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+1+t_3)^{s_3}}$ $(i = 1,\ldots,n)$,

hold with

$$\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i = 1,\ldots,n),$$

then the following inequality holds

$$\begin{aligned} I_f(i,n,t_2,s_2,t_3,s_3,\mathbf{u}) &:= \sum_{i=1}^{n} \frac{u_i}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq I_f(i,n,t_1,s_1,t_3,s_3,\mathbf{u}) &:= \sum_{i=1}^{n} \frac{u_i}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{3.159}$$

Proof. (a) Let us consider that $p_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $r_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, so as given in the proof of Corollary 3.49 for $q_i > 0$ where $(i = 1,2,\ldots,n)$, we get $\mathbf{y} = \mathbf{r}/\mathbf{q}$ is decreasing $\Leftrightarrow \frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+1+t_3)^{s_3}}$, for $(i = 1,\ldots,n)$, similarly we can prove that $\mathbf{x} = \mathbf{p}/\mathbf{q}$ is also decreasing $\Leftrightarrow \frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \leq \frac{(i+t_3)^{s_3}}{(i+1+t_3)^{s_3}}$ for $(i = 1,\ldots,n)$. Therefore, all the assumptions of Theorem 3.18 are true, then by using (3.156) we get (3.159). □

Remark 3.8 *We can give Theorem 3.15, Theorem 3.16, Theorem 3.17, Theorem 3.18, Corollary 3.49 and Corollary 3.50 for $\mathbf{u} := \mathbf{1}$ as special case, some of them has been given in [106].*

In the following results the functions G_2, G_3, G_4 and G_5 represent the functions G_1, G_2, G_3 and G_4 as defined in (2.47), (2.48), (2.49) and (2.50) respectively and the function G_1 represent the function G as defined in (1.180).

The following theorem gives the equivalent statements between continuous convex functions and Green functions via majorization inequality and "useful" Csiszár divergence.

Theorem 3.19 *Assume $J \subset \mathbb{R}$ be an interval, p_i, r_i $(i = 1,\ldots,n)$ be real numbers and q_i, u_i $(i = 1,\ldots,n)$ be positive real numbers such that satisfying*

$$\sum_{i=1}^{n} u_i r_i = \sum_{i=1}^{n} u_i p_i, \tag{3.160}$$

with $\frac{p_i}{q_i}, \frac{r_i}{q_i} \in J$ $(i = 1,\ldots,n)$. If $\frac{\mathbf{r}}{\mathbf{q}}$ is decreasing and G_d $(d = 1,2,3,4,5)$ be defined as in (2.47)-(2.50) and (1.180), then we have following equivalent statements.

(i) For every continuous convex function $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$*, we have*

$$I_f(\mathbf{p},\mathbf{q},\mathbf{u}) - I_f(\mathbf{r},\mathbf{q},\mathbf{u}) \geq 0. \tag{3.161}$$

(ii) For all $v \in [\vartheta_1, \vartheta_2]$*, we have*

$$I_{G_d}(\mathbf{p},\mathbf{q},\mathbf{u}) - I_{G_d}(\mathbf{r},\mathbf{q},\mathbf{u}) \geq 0, \ \ d = 1,2,3,4,5. \tag{3.162}$$

Moreover, if we change the sign of inequality in both inequalities (3.161) *and* (3.162)*, then the above result still holds.*

Proof. The scheme of proof is similar for each $d = 1,2,3,4,5$, therefore we will only give the proof for $d = 5$.

$(i) \Rightarrow (ii)$: Let statement (i) holds. As the function $G_5 : [\vartheta_1, \vartheta_2] \times [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is convex and continuous, so it will satisfy the condition (3.161), i.e.,

$$I_{G_5}(\mathbf{p},\mathbf{q},\mathbf{u}) - I_{G_5}(\mathbf{r},\mathbf{q},\mathbf{u}) \geq 0.$$

$(ii) \Rightarrow (i)$: Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a convex function, then without loss of generality we can assume that such that $f \in C^2([\vartheta_1, \vartheta_2])$, and further, assume that the statement (ii) holds. Then by Lemma 2.2, we have

$$f(x_i) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1)f'(\vartheta_1) - (\vartheta_2 - x_i)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_5(x_i, v)f''(v)dv, \tag{3.163}$$

$$f(y_i) = f(\vartheta_1) + (\vartheta_2 - \vartheta_1)f'(\vartheta_1) - (\vartheta_2 - y_i)f'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2} G_5(y_i, v)f''(v)dv. \tag{3.164}$$

From (3.163) and (3.164), we get

$$\begin{aligned} I_f(\mathbf{p},\mathbf{q},\mathbf{u}) - I_f(\mathbf{r},\mathbf{q},\mathbf{u}) &= \sum_{i=1}^{n} u_i q_i f\left(\frac{p_i}{q_i}\right) - \sum_{i=1}^{n} u_i q_i f\left(\frac{r_i}{q_i}\right) \\ &= -\sum_{i=1}^{n} u_i q_i \left(\vartheta_2 - \frac{p_i}{q_i}\right) f'(\vartheta_2) + \sum_{i=1}^{n} u_i q_i \left(\vartheta_2 - \frac{r_i}{q_i}\right) f'(\vartheta_2) \\ &+ \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{n} u_i q_i G_5\left(\frac{p_i}{q_i}, v\right) - \sum_{i=1}^{n} u_i q_i G_5\left(\frac{r_i}{q_i}, v\right)\right] f''(v)dv. \end{aligned} \tag{3.165}$$

Using (3.160), we have

$$I_f(\mathbf{p},\mathbf{q},\mathbf{u}) - I_f(\mathbf{r},\mathbf{q},\mathbf{u}) = \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{n} u_i q_i G_5\left(\frac{p_i}{q_i}, v\right) - \sum_{i=1}^{n} u_i q_i G_5\left(\frac{r_i}{q_i}, v\right)\right] f''(v)dv. \tag{3.166}$$

As f is convex function so $f''(v) \geq 0$ for all $v \in [\vartheta_1, \vartheta_2]$. Hence using (3.162) in (3.166), we get (3.161).

Note that the condition for the existence of second derivative of f is not necessary ([144, p.172]). As it is possible to approximate uniformly a continuous convex function by convex polynomials, so we can directly eliminate this differentiability condition. □

The following theorem gives equivalent statements between continuous convex functions and Green functions via majorization inequality and "useful" Zipf-Mandelbrot law.

Theorem 3.20 *Assume* $n \in \{1,2,3,\ldots\}$, $t_1, t_2 \geq 0$ *and* $s_1, s_2 > 0$ *such that satisfying*

$$\sum_{i=1}^{n} \frac{u_i}{(i+t_2)^{s_2}} = \frac{H_{n,t_2,s_2}}{H_{n,t_1,s_1}} \sum_{i=1}^{n} \frac{u_i}{(i+t_1)^{s_1}}, \tag{3.167}$$

with

$$\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i=1,\ldots,n).$$

If $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i=1,\ldots,n)$ *and* G_d $(d=1,2,3,4,5)$ *be defined as in (2.47)-(2.50) and (1.180), then we have following equivalent statements.*

(i) For every continuous convex function $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$, *we have*

$$I_f(i,n,t_1,s_1,\mathbf{q},\mathbf{u}) - I_f(i,n,t_2,s_2,\mathbf{q},\mathbf{u}) \geq 0. \tag{3.168}$$

(ii) For all $v \in [\vartheta_1, \vartheta_2]$, *we have*

$$I_{G_d}(i,n,t_1,s_1,\mathbf{q},\mathbf{u}) - I_{G_d}(i,n,t_2,s_2,\mathbf{q},\mathbf{u}) \geq 0, \ \ d=1,2,3,4,5. \tag{3.169}$$

Moreover, if we change the sign of inequality in both inequalities (3.168) *and* (3.169), *then the above result still holds.*

Proof. $(i) \Rightarrow (ii)$: The proof is similar to the proof of Theorem 3.19.

$(ii) \Rightarrow (i)$: Let $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a convex function so without loss of generality we can assume that $f \in C^2([\vartheta_1, \vartheta_2])$, and further, assume that the statement (ii) holds. Then by Lemma 2.2, we have (3.163) and (3.164).

From (3.163) and (3.164), we get

$$\begin{aligned} I_f(i,n,t_1,s_1,\mathbf{q},\mathbf{u}) - I_f(i,n,t_2,s_2,\mathbf{q},\mathbf{u}) &= \sum_{i=1}^{n} u_i q_i f(\lambda_i) - \sum_{i=1}^{n} u_i q_i f(\mu_i) \\ &= -\sum_{i=1}^{n} u_i q_i (\vartheta_2 - \lambda_i) f'(\vartheta_2) + \sum_{i=1}^{n} u_i q_i (\vartheta_2 - \mu_i) f'(\vartheta_2) \\ &+ \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{n} u_i q_i G_5(\lambda_i, v) - \sum_{i=1}^{n} u_i q_i G_5(\mu_i, v) \right] f''(v) dv, \end{aligned}$$

where,

$$\lambda_i := \frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}, \ \ and \ \ \mu_i := \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}.$$

Using (3.167), we have

$$
\begin{aligned}
& I_f(i,n,t_1,s_1,\mathbf{q},\mathbf{u}) - I_f(i,n,t_2,s_2,\mathbf{q},\mathbf{u}) \\
&= \int_{\vartheta_1}^{\vartheta_2} \left[\sum_{i=1}^{n} u_i q_i G_5(\lambda_i, v) - \sum_{i=1}^{n} u_i q_i G_5(\mu_i, v) \right] f''(v) dv.
\end{aligned} \tag{3.170}
$$

As f is convex function, therefore $f''(v) \geq 0$ for all $v \in [\vartheta_1, \vartheta_2]$. Hence using (3.169) in (3.170), we get (3.168). □

We consider "useful" Csiszár functional [64, 63] in integral form:

Definition 3.36 ***("Useful" Csiszár divergence as integral form)*** *Assume $J := [\alpha, \beta] \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function with densities $p : [a,b] \to J$, $q : [a,b] \to (0,\infty)$ and associated with the utility density $u : [a,b] \to J$ such that*

$$
\frac{p(x)}{q(x)} \in J, \quad \forall x \in [a,b],
$$

then we denote "useful" Csiszár divergence in integral form as

$$
\hat{I}_f(p,q,u) := \int_a^b u(t)q(t) f\left(\frac{p(t)}{q(t)}\right) dt. \tag{3.171}
$$

Remark 3.9 *One can easily seen that if we substitute $u(t) = 1$ for all $t \in [a,b]$, then (3.171) becomes*

$$
\hat{I}_f(p,q,1) := \hat{I}_f(p,q) = \int_a^b q(t) f\left(\frac{p(t)}{q(t)}\right) dt.
$$

Theorem 3.21 *Assume $J := [0,\infty) \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ be a convex function and $p,q,r,u : [a,b] \to (0,\infty)$ such that*

$$
\int_a^{\upsilon} u(t)r(t)dt \leq \int_a^{\upsilon} u(t)p(t)dt, \quad \upsilon \in [a,b] \tag{3.172}
$$

and

$$
\int_a^b u(t)r(t)dt = \int_a^b u(t)p(t)dt, \tag{3.173}
$$

with

$$
\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J, \quad \forall t \in [a,b].
$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function on $[a,b]$, then

$$
\hat{I}_f(r,q,u) \leq \hat{I}_f(p,q,u). \tag{3.174}
$$

(ii) If $\frac{p(t)}{q(t)}$ *is an increasing function on* $[a,b]$*, then the inequality is reversed, i.e.*

$$\hat{I}_f(r,q,u) \geq \hat{I}_f(p,q,u). \tag{3.175}$$

If f *is strictly convex function and* $p(t) \neq r(t)$ $(a.e.)$*, then strict inequality holds in* (3.174) *and* (3.175)*.*
If f *is concave function then the reverse inequalities hold in (3.174) and (3.175). If* f *is strictly concave and* $p(t) \neq r(t)$ $(a.e.)$*, then the strict reverse inequalities hold in (3.174) and (3.175).*

Proof. (i): We use Theorem 1.20 (i) with substitutions $x(t) := \frac{p(t)}{q(t)}$, $y(t) := \frac{r(t)}{q(t)}$, $w(t) := u(t)q(t) > 0\ \forall t \in [a,b]$ and also using the fact that $\frac{r(t)}{q(t)}$ is a decreasing function then we get (3.174).
(ii) We can prove with the similar substitutions as in the first part by using Theorem 1.20 (ii) that is the fact that $\frac{p(t)}{q(t)}$ is an increasing function. □

Remark 3.10 *We can give Theorem 3.21 for* $u(t) := 1$ *for all* $t \in [a,b]$ *as special case which has been given in [103].*

Here, we present several special cases of the previous results as applications.
The first case corresponds to the entropy of a continuous probability density (see [120, p.506]):

Definition 3.37 ***(Integral Shannon's entropy)*** *Let* $p : [a,b] \to (0,\infty)$ *be a positive probability density, then the Shannon entropy of* $p(x)$ *is defined by*

$$H\left(p(x),u(x)\right) := -\int_a^b u(x)\,p(x)\log p(x)dx, \tag{3.176}$$

associated with the utility density $u : [a,b] \to \mathbb{R}$*, whenever the integral exists.*

Note that there is no problem with the definition in the case of a zero probability, since

$$\lim_{x\to 0} x\log x = 0. \tag{3.177}$$

Corollary 3.51 *Assume* $p,q,r,u : [a,b] \to (0,\infty)$ *be functions such that satisfying (3.172) and (3.173) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \quad \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ *is a decreasing function and the base of* log *is greater than 1, then we have estimates for the Shannon entropy of* $q(t)$ *associated with utility density* $u(t)$

$$\int_a^b u(t)q(t)\log\left(\frac{r(t)}{q(t)}\right) \geq H(q(t),u(t)). \tag{3.178}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.178).*

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function and the base of log *is greater than 1, then we have estimates for the Shannon entropy of $q(t)$ associated with utility density $u(t)$*

$$H\left(q(t),u(t)\right) \leq \int_a^b u(t)q(t)\log\left(\frac{p(t)}{q(t)}\right). \tag{3.179}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.179).*

Proof. (i): Substitute $f(x) := -\log x$ and $p(t) := 1, \forall t \in [a,b]$ in Theorem 3.21 (i) then we get (3.178).
(ii) We can prove by switching the role of $p(t)$ with $r(t)$ i.e., $r(t) := 1 \forall t \in [a,b]$ and $f(x) := -\log x$ in Theorem 3.21 (ii) then we get (3.179). □

The second case corresponds to the relative entropy or the Kullback-Leibler divergence between two probability densities associated with the utility density $u(t)$:

Definition 3.38 ***(Integral Kullback-Leibler divergence)*** *Let $p,q : [a,b] \to (0,\infty)$ be a positive probability densities, then the Kullback-Leibler (K-L) divergence between $p(t)$ and $q(t)$ is defined by*

$$L\left(p(t),q(t),u(t)\right) := \int_a^b u(t)p(t)\log\left(\frac{p(t)}{q(t)}\right)dt,$$

associated with the utility density $u : [a,b] \to \mathbb{R}$.

Corollary 3.52 *Assume $p,q,r,u : [a,b] \to (0,\infty)$ be functions such that satisfying (3.172) and (3.173) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function and the base of log *is greater than 1, then*

$$\hat{I}_{(-\log x)}(r,q,u) \geq \hat{I}_{(-\log x)}(p,q,u). \tag{3.180}$$

If the base of log *is in between 0 and 1, then the reverse inequality holds in (3.180).*

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function and the base of log *is greater than 1, then*

$$\hat{I}_{(-\log x)}(r,q,u) \leq \hat{I}_{(-\log x)}(p,q,u). \tag{3.181}$$

If the base of log *is in between 0 and 1 then the reverse inequality holds in (3.181).*

Proof. (i): Substitute $f(x) := -\log x$ in Theorem 3.21 (i) then we get (3.180).
(ii) We can prove with substitution $f(x) := -\log x$ in Theorem 3.21 (ii). □

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence.

Definition 3.39 ***(Variational distance)*** *Let $p,q:[a,b]\to(0,\infty)$ be a positive probability densities, then variation distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{I}_v(p(t),q(t),u(t)) := \int_a^b u(t)\,|p(t)-q(t)|\,dt,$$

associated with the utility density $u:[a,b]\to\mathbb{R}$.

Corollary 3.53 *Assume $p,q,r,u:[a,b]\to(0,\infty)$ be functions such that satisfying (3.172) and (3.173) with*

$$\frac{p(t)}{q(t)},\frac{r(t)}{q(t)} \in J := (0,\infty),\ \ \forall t\in[a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{I}_v(r(t),q(t),u(t)) \le \hat{I}_v(p(t),q(t),u(t)). \tag{3.182}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{I}_v(r(t),q(t),u(t)) \ge \hat{I}_v(p(t),q(t),u(t)). \tag{3.183}$$

Proof. (i): Since $f(x):=|x-1|$ be a convex function for $x\in\mathbb{R}^+$, therefore substitute $f(x):=|x-1|$ in Theorem 3.21 (i) then

$$\int_a^b u(t)q(t)\left|\frac{r(t)}{q(t)}-1\right|dt \le \int_a^b u(t)q(t)\left|\frac{p(t)}{q(t)}-1\right|dt,$$

$$\int_a^b u(t)q(t)\frac{|r(t)-q(t)|}{|q(t)|}dt \le \int_a^b u(t)q(t)\frac{|p(t)-q(t)|}{|q(t)|}dt,$$

since $q(t)>0$ then we get (3.182).
(ii) We can prove with substitution $f(x):=|x-1|$ in Theorem 3.21 (ii). □

Definition 3.40 ***(Hellinger distance)*** *Let $p,q:[a,b]\to(0,\infty)$ be a positive probability densities, then the Hellinger distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{I}_H(p(t),q(t),u(t)) := \int_a^b u(t)\left[\sqrt{p(t)}-\sqrt{q(t)}\right]^2 dt,$$

associated with the utility density $u:[a,b]\to\mathbb{R}$.

Corollary 3.54 *Assume $p,q,r,u:[a,b]\to(0,\infty)$ be functions such that satisfying (3.172) and (3.173) with*

$$\frac{p(t)}{q(t)},\frac{r(t)}{q(t)} \in J := (0,\infty),\ \ \forall t\in[a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{I}_H(r(t),q(t),u(t)) \leq \hat{I}_H(p(t),q(t),u(t)). \tag{3.184}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{I}_H(r(t),q(t),u(t)) \geq \hat{I}_H(p(t),q(t),u(t)). \tag{3.185}$$

Proof. (i): Since $f(x) := (\sqrt{x}-1)^2$ be a convex function for $x \in \mathbb{R}^+$, therefore substitute $f(x) := (\sqrt{x}-1)^2$ in Theorem 3.21 (i) then

$$\int_a^b u(t)q(t)\left[\sqrt{\frac{r(t)}{q(t)}}-1\right]^2 dt \leq \int_a^b u(t)q(t)\left[\sqrt{\frac{p(t)}{q(t)}}-1\right]^2 dt,$$

since $q(t) > 0$ then we get (3.184).
(ii) We can prove with substitution $f(x) := (\sqrt{x}-1)^2$ in Theorem 3.21 (ii). □

Definition 3.41 ***(Bhattacharyya distance)*** *Let $p,q : [a,b] \to (0,\infty)$ be a positive probability densities, then the Bhattacharyya distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{I}_B(p(t),q(t),u(t)) := \int_a^b u(t)\sqrt{p(t)\,q(t)}dt,$$

associated with the utility density $u : [a,b] \to \mathbb{R}$.

Corollary 3.55 *Assume $p,q,r,u : [a,b] \to (0,\infty)$ be functions such that satisfying (3.172) and (3.173) with*

$$\frac{p(t)}{q(t)}, \frac{r(t)}{q(t)} \in J := (0,\infty), \ \forall t \in [a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{I}_B(p(t),q(t),u(t)) \leq \hat{I}_B(r(t),q(t),u(t)). \tag{3.186}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{I}_B(p(t),q(t),u(t)) \geq \hat{I}_B(r(t),q(t),u(t)). \tag{3.187}$$

Proof. (i): Since $f(x) := -\sqrt{x}$ be a convex function for $x \in \mathbb{R}^+$, therefore substitute $f(x) := -\sqrt{x}$ in Theorem 3.21 (i) then

$$\int_a^b u(t)q(t)\left(-\sqrt{\frac{r(t)}{q(t)}}\right) dt \leq \int_a^b u(t)q(t)\left(-\sqrt{\frac{p(t)}{q(t)}}\right) dt,$$

we get (3.186).
(ii) We can prove with substitution $f(x) := -\sqrt{x}$ in Theorem 3.21 (ii). □

Definition 3.42 ***(Jeffreys distance)*** *Let $p,q:[a,b]\to(0,\infty)$ be a positive probability densities, then the Jeffreys distance between $p(t)$ and $q(t)$ is defined by*

$$\hat{I}_J\left(p(t),q(t),u(t)\right):=\int_a^b u(t)\left[p(t)-q(t)\right]\ln\left[\frac{p(t)}{q(t)}\right]dt,$$

associated with the utility density $u:[a,b]\to\mathbb{R}$.

Corollary 3.56 *Assume $p,q,r,u:[a,b]\to(0,\infty)$ be functions such that satisfying (3.172) and (3.173) with*

$$\frac{p(t)}{q(t)},\ \frac{r(t)}{q(t)}\in J:=(0,\infty),\ \ \forall t\in[a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{I}_J\left(r(t),q(t),u(t)\right)\le\hat{I}_J\left(p(t),q(t),u(t)\right). \tag{3.188}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{I}_J\left(r(t),q(t),u(t)\right)\ge\hat{I}_J\left(p(t),q(t),u(t)\right). \tag{3.189}$$

Proof. (i): Since $f(x):=(x-1)\ln x$ be a convex function for $x\in\mathbb{R}^+$, therefore substitute $f(x):=(x-1)\ln x$ in Theorem 3.21 (i) then

$$\begin{aligned}&\int_a^b u(t)q(t)\left(\frac{r(t)}{q(t)}-1\right)\ln\left(\frac{r(t)}{q(t)}\right)dt\\ &\le\int_a^b u(t)q(t)\left(\frac{p(t)}{q(t)}-1\right)\ln\left(\frac{p(t)}{q(t)}\right)dt,\end{aligned}$$

we get (3.188).
(ii) We can prove with substitution $f(x):=(x-1)\ln x$ in Theorem 3.21 (ii). □

Definition 3.43 ***(Triangular discrimination)*** *Let $p,q:[a,b]\to(0,\infty)$ be a positive probability densities, then the triangular discrimination between $p(t)$ and $q(t)$ is defined by*

$$\hat{I}_\Delta\left(p(t),q(t),u(t)\right):=\int_a^b u(t)\frac{\left[p(t)-q(t)\right]^2}{p(t)+q(t)}dt,$$

associated with the utility density $u:[a,b]\to\mathbb{R}$.

Corollary 3.57 *Assume $p,q,r,u:[a,b]\to(0,\infty)$ be functions such that satisfying (3.172) and (3.173) with*

$$\frac{p(t)}{q(t)},\ \frac{r(t)}{q(t)}\in J:=(0,\infty),\ \ \forall t\in[a,b].$$

(i) If $\frac{r(t)}{q(t)}$ is a decreasing function, then

$$\hat{I}_\Delta(r(t),q(t),u(t)) \leq \hat{I}_\Delta(p(t),q(t),u(t)). \tag{3.190}$$

(ii) If $\frac{p(t)}{q(t)}$ is an increasing function, then the inequality is reversed, i.e.

$$\hat{I}_\Delta(r(t),q(t),u(t)) \geq \hat{I}_\Delta(p(t),q(t),u(t)). \tag{3.191}$$

Proof. (i): Since $f(x) := \frac{(x-1)^2}{x+1}$ be a convex function for $x \geq 0$, therefore substitute $f(x) := \frac{(x-1)^2}{x+1}$ in Theorem 3.21 (i) then

$$\int_a^b u(t)q(t)\frac{(r(t)/q(t)-1)^2}{r(t)/q(t)+1}\,dt \leq \int_a^b u(t)q(t)\frac{(p(t)/q(t)-1)^2}{p(t)/q(t)+1}\,dt,$$

$$\int_a^b u(t)q(t)\frac{((r(t)-q(t))/q(t))^2}{(r(t)+q(t))/q(t)}\,dt \leq \int_a^b u(t)q(t)\frac{((p(t)-q(t))/q(t))^2}{(p(t)+q(t))/q(t)}\,dt,$$

we get (3.190).
(ii) We can prove with substitution $f(x) := \frac{(x-1)^2}{x+1}$ in Theorem 3.21 (ii). □

Remark 3.11 *We can give all the results of section 5 for $u(t) = 1$ for all $t \in [a,b]$ as a special case, which has been given in [103].*

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Index

Author Index

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