Chapter 1

Definitions and Basic Results

1.1 Hilbert-type Inequalities with Conjugate Exponents

Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. The Hilbert inequality asserts that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q \tag{1.1}$$

holds for all non-negative measurable functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$. After its discovery at the beginning of the 20th century, the Hilbert inequality was studied by numerous authors, who improved and generalized it in many different directions. This inequality is still of interest to numerous authors. The applications in diverse fields of mathematics have certainly contributed to its importance. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to a classical monograph [47], while some recent results are collected in monograph [63].

In this book we refer to the following multidimensional extension of inequality (1.1) established by Krnić *et al.* (see [63], [99]).

Theorem 1.1 *Suppose* $(\Omega_i,\Sigma_i,\mu_i)$ *are* σ*-finite measure spaces,* $\sum_{i=1}^n \frac{1}{p_i}=1$, $p_i>1$, *and* $K:\Omega\to\mathbb{R}$, $\phi_{ij}:\Omega_j\to\mathbb{R}$, $f_i:\Omega_i\to\mathbb{R}$, $i,j=1,2,\dots,n$, *are non-negative measurable functions. If* $\prod_{i,j=1}^n \phi_{ij}(x_j)=1$, *then the following inequalities hold and are equivalent*

$$\int_{\Omega} K(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mu(\mathbf{x}) \leq \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i} \tag{1.2}$$

and

$$\left[\int_{\Omega_n}\left(\frac{1}{(\phi_{nn}\omega_n)(x_n)}\int_{\hat{\Omega}^n}K(\mathbf{x})\prod_{i=1}^{n-1}f_i(x_i)d\hat{\mu}^n(\mathbf{x})\right)^P d\mu(x_n)\right]^{\frac{1}{P}} \le \prod_{i=1}^{n-1}\|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{1.3}$$

where $\frac{1}{P}=\sum_{i=1}^{n-1}\frac{1}{p_i}$, $\mathbf{\Omega}=\prod_{i=1}^{n}\Omega_i$, $\hat{\mathbf{\Omega}}^i=\prod_{j=1,j\neq i}^{n}\Omega_j$, $\mathbf{x}=(x_1,x_2,\dots,x_n)$, $d\mu(\mathbf{x})=\prod_{i=1}^{n}d\mu_i(x_i)$, $d\hat{\mu}^i(\mathbf{x})=\prod_{j=1,j\neq i}^{n}d\mu_j(x_j)$, *and*

$$\omega_i(x_i)=\left[\int_{\hat{\Omega}^i}K(\mathbf{x})\prod_{j=1,j\neq i}^{n}\phi_{ij}^{p_i}(x_j)d\hat{\mu}^i(\mathbf{x})\right]^{\frac{1}{p_i}}. \tag{1.4}$$

The above notation will be used throughout the whole monograph. In addition, $\|\cdot\|_r$ stands for the usual norm in $L^r(\Omega)$, that is $\|f\|_r=[\int_\Omega|f(x)|^r d\mu(x)]^{\frac{1}{r}}$, $r>1$. Inequalities following from (1.2) are usually referred to as the Hilbert-type inequalities since (1.1) is a particular case of (1.2). Further, inequalities related to (1.3) are usually called Hardy-Hilbert-type inequalities since (1.3) implies the classical Hardy inequality, which will be discussed later. Inequalities (1.2) and (1.3) are closely connected in the sense that one implies the other, hence they are sometimes both referred to as the Hilbert-type inequalities, for brevity.

Perić and Vuković [77], developed a unified treatment of the Hilbert and Hardy-Hilbert type inequalities with general homogeneous kernel. Further, regarding the notations from Theorem 1.1, we assume that $\Omega_i=\mathbb{R}_+$, equipped with the non-negative Lebesgue measures $d\mu_i(x_i)=dx_i$, $i=1,2,\dots,n$. In addition, we have $\mathbf{\Omega}=\mathbb{R}_+^n$ and $d\mathbf{x}=dx_1dx_2\dots dx_n$.

Recall that the function $K:\mathbb{R}_+^n\to\mathbb{R}$ is said to be homogeneous of degree $-s$, $s>0$, if $K(t\mathbf{x})=t^{-s}K(\mathbf{x})$ for all $t>0$. Furthermore, for $\mathbf{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}_+^n$, we define

$$k_i(\mathbf{a})=\int_{\mathbb{R}_+^{n-1}}K(\hat{\mathbf{u}}^i)\prod_{j=1,j\neq i}^{n}u_j^{a_j}\hat{d}^i\mathbf{u}, \quad i=1,2,\dots,n, \tag{1.5}$$

where $\hat{\mathbf{u}}^i=(u_1,\dots,u_{i-1},1,u_{i+1},\dots,u_n)$, $\hat{d}^i\mathbf{u}=du_1\dots du_{i-1}du_{i+1}\dots du_n$, and provided that the above integral converges.

Utilizing Theorem 1.1 one obtains the following equivalent inequalities with general homogeneous kernel of degree $-s$:

$$\int_{\mathbb{R}_+^n}K(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mathbf{x}\le\prod_{i=1}^{n}k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n}\|x_i^{(n-1-s)/p_i+\alpha_i}f_i\|_{p_i} \tag{1.6}$$

and

$$\left[\int_{\mathbb{R}_+}x_n^{(1-P)(n-1-s)-P\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}}K(\mathbf{x})\prod_{i=1}^{n-1}f_i(x_i)\hat{d}^n\mathbf{x}\right)^P dx_n\right]^{\frac{1}{P}} \le\prod_{i=1}^{n}k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n-1}\|x_i^{(n-1-s)/p_i+\alpha_i}f_i\|_{p_i}, \tag{1.7}$$

where A_{ij}, $i,j = 1,2,\ldots,n$, are real parameters such that $\sum_{i=1}^{n} A_{ij} = 0$ for $j = 1,2,\ldots,n$, $\alpha_i = \sum_{j=1}^{n} A_{ij}$, $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $i = 1,2,\ldots,n$, and $k_i(\cdot)$, $i = 1,2,\ldots,n$, is defined by (1.5).
To obtain a case of the inequalities with the best possible constants it is natural to impose the following conditions on parameters A_{ij} :

$$p_j A_{ji} = s - n - p_i(\alpha_i - A_{ii}), \quad j \neq i, \quad i,j \in \{1,2,\ldots,n\}. \tag{1.8}$$

In that case the constant factors from inequalities (1.6) and (1.7) are simplified to the following form:

$$L^* = k_1(\widetilde{\mathbf{A}}), \tag{1.9}$$

where $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$ and

$$\widetilde{A}_i = p_1 A_{1i} \quad \text{for} \quad i \neq 1 \quad \text{and} \quad \widetilde{A}_1 = p_n A_{n1}. \tag{1.10}$$

Further, by using (1.8) and (1.9), the inequalities (1.6) and (1.7) with the parameters A_{ij}, satisfying the relation (1.8) become

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x} \leq L^* \prod_{i=1}^{n} \|x_i^{-\widetilde{A}_i - 1/p_i} f_i\|_{p_i} \tag{1.11}$$

and

$$\begin{aligned} &\left[\int_{\mathbb{R}_+} x_n^{(1-P)(-1-p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, d\hat{}^n\mathbf{x} \right)^P dx_n \right]^{\frac{1}{P}} \\ &\leq L^* \prod_{i=1}^{n-1} \|x_i^{-\widetilde{A}_i - 1/p_i} f_i\|_{p_i}. \end{aligned} \tag{1.12}$$

Theorem 1.2 ([63]) *Let $K : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, such that for every $i = 2,3,\ldots,n$,*

$$K(1, t_2, \ldots, t_i, \ldots, t_n) \leq CK(1, t_2, \ldots, 0, \ldots, t_n), \ -1 \leq t_i \leq 1, \tag{1.13}$$

where C is a positive constant. Let the parameters $\widetilde{A}_i$, $i = 1,\ldots,n$, be defined by (1.10) *and $0 < \varepsilon < \min_{1 \leq i \leq n}\{p_i + p_i\widetilde{A}_i\}$. If the parameters A_{ij} satisfy the conditions $\sum_{i=1}^{n} A_{ij} = 0$ for $j = 1,2,\ldots,n$, and* (1.8)*, then the constant L^* is the best possible in inequalities* (1.11) *and* (1.12).

The following result based on Theorem 1.1 can be seen in [88]. Let $K : \mathbb{R}_+^n \to \mathbb{R}$ and A_{ij}, $i,j = 1,2,\ldots,n$, be as in Theorem 1.2. If $u_i : (a_i, b_i) \to (0,\infty)$, $i = 1,\ldots,n$ are strictly increasing differentiable functions such that $u_i(a_i) = 0$ and $u_i(b_i) = \infty$, then the following inequalities hold and are equivalent

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} K(u_1(t_1), \ldots, u_n(t_n)) \prod_{i=1}^{n} f_i(t_i)\, dt_1 \cdots dt_n$$

$$< L\prod_{i=1}^{n}\left[\int_0^\infty (u_i(t_i))^{-1-p_i\widetilde{A}_i}(u_i'(t_i))^{1-p_i}f_i^{p_i}(t_i)dt_i\right]^{\frac{1}{p_i}} \tag{1.14}$$

and

$$\int_{a_n}^{b_n}(u_n(t_n))^{(1-P)(-1-p_n\widetilde{A}_n)}\left[\int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}}K(u_1(t_1),\ldots,u_n(t_n))\prod_{i=1}^{n-1}f_i(t_i)dt_1\cdots dt_{n-1}\right]^P dt_n$$
$$< L^P\prod_{i=1}^{n-1}\left[\int_0^\infty (u_i(t_i))^{-1-p_i\widetilde{A}_i}(u_i'(t_i))^{1-p_i}f_i^{p_i}(t_i)dt_i\right]^{\frac{1}{p_i}}, \tag{1.15}$$

where the constants $L = k(\widetilde{A}_2,\ldots,\widetilde{A}_n)$ and L^P are the best possible in inequalities (1.14) and (1.15).

Since the case $n = 2$ of inequalities (1.2) and (1.3) will be of special interest to us, we state it as a separate result. The proof follows directly using substitutions $p_1 = p$, $p_2 = q$, $\phi_{11} = \varphi$ and $\phi_{22} = \psi$. Observe that from $\phi_{11}\phi_{21} = 1$ and $\phi_{12}\phi_{22} = 1$ we have $\phi_{21} = 1/\varphi$ and $\phi_{12} = 1/\psi$ (for more details see e.g. [66]).

Theorem 1.3 *Let $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, and let Ω be a measure space with positive σ-finite measures μ_1 and μ_2. Let $K : \Omega\times\Omega \to \mathbb{R}$ and $\varphi,\psi : \Omega \to \mathbb{R}$ be non-negative measurable functions. If the functions F and G are defined by*

$$F^p(x) = \int_\Omega K(x,y)\psi^{-p}(y)d\mu_2(y), \quad G^q(y) = \int_\Omega K(x,y)\varphi^{-q}(x)d\mu_1(x), \tag{1.16}$$

then for all non-negative measurable functions f and g on Ω the inequalities

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \le \|\varphi F f\|_p\|\psi G g\|_q \tag{1.17}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)$$
$$\le \|\varphi F f\|_p^p \tag{1.18}$$

hold and are equivalent.
If $0 < p < 1$, then the reverse inequalities in (1.17) *and* (1.18) *are valid, as well as the inequality*

$$\int_\Omega F^{1-q}(x)\varphi^{-q}(x)\left[\int_\Omega K(x,y)g(y)d\mu_2(y)\right]^q d\mu_1(x)$$
$$\le \|\psi G g\|_q^q. \tag{1.19}$$

Remark 1.1 The equality in the previous theorem is possible if and only if it holds in the Hölder inequality, that is, if

$$\left[f(x)\frac{\varphi(x)}{\psi(y)}\right]^p = C\left[g(y)\frac{\psi(y)}{\varphi(x)}\right]^q, \quad \text{a.e. on } \Omega,$$

where C is a positive constant. In that case we have

$$f(x) = C_1\varphi^{-q}(x) \quad \text{and} \quad g(y) = C_2\psi^{-p}(y) \quad \text{a.e. on } \Omega, \tag{1.20}$$

for some constants C_1 and C_2, which is possible if and only if

$$\int_\Omega F(x)\varphi^{-q}(x)d\mu_1(x) < \infty \quad \text{and} \int_\Omega G(y)\psi^{-p}(y)d\mu_2(y) < \infty. \tag{1.21}$$

Otherwise, the inequalities in Theorem 1.3 are strict.

For homogeneous function $K(x,y)$ we define $k(\alpha)$ (see also definition (1.5)) as

$$k(\alpha) = \int_0^\infty K(1,u)u^{-\alpha}du, \tag{1.22}$$

provided that the above integral converges.
In the following theorem the integrals are taken over an arbitrary interval of non-negative real numbers, i.e. $(a,b) \subseteq \mathbb{R}_+$, $0 \le a < b \le \infty$, and the weight functions are chosen to be power functions.

Theorem 1.4 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:(a,b)\times(a,b)\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both variables. If A_1 and A_2 are real parameters such that $A_1 \in (\frac{1-s}{q},\frac{1}{q})$, $A_2 \in (\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ the inequalities*

$$\begin{aligned}
&\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy \\
&\le \left[\int_a^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_a^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}
\end{aligned} \tag{1.23}$$

and

$$\begin{aligned}
&\int_a^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)^{1-p}y^{(p-1)(s-1)+p(A_1-A_2)} \\
&\quad\times\left[\int_a^b K(x,y)f(x)dx\right]^p dy \\
&\le \int_a^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx
\end{aligned} \tag{1.24}$$

hold and are equivalent, where

$$\varphi_1(\alpha,x) = \left(\frac{a}{x}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du + \left(\frac{x}{b}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\varphi_2(\alpha,y)=\left(\frac{a}{y}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\left(\frac{y}{b}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$, and $K(x,y)$ is strictly decreasing in x and strictly increasing in y, then the reverse inequalities in (1.23) *and* (1.24) *are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality*

$$\int_a^\infty \left(k(pA_2)-\varphi_1(pA_2,x)\right)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_a^\infty K(x,y)g(y)dy\right]^q dx$$
$$\leq \int_a^\infty \left(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\right)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Moreover, if $0<p<1$, $a=0$, and $K(x,y)$ is strictly increasing in x and strictly decreasing in y, then the reverse inequalities in (1.23) *and* (1.24) *hold for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality*

$$\int_0^b \left(k(pA_2)-\varphi_1(pA_2,x)\right)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^b K(x,y)g(y)dy\right]^q dx$$
$$\leq \int_0^b \left(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\right)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Setting $a=0$, $b=\infty$ in the previous theorem, one obtains the corresponding inequalities for an arbitrary non-negative homogeneous function of degree $-s$.

Corollary 1.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:\mathbb{R}_+\to\mathbb{R}$ the inequalities*

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy$$
$$\leq L\left[\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \tag{1.25}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_0^\infty K(x,y)f(x)dx\right]^p dy$$
$$\leq L^p\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx \tag{1.26}$$

hold and are equivalent, where $L=k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1)$.
If $0<p<1$, then the reverse inequalities in (1.25) *and* (1.26) *are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality*

$$\int_0^\infty x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^\infty K(x,y)g(y)dy\right]^q dx$$

$$\leq L^q \int_0^\infty y^{1-s+q(A_2-A_1)} g^q(y)dy. \tag{1.27}$$

Inequalities (1.25) *and* (1.26)*, as well as their reverse inequalities are equivalent. Moreover, equality in the above relations holds if and only if* $f = 0$ *or* $g = 0$ *a.e. on* $\mathbb{R}_+$.

Considering inequalities in Corollary 1.1 with parameters A_1 and A_2 fulfilling condition

$$pA_2 + qA_1 = 2 - s, \tag{1.28}$$

the constant L reduces to $L = k(pA_2)$. It has been shown that such constant is the best possible in the corresponding inequalities.

The following result contains a generalized discrete Hilbert-type inequalities in both equivalent forms. Krnić *et al.* (see [65]) considered the weight functions involving real differentiable functions. By $H(r)$, $r > 0$, is denoted the set of all non-negative differentiable functions $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(i) u is strictly increasing on $\mathbb{R}_+$ and there exists $x_0 \in \mathbb{R}_+$ such that $u(x_0) = 1$,

(ii) $\lim_{x\to\infty} u(x) = \infty$, $\frac{u'(x)}{[u(x)]^r}$ is decreasing on $\mathbb{R}_+$.

Theorem 1.5 *Let* $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, *and let* $s > 0$. *Further, suppose that* $A_1 \in \left(\max\{\frac{1-s}{q}, 0\}, \frac{1}{q}\right)$, $A_2 \in \left(\max\{\frac{1-s}{p}, 0\}, \frac{1}{p}\right)$, $u \in H(qA_1)$ *and* $v \in H(pA_2)$. *If* $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is a non-negative homogeneous function of degree* $-s$, *strictly decreasing in each argument, then the inequalities*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty K(u(m), v(n)) a_m b_n$$
$$\leq L \left[\sum_{m=1}^\infty [u(m)]^{1-s+p(A_1-A_2)} [u'(m)]^{1-p} a_m^p \right]^{\frac{1}{p}}$$
$$\times \left[\sum_{n=1}^\infty [v(n)]^{1-s+q(A_2-A_1)} [v'(n)]^{1-q} b_n^q \right]^{\frac{1}{q}} \tag{1.29}$$

and

$$\sum_{n=1}^\infty [v(n)]^{(s-1)(p-1)+p(A_1-A_2)} v'(n) \left[\sum_{m=1}^\infty K(u(m), v(n)) a_m \right]^p$$
$$\leq L^p \sum_{m=1}^\infty [u(m)]^{(1-s)+p(A_1-A_2)} [u'(m)]^{1-p} a_m^p \tag{1.30}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, *where*

$$L = k^{\frac{1}{p}}(pA_2) k^{\frac{1}{q}}(2 - s - qA_1). \tag{1.31}$$

Moreover, inequalities (1.29) *and* (1.30) *are equivalent.*

If the parameters A_1 and A_2 satisfy (1.28), that is, $pA_2 + qA_1 = 2 - s$, then the constant L from Theorem 1.5 becomes

$$L^* = k(pA_2). \tag{1.32}$$

Moreover, it has been shown that the constant L^* is the best possible in the following inequalities

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(u(m),v(n))a_m b_n \le L^* \left[\sum_{m=1}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}b_n^q\right]^{\frac{1}{q}} \tag{1.33}$$

and

$$\sum_{n=1}^{\infty}[v(n)]^{(p-1)(1-pqA_2)}v'(n)\left[\sum_{m=1}^{\infty}K(u(m),v(n))a_m\right]^p \le (L^*)^p \sum_{m=1}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p. \tag{1.34}$$

1.2 Hilbert-type Inequalities with Non-conjugate Exponents

First, we introduce n-dimensional extension of conjugate exponents. Let $i = 1,2,\dots n$ and let p_i, p_i', q_i, λ satisfy

$$\begin{aligned}
&p_i > 1, \quad \frac{1}{p_i} + \frac{1}{p_i'} = 1,\\
&\sum_{i=1}^{n}\frac{1}{p_i} \ge 1,\\
&\lambda = \frac{1}{n-1}\sum_{i=1}^{n}\frac{1}{p_i'} \quad \text{and} \quad \frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1,\dots,n,\\
&\frac{1}{q_i} > 0, \quad i = 1,\dots,n.
\end{aligned} \tag{1.35}$$

It follows from these conditions that

$$\frac{1}{q_i} + (1-\lambda) = \frac{1}{p_i}, \quad i = 1,\dots,n, \tag{1.36}$$

and

$$\sum_{i=1}^{n} \frac{1}{q_i} + (1-\lambda) = 1. \tag{1.37}$$

Observe that for $\lambda = 1$ the above parameters reduce to the conjugate case, that is, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ and $p_i = q_i$, $i = 1,2,\ldots,n$.

The following extension from [27] may also be regarded as a non-conjugate version of Theorem 1.1.

Let Ω_i be a measure space with σ-finite measure μ_i, $i = 1,2,\ldots,n$. Further, suppose that $K : \mathbf{\Omega} \to \mathbb{R}$ and $\phi_{ij} : \Omega \to \mathbb{R}$, $i,j = 1,\ldots,n$, are non-negative measurable functions such that $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$. If the functions ω_i, $i = 1,2,\ldots,n$, are defined by

$$\omega_i(x_i) = \left[\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) d\hat{\mu}^i(\mathbf{x}) \right]^{\frac{1}{q_i}} \tag{1.38}$$

then for all non-negative measurable functions $f_i : \Omega \to \mathbb{R}$, $i = 1,2,\ldots,n$, the inequalities

$$\int_{\mathbf{\Omega}} K^{\lambda}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mu(\mathbf{x}) \leq \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i} \tag{1.39}$$

and

$$\left[\int_{\Omega_n} \left(\frac{1}{(\phi_{nn}\omega_n)(x_n)} \int_{\hat{\mathbf{\Omega}}^n} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{\mu}^n(\mathbf{x}) \right)^{p'_n} d\mu(x_n) \right]^{\frac{1}{p'_n}} \leq \prod_{i=1}^{n-1} \|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{1.40}$$

hold and are equivalent.

Remark 1.2 Equality in the previous inequalities is possible if and only if it holds in Hölder's inequality. It means that the functions

$$K(\mathbf{x})\phi_{ii}{}^{p_i}(x_i) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j)\omega_i{}^{p_i - q_i}(x_i) f_i{}^{p_i}(x_i), \quad i = 1,2,\ldots n,$$

and $\prod_{i=1}^{n}(\phi_{ii}\omega_i f_i)^{p_i}(x_i)$ are proportional (see also [27]). Hence, we obtain that the equality in mentioned inequalities can be achieved only if the functions f_i and the kernel K are defined by $f_i(x_i) = C_i \phi_{ii}(x_i)^{\frac{q_i}{1-\lambda q_i}} \omega_i(x_i)^{(1-\lambda)q_i}$ and $K(\mathbf{x}) = C\prod_{i=1}^{n} \omega_i{}^{q_i}(x_i)$, $i = 1,2,\ldots n$, where C and C_i are arbitrary constants. It is possible only if the functions

$$\frac{\prod_{j=1, j\neq i}^{n} \phi_{jj}{}^{\frac{\lambda q_j}{1-\lambda q_j}}(x_j)}{\prod_{j=1, j\neq i}^{n} \phi_{ij}{}^{\lambda q_j}(x_j)}, \quad i = 1,2,\ldots,n$$

are adequate constants, and

$$\int_{\Omega} \omega_i^{q_i}(x_i)\phi_{ii}^{\frac{q_i}{1-\lambda q_i}}(x_i)d\mu_i(x_i) < \infty, \quad i = 1,2,\ldots n.$$

Otherwise, the inequalities (1.39) and (1.40) are strict.

Now, suppose that the kernel $K : \mathbb{R}_+^n \to \mathbb{R}$ is homogeneous of degree $-s$, $s > 0$. Taking into account the notation from Theorem 1.1, we assume that $\Omega_i = \mathbb{R}_+$, equipped with the non-negative Lebesgue measures $d\mu_i(x_i) = dx_i$, $i = 1,2,\ldots,n$. In addition, we have $\Omega = \mathbb{R}_+^n$ and $d\mathbf{x} = dx_1dx_2\ldots dx_n$. If the parameters A_{ij} appearing in functions $\phi_{ij}(x_j) = x_j^{A_{ij}}$ satisfy relations $\sum_{i=1}^n A_{ij} = 0$, $j = 1,\ldots,n$, then the condition $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$ is fulfilled. Setting the power weight functions in the inequalities (1.39) and (1.40), one obtains the following equivalent inequalities

$$\begin{aligned} &\int_{\mathbb{R}_+^n} K^{\lambda}(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x} \\ &\leq \prod_{i=1}^n k_i^{\frac{1}{q_i}}(q_i\mathbf{A}_i)\prod_{i=1}^n \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}, \end{aligned} \tag{1.41}$$

and

$$\begin{aligned} &\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda}(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)dx_1\cdots dx_{n-1}\right)^{p_n'} dx_n\right]^{1/p_n'} \\ &\leq \prod_{i=1}^n k_i^{\frac{1}{q_i}}(q_i\mathbf{A}_i)\prod_{i=1}^{n-1} \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}, \end{aligned} \tag{1.42}$$

where $\alpha_i = \sum_{j=1}^n A_{ij}$, $q_i\mathbf{A}_i = (q_iA_{i1},\ldots,q_iA_{in})$ and $k_i(\cdot)$ is defined by (1.5).

To conclude this section, we restate conditions in (1.35) for the case when $n = 2$. Let p and q be real parameters, such that

$$p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} \geq 1, \tag{1.43}$$

and let p' and q' respectively be their conjugate exponents, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Further, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \tag{1.44}$$

and note that $0 < \lambda \leq 1$ for all p and q as in (1.43). Especially, $\lambda = 1$ holds if and only if $q = p'$, that is, only when p and q are mutually conjugate. Otherwise, we have $0 < \lambda < 1$.

The two-dimensional version of inequalities (1.39) and (1.40) can be found in [36].

www.element.hr

Theorem 1.6 *Let p, q, and λ be real parameters as in* (1.43) *and* (1.44)*, and let Ω_1 and Ω_2 be measure spaces with positive σ-finite measures μ_1 and μ_2 respectively. Let K be a non-negative measurable function on $\Omega_1 \times \Omega_2$, φ a measurable, a.e. positive function on Ω_1, and ψ a measurable, a.e. positive function on Ω_2. If the functions F on Ω_1 and G on Ω_2 are defined by*

$$F(x) = \left[\int_{\Omega_2} K(x,y)\psi^{-q'}(y)\,d\mu_2(y)\right]^{\frac{1}{q'}}, \quad x \in \Omega_1, \tag{1.45}$$

and

$$G(y) = \left[\int_{\Omega_1} K(x,y)\varphi^{-p'}(x)\,d\mu_1(x)\right]^{\frac{1}{p'}}, \quad y \in \Omega_2, \tag{1.46}$$

then for all non-negative measurable functions f on Ω_1 and g on Ω_2 the inequalities

$$\int_{\Omega_1}\int_{\Omega_2} K^{\lambda}(x,y)f(x)g(y)\,d\mu_1(x)d\mu_2(y) \le \|\varphi F f\|_p \|\psi G g\|_q \tag{1.47}$$

and

$$\left\{\int_{\Omega_2}\left[(\psi G)^{-1}(y)\int_{\Omega_1} K^{\lambda}(x,y)f(x)\,d\mu_1(x)\right]^{q'} d\mu_2(y)\right\}^{\frac{1}{q'}} \le \|\varphi F f\|_p \tag{1.48}$$

hold and are equivalent.

Applying Theorem 1.6 to non-negative homogeneous functions $K : \Omega \subseteq \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with a negative degree of homogeneity, one obtains the following result. In this way Theorem 1.4 from previous section can be extended to the case of non-conjugate exponents.

Theorem 1.7 *Let p, q, and λ be as in* (1.43) *and* (1.44)*, and let $K : (a,b)\times(a,b) \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$, strictly decreasing in both arguments. Further, suppose that A_1 and A_2 are real parameters such that $A_1 \in \left(\frac{1-s}{p'}, \frac{1}{p'}\right)$, $A_2 \in \left(\frac{1-s}{q'}, \frac{1}{q'}\right)$. If the functions φ_1 and φ_2 are defined as in the statement of Theorem* 1.4*, then for all non-negative measurable functions f and g on (a,b) the inequalities*

$$\begin{aligned}&\int_a^b \int_a^b K^{\lambda}(x,y)f(x)g(y)dxdy \\ &\le \left[\int_a^b \left(k(q'A_2) - \varphi_1(q'A_2,x)\right)^{\frac{p}{q'}} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x)dx\right]^{\frac{1}{p}} \\ &\quad\times\left[\int_a^b \left(k(2-s-p'A_1) - \varphi_2(2-s-p'A_1,y)\right)^{\frac{q}{p'}} y^{\frac{q}{p'}(1-s)+q(A_2-A_1)} g^q(y)dy\right]^{\frac{1}{q}}\end{aligned} \tag{1.49}$$

and

$$\left[\int_a^b y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)}\left(k(2-s-p'A_1)-\varphi_2(2-s-p'A_1,y)\right)^{-\frac{q'}{p'}}\right.$$
$$\left.\times\left(\int_a^b K^{\lambda}(x,y)f(x)dx\right)^{q'}dy\right]^{\frac{1}{q'}}$$
$$\leq\left[\int_a^b \left(k(q'A_2)-\varphi_1(q'A_2,x)\right)^{\frac{p}{q'}}x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}} \tag{1.50}$$

hold and are equivalent. The function $k(\cdot)$ *is defined by* (1.22).

Setting $a=0$, $b=\infty$ in Theorem 1.7, one obtains the corresponding equivalent Hilbert-type and Hardy-Hilbert-type inequalities.

Corollary 1.2 *Assume that* p, q, *and* λ *are as in* (1.43) *and* (1.44), *and* $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ *is a non-negative homogeneous function of degree* $-s$, $s>0$. *Then the inequalities*

$$\int_0^{\infty}\int_0^{\infty}K^{\lambda}(x,y)f(x)g(y)dxdy$$
$$\leq L'\left[\int_0^{\infty}x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^{\infty}y^{\frac{q}{p'}(1-s)+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \tag{1.51}$$

and

$$\left[\int_0^{\infty}y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)}\left(\int_0^{\infty}K^{\lambda}(x,y)f(x)dx\right)^{q'}dy\right]^{\frac{1}{q'}}$$
$$\leq L'\left[\int_0^{\infty}x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}} \tag{1.52}$$

hold for all parameters $A_1\in\left(\frac{1-s}{p'},\frac{1}{p'}\right)$, $A_2\in\left(\frac{1-s}{q'},\frac{1}{q'}\right)$, *and for all non-negative measurable functions* f *and* g *on* $\mathbb{R}_+$, *where* $L'=k^{\frac{1}{q'}}(q'A_2)k^{\frac{1}{p'}}(2-s-p'A_1)$. *Moreover, these inequalities are equivalent.*

1.3 Hardy-type Inequalities

In 1925, Hardy stated and proved in [47] the following integral inequality:

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx<\left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)dx, \tag{1.53}$$