Chapter 1

Zipf-Mandelbrot law, properties and its generalizations

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Abstract. Despite a wide spread applications of Zipf-Mandelbrot law, there is quite small amount of results concerning analytical properties on distribution law. On the first stage, we examine some monotonicity properties of the law, we derive the whole variety of its lower and upper estimations. We then further refine our results using some well-known inequalities such as Hölder and Lyapunov inequality.

On the second stage we consider the case when total mass of Zipf-Mandelbrot law is spread all over positive integer, and then we come to Hurwitz $\zeta-$function. As we show, it is very natural first to examine properties of Hurwitz $\zeta-$function to derive properties of Zipf-Mandelbrot law. Using some well-known inequalities such as Chebyshev's and Lyapunov's inequality we are able to deduce a whole variety of theoretical characterizations that include, among others, log-convexity, log-subadditivity, exponential convexity.

On the third stage, we generalize Zipf-Mandelbrot law using maximization of Shannon entropy, as we get hybrid Zipf-Mandelbrot law. It is interesting that examination of its densities provides some new insights of Lerch's transcendent.

1.1 Some classical inequalities and Zipf-Mandelbrot law

1.1.1 Introduction

For $N \in \mathbb{N},\ q \geq 0,\ s > 0,\ k \in \{1,2,\ldots,N\}$, Zipf-Mandelbrot probability mass function is defined with

$$f(k,N,q,s) = \frac{1/(k+q)^s}{H_{N,q,s}}, \tag{1.1}$$

where

$$H_{N,q,s} = \sum_{i=1}^{N} \frac{1}{(i+q)^s}, \tag{1.2}$$

$N \in \mathbb{N},\ q \geq 0,\ s > 0,\ k \in \{1,2,\ldots,N\}$ (see [5]).

Proposition 1.1 *For $s > t > 0$*

$$(Nf(k,N,q,s))^{1/s} \leq (Nf(k,N,q,t))^{1/t}. \tag{1.3}$$

Proof. In [6] it is proved, after $\frac{1}{Nf(k,N,q,s)}$ is interpreted as power mean depending on s, that $s \mapsto Nf(k,N,q,s)$ is a decreasing function. □

Denote $m = \frac{k+q}{N+q}$, $M = \frac{k+q}{1+q}$ and observe $m = \min\{x_i :\ i = 1,\ldots,N\}$, $M = \max\{x_i :\ i = 1,\ldots,N\}$.
Further, for $s,\ t > 0$ let

$$\mu = \frac{M^s - m^s}{M^t - m^t}$$

and

$$B_{t,s} = \left(\frac{\mu t}{s}\right)^{\frac{1}{t}} \left\{ \frac{m^s M^t - m^t M^s}{(1-s/t)(M^t - m^t)} \right\}^{\frac{1}{s}-\frac{1}{t}}. \tag{1.4}$$

Theorem 1.1 *For probability mass function* (1.39) *we have following inequalities, for* $0 < t < s$

a)

$$\frac{N^{\frac{s}{t}-1}}{B_{t,s}^s} (f(k,N,q,t))^{s/t} \leq f(k,N,q,s) \leq N^{\frac{s}{t}-1} (f(k,N,q,t))^{s/t}, \tag{1.5}$$

b)

$$\frac{M^t - m^t}{f(k,N,q,s)} - \frac{M^s - m^s}{f(k,N,q,t)} \leq N\left(M^t m^s - M^s m^t\right). \tag{1.6}$$

Proof.
a) It follows, for $0 < t < s$,

$$(Nf(k,N,q,s))^{1/s} \leq (Nf(k,N,q,t))^{1/t},$$

hence

$$f(k,N,q,s) \leq N^{\frac{s}{t}-1}\,(f(k,N,q,t))^{s/t}.$$

Now we prove left hand side inequality. First, observe here that $m = \min\{x_i : i = 1,\ldots,N\}$, $M = \max\{x_i : i = 1,\ldots,N\}$.
Using *Beesack inequality (see [2], p.* 334; *[13]*, p. 110)

$$M_N^{[s]}(x_{\overline{1,N}}) \leq B_{t,s} M_N^{[t]}(x_{\overline{1,N}}),\ 0 < t < s, \tag{1.7}$$

where

$$B_{t,s} = \left(\frac{\mu t}{s}\right)^{\frac{1}{t}} \left\{\frac{m^s M^t - m^t M^s}{(1-s/t)(M^t - m^t)}\right\}^{\frac{1}{s}-\frac{1}{t}}.$$

It follows

$$f(k,N,q,s) \geq \frac{N^{\frac{s}{t}-1}}{B_{t,s}^s}\,(f(k,N,q,t))^{s/t}.$$

b) From *Goldman inequality* (see [13], p. 109.), $0 < t < s$,

$$\left(M^t - m^t\right)\{M_N^{[s]}(x_{\overline{1,N}})\}^s - (M^s - m^s)\{M_N^{[t]}(x_{\overline{1,N}})\}^t \leq M^t m^s - M^s m^t.$$

Hence, for $0 < t < s$,

$$\frac{M^t - m^t}{f(k,N,q,s)} - \frac{M^s - m^s}{f(k,N,q,t)} \leq N\left(M^t m^s - M^s m^t\right).$$

□

Remark 1.1 Another type of a lower bound for $f(k,N,q,s)$ can be derived from another Beesack inequality (see [2], p. 336; [13], p. 111):

$$M_N^{[s]}(x_{\overline{1,N}}) \leq C_{t,s} + M_N^{[t]}(x_{\overline{1,N}}),$$

where

$$C_{t,s} = \left\{\frac{m^s M^t}{M^t - m^t} + \frac{s-t}{t}\left(\frac{\mu t}{s}\right)^{\frac{s}{s-t}}\right\}^{\frac{1}{s}},$$

concluding

$$f(k,N,q,s) \geq \frac{1}{N} \cdot \frac{1}{\left(C_{t,s} + [Nf(k,N,q,t)]^{-\frac{1}{t}}\right)^s}.$$

1.1.2 Zipf law estimations

If we take $q = 0$ in probability mass function (1.39) we get Zipf law with probability mass function

$$f(k,N,s) = \frac{1}{k^s H_{N,s}} \tag{1.8}$$

where

$$H_{N,s} = \sum_{i=1}^{N} \frac{1}{i^s}. \tag{1.9}$$

For $s = 1$ $H_N = H_{N,1}$ we get $N-$th harmonic number.

1° (case $t = 1$)

Using Proposition 1.2 for $q = 0,\ t = 1$ and $s > 1$ we have

$$(Nf(k,N,s))^{\frac{1}{s}} \le Nf(k,N,1)$$

i.e.

$$f(k,N,s) \le \frac{N^{s-1}}{k^s H_N^s}. \tag{1.10}$$

We can derive further bounds using well-known inequalities for harmonic numbers. Using Schlömlich-Lemonnier inequalities (see [12], p. 118)

$$\ln(N+1) < H_N < 1 + \ln(N+1) \tag{1.11}$$

and (1.10) we get

$$f(k,N,s) < N^{s-1} k^{-s} \ln^{-s}(N+1).$$

Also, using (see [12], p. 120)

$$r(1-(N+1)^{-1/r}) < H_n < r(N^{1/r} - 1) + 1 \tag{1.12}$$

we have

$$f(k,N,s) < N^{s-1}(rk(1-(N+1)^{-1/r}))^{-s}.$$

Similarly, we have a list of inequalities with Euler constant $\gamma = \lim_{N\to}(H_N - \ln N)$ (see [12], p. 120):

$$\gamma + \ln N + \tfrac{1}{2N} - \tfrac{1}{8N^2} < H_N < \gamma + \ln N + \tfrac{1}{2N} \tag{1.13}$$

$$\gamma + \ln N + \tfrac{1}{2(N+1)} < H_N < \gamma + \ln N + \tfrac{1}{2(N-1)} \tag{1.14}$$

$$\gamma + \ln(N+1/2) + \tfrac{1}{24(N+1)^2} < H_N < \gamma + \ln(N+1/2) + \tfrac{1}{24N^2} \tag{1.15}$$

$$\begin{aligned} &\gamma + \ln(N+1/2) + \tfrac{1}{24(N+1/2)^2} - \tfrac{7}{960N^4} < H_N \\ &< \gamma + \ln(N+1/2) + \tfrac{1}{24(N+1/2)^2} - \tfrac{7}{960(N+1)^4}. \end{aligned} \tag{1.16}$$

Now, using (1.10) and left-hand side inequalities in (1.13)-(1.16) we get

$$f(k,N,s) < k^{-s} N^{s-1} \left(\gamma + \ln N + \tfrac{1}{2N} - \tfrac{1}{8N^2}\right)^{-s}$$

$$f(k,N,s) < k^{-s}N^{s-1}\left(\gamma+\ln N+\tfrac{1}{2(N+1)}\right)^{-s}$$
$$f(k,N,s) < k^{-s}N^{s-1}\left(\gamma+\ln(N+1/2)+\tfrac{1}{24(N+1)^2}\right)^{-s}$$
$$f(k,N,s) < k^{-s}N^{s-1}\left(\gamma+\ln(N+1/2)+\tfrac{1}{24(N+1/2)^2}-\tfrac{7}{960N^4}\right)^{-s}$$

Similarly, using Proposition 1.2 for $q=0,\ t=1$ and $0<s<1$ we have

$$(Nf(k,N,s))^{\frac{1}{s}} \geq Nf(k,N,1)$$

i.e.

$$f(k,N,s) \geq \frac{N^{s-1}}{k^s H_N^s}. \tag{1.17}$$

and then using (1.13)-(1.16) we will get lower bounds

$$f(k,N,s) > k^{-s}N^{s-1}\left(\gamma+\ln N+\tfrac{1}{2N}\right)^{-s}$$
$$f(k,N,s) > k^{-s}N^{s-1}\left(\gamma+\ln N+\tfrac{1}{2(N-1)}\right)^{-s}$$
$$f(k,N,s) > k^{-s}N^{s-1}\left(\gamma+\ln(N+1/2)+\tfrac{1}{24N^2}\right)^{-s}$$
$$f(k,N,s) > k^{-s}N^{s-1}\left(\gamma+\ln(N+1/2)+\tfrac{1}{24(N+1/2)^2}-\tfrac{7}{960(N+1)^4}\right)^{-s}$$

2° (case $t=2$)
Using Proposition 1.2 for $q=0,\ t=2$ and $s>2$ we have

$$(Nf(k,N,s))^{\frac{1}{s}} \leq Nf(k,N,1) = \left(Nk^{-2}H_{N,2}\right)^{\frac{1}{2}}$$

i.e.

$$f(k,N,s) \leq N^{\frac{s}{2}-1}k^{-s}H_{N,2}^{-\frac{s}{2}}. \tag{1.18}$$

Appling Proposition 1.2 for $q=0,\ t=2$ and $0<s<2$ we get reversed inequality

$$f(k,N,s) \geq N^{\frac{s}{2}-1}k^{-s}H_{N,2}^{-\frac{s}{2}}. \tag{1.19}$$

Now we use the next estimations for $H_{N,2}$ (see [12] p. 121–122; [**?**])

$$\frac{\pi^2}{6}-\frac{N+1/2}{N^2+N+d} < H_{N,2} < \frac{\pi^2}{6}-\frac{N+1/2}{N^2+N+1/3},\ d=0.324555 \tag{1.20}$$

and (see [12] p. 122)

$$H_{N,2} \geq \tfrac{8}{5}-\tfrac{1}{N+\frac{2}{3}},\ N\geq 1 \tag{1.21}$$
$$H_{N,2} \geq \tfrac{13}{8}-\tfrac{1}{N+\frac{3}{5}},\ N\geq 1 \tag{1.22}$$
$$H_{N,2} \geq \tfrac{13}{8}-\tfrac{1}{N+\frac{2}{3}},\ N\geq 2 \tag{1.23}$$

$$H_{N,2} \leq \tfrac{10N-1}{6N+3},\ N \geq 1 \tag{1.24}$$
$$H_{N,2} < 2 - \tfrac{1}{N},\ N \geq 2. \tag{1.25}$$

Hence, for $s > 2$

$$f(k,N,s) < N^{\frac{s}{2}-1}k^{-s}\left(\tfrac{\pi^2}{6} - \tfrac{N+1/2}{N^2+N+d}\right)^{-\frac{s}{2}},\ N \geq 1;$$
$$f(k,N,s) \leq N^{\frac{s}{2}-1}k^{-s}\left(\tfrac{8}{5} - \tfrac{1}{N+\frac{2}{3}}\right)^{-\frac{s}{2}},\ N \geq 1;$$
$$f(k,N,s) \leq N^{\frac{s}{2}-1}k^{-s}\left(\tfrac{13}{8} - \tfrac{1}{N+\frac{3}{5}}\right)^{-\frac{s}{2}},\ N \geq 1;$$
$$f(k,N,s) \leq N^{\frac{s}{2}-1}k^{-s}\left(\tfrac{13}{8} - \tfrac{1}{N+\frac{2}{3}}\right)^{-\frac{s}{2}},\ N \geq 2,$$

and for $0 < s < 2$

$$f(k,N,s) > N^{\frac{s}{2}-1}k^{-s}\left(\tfrac{\pi^2}{6} - \tfrac{N+1/2}{N^2+N+1/3}\right)^{-\frac{s}{2}},\ N \geq 1;$$
$$f(k,N,s) \geq N^{\frac{s}{2}-1}k^{-s}\left(\tfrac{10N-1}{6N+3}\right)^{-\frac{s}{2}},\ N \geq 1;$$
$$f(k,N,s) \leq N^{\frac{s}{2}-1}k^{-s}\left(2 - \tfrac{1}{N}\right)^{-\frac{s}{2}},\ N \geq 2.$$

1.1.3 Zipf law and Goldman inequality

From Goldman inequality we derived (1.6). For $q = 0,\ 0 < t < s$, (now $m = k/N,\ M = k$)

$$\frac{k^t - (\frac{k}{N})^t}{f(k,N,s)} - \frac{k^s - (\frac{k}{N})^s}{f(k,N,t)} \leq N\left(k^t\left(\frac{k}{N}\right)^s - k^s\left(\frac{k}{N}\right)^t\right) \tag{1.26}$$

1° for $s > t = 1$ we have then

$$\frac{k - \frac{k}{N}}{f(k,N,s)} - \frac{k^s - (\frac{k}{N})^s}{f(k,N,1)} \leq N\left(k\left(\frac{k}{N}\right)^s - k^s\frac{k}{N}\right)$$

i.e.

$$f(k,N,s) \geq \frac{1}{k^s} \cdot \frac{N^{s-1}(N-1)}{N - N^s + (N^s-1)H_N}. \tag{1.27}$$

Using (1.13)-(1.16) we get the following sequence of lower bounds for $f(k,N,s),\ s > 1$,

$$f(k,N,s) > \tfrac{1}{k^s} \cdot \tfrac{N^{s-1}(N-1)}{N-N^s+(N^s-1)\left(\gamma+\ln N+\frac{1}{2N}\right)},\ N > 1;$$
$$f(k,N,s) > \tfrac{1}{k^s} \cdot \tfrac{N^{s-1}(N-1)}{N-N^s+(N^s-1)\left(\gamma+\ln N+\frac{1}{2(N-1)}\right)},\ N > 1;$$

$$f(k,N,s) > \frac{1}{k^s} \cdot \frac{N^{s-1}(N-1)}{N-N^s+(N^s-1)\left(\gamma+\ln(N+1/2)+\frac{1}{24N^2}\right)},\ N>1;$$

$$f(k,N,s) > \frac{1}{k^s} \cdot \frac{N^{s-1}(N-1)}{N-N^s+(N^s-1)\left(\gamma+\ln(N+1/2)+\frac{1}{24(N+1/2)^2}-\frac{7}{960(N+1)^4}\right)}.\ N>1;$$

2° for $0<t<s=1$ in (1.26)

$$f(k,N,t) \leq \frac{1}{k^t} \cdot \frac{N^{t-1}(N-1)}{N-N^t+(N^t-1)H_N}. \tag{1.28}$$

Using (1.13)-(1.16) we get the following sequence of upper bounds for $f(k,N,t),\ t<1,$

$$f(k,N,t) < \frac{1}{k^t} \cdot \frac{N^{t-1}(N-1)}{N-N^t+(N^t-1)\left(\gamma+\ln N+\frac{1}{2N}-\frac{1}{8N^2}\right)},\ N>1;$$

$$f(k,N,t) < \frac{1}{k^t} \cdot \frac{N^{s-1}(N-1)}{N-N^s+(N^s-1)\left(\gamma+\ln N+\frac{1}{2(N+1)}\right)},\ N>1;$$

$$f(k,N,t) < \frac{1}{k^t} \cdot \frac{N^{t-1}(N-1)}{N-N^t+(N^t-1)\left(\gamma+\ln(N+1/2)+\frac{1}{24(N+1)^2}\right)},\ N>1;$$

$$f(k,N,t) < \frac{1}{k^t} \cdot \frac{N^{t-1}(N-1)}{N-N^t+(N^t-1)\left(\gamma+\ln(N+1/2)+\frac{1}{24(N+1/2)^2}-\frac{7}{960(N+1)^4}\right)}\ N>1.$$

3° For $s>t=2$ in (1.26)

$$\frac{k^2-(\frac{k}{N})^2}{f(k,N,s)} - \frac{k^s-(\frac{k}{N})^s}{f(k,N,2)} \leq N\left(k^2\left(\frac{k}{N}\right)^s - k^s\left(\frac{k}{N}\right)^2\right)$$

i.e.

$$f(k,N,s) \geq \frac{1}{k^s} \cdot \frac{N^{s-2}(N^2-1)}{N-N^{s-1}+(N^s-1)H_{N,2}},\ N>1. \tag{1.29}$$

Combining (1.30) with (1.20), (1.24) and (1.25) we get the sequence of inequalities

$$f(k,N,s) > \frac{1}{k^s} \cdot \frac{N^{s-2}(N^2-1)}{N-N^{s-1}+(N^s-1)\left(\frac{\pi^2}{6}-\frac{N+1/2}{N^2+N+1/3}\right)},\ N>1;$$

$$f(k,N,s) \geq \frac{1}{k^s} \cdot \frac{N^{s-2}(N^2-1)}{N-N^{s-1}+(N^s-1)\left(\frac{10N-1}{6N+3}\right)},\ N>1;$$

$$f(k,N,s) \geq \frac{1}{k^s} \cdot \frac{N^{s-2}(N^2-1)}{N-N^{s-1}+(N^s-1)\left(2-\frac{1}{N}\right)},\ N>2.$$

4° For $t>s=2$ in (1.26)

$$\frac{k^t-(\frac{k}{N})^t}{f(k,N,2)} - \frac{k^2-(\frac{k}{N})^2}{f(k,N,t)} \leq N\left(k^t\left(\frac{k}{N}\right)^2 - k^2\left(\frac{k}{N}\right)^t\right)$$

i.e.

$$f(k,N,t) \leq \frac{1}{k^t} \cdot \frac{N^{t-2}(N^2-1)}{N - N^{t-1} + (N^t-1)H_{N,2}}, \; N > 1. \tag{1.30}$$

Combining (1.30) with (1.20), (1.21), (1.22) and (1.23) we get the sequence of inequalities

$$\begin{aligned}
f(k,N,t) &< \tfrac{1}{k^t} \cdot \frac{N^{t-2}(N^2-1)}{N-N^{t-1}+(N^t-1)\left(\frac{\pi^2}{6}-\frac{N+1/2}{N^2+N+d}\right)}, \; N > 1;\\
f(k,N,s) &\leq \tfrac{1}{k^t} \cdot \frac{N^{t-2}(N^2-1)}{N-N^{t-1}+(N^t-1)\left(\frac{8}{5}-\frac{1}{N+\frac{2}{3}}\right)}, \; N > 1;\\
f(k,N,t) &\leq \tfrac{1}{k^t} \cdot \frac{N^{t-2}(N^t-1)}{N-N^{t-1}+(N^t-1)\left(\frac{13}{8}-\frac{1}{N+\frac{3}{5}}\right)}, \; N > 1.\\
f(k,N,t) &\leq \tfrac{1}{k^t} \cdot \frac{N^{t-2}(N^t-1)}{N-N^{t-1}+(N^t-1)\left(\frac{13}{8}-\frac{1}{N+\frac{2}{3}}\right)}, \; N \geq 2.
\end{aligned}$$

1.1.4 Further bounds via Lyapunov and Hölder inequality

Theorem 1.2 *For probability mass function* (1.39) *we have the following inequality, for* $0 < r < s < t$

$$\frac{[Nf(k,N,q,t)]^{-\frac{1}{t}} - [Nf(k,N,q,r)]^{-\frac{1}{r}}}{[Nf(k,N,q,t)]^{-\frac{1}{t}} - [Nf(k,N,q,s)]^{-\frac{1}{s}}} \leq \frac{s(t-r)}{r(t-s)}. \tag{1.31}$$

Proof. Using Lyapunov inequality (see [12], p. 34, [13] p. 117). For $0 < r < s < t$

$$\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^s\right)^{t-r} \leq \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^r\right)^{t-s} \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^t\right)^{s-r} \tag{1.32}$$

We can rewrite this as

$$[Nf(k,N,q,s)]^{-\frac{1}{s}} \leq \left\{[Nf(k,N,q,r)]^{-\frac{1}{r}}\right\}^{\frac{r}{s}\frac{t-s}{t-r}} \left\{[Nf(k,N,q,t)]^{-\frac{1}{t}}\right\}^{\frac{t}{s}\frac{s-r}{t-r}} \tag{1.33}$$

Applying A-G inequality on right-hand side of (1.59) we have

$$[Nf(k,N,q,s)]^{-\frac{1}{s}} \leq \frac{r}{s}\frac{t-s}{t-r}[Nf(k,N,q,r)]^{-\frac{1}{r}} + \frac{t}{s}\frac{s-r}{t-r}[Nf(k,N,q,t)]^{-\frac{1}{t}}$$

which we can rewrite as

$$\frac{[Nf(k,N,q,t)]^{-\frac{1}{t}} - [Nf(k,N,q,r)]^{-\frac{1}{r}}}{[Nf(k,N,q,t)]^{-\frac{1}{t}} - [Nf(k,N,q,s)]^{-\frac{1}{s}}} \leq \frac{s(t-r)}{r(t-s)}.$$

□

Theorem 1.3 *For $\alpha > 1$, let (α, β) be a pair of Hölder conjugates. Then for $r,\ s > 0$ we have*

$$f(k,N,q,s+r) \geq f(k,N,q,s\alpha)^{\frac{1}{\alpha}} f(k,N,q,r\beta)^{\frac{1}{\beta}}. \tag{1.34}$$

Proof. Using Hölder inequality for sequences $\left\{\left(\frac{k+q}{i+q}\right)^r : i = 1,\dots,N\right\}$ and $\left\{\left(\frac{k+q}{i+q}\right)^s : i = 1,\dots,N\right\}$, we have

$$\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{r+s} \leq \left(\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{r\alpha}\right)^{1/\alpha}\left(\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{s\beta}\right)^{1/\beta}$$

i.e.

$$(f(k,N,q,s+r))^{-1} \leq f(k,N,q,s\alpha)^{-\frac{1}{\alpha}} f(k,N,q,r\beta)^{-\frac{1}{\beta}}.$$

□

Let

$$m = \begin{cases} \left(\frac{k+q}{N+q}\right)^{s-\frac{r\beta}{\alpha}}, & s\alpha > r\beta \\ \left(\frac{k+q}{1+q}\right)^{s-\frac{r\beta}{\alpha}}, & s\alpha < r\beta \end{cases} \tag{1.35}$$

and

$$M = \begin{cases} \left(\frac{k+q}{1+q}\right)^{s-\frac{r\beta}{\alpha}}, & s\alpha > r\beta \\ \left(\frac{k+q}{N+q}\right)^{s-\frac{r\beta}{\alpha}}, & s\alpha < r\beta. \end{cases} \tag{1.36}$$

Theorem 1.4 *For $\alpha > 1$, let (α, β) be a pair of Hölder conjugates. Then for $r,\ s > 0$ we have*

$$\frac{M-m}{f(k,N,q,s\alpha)} + \frac{mM^{\alpha} - Mm^{\alpha}}{f(k,N,q,r\beta)} \leq \frac{M^{\alpha} - m^{\alpha}}{f(k,N,q,r+s)}, \tag{1.37}$$

where m and M are defined with (1.35) *and* (1.36) *respectively.*

Proof. Follows from a conversion of the Hölder inequality and a discreet version of the linear functional in Theorem 4.14, [13], p. 114, applied for sequences

$$\left\{\left(\frac{k+q}{i+q}\right)^r : i = 1,\dots,N\right\} \text{ and } \left\{\left(\frac{k+q}{i+q}\right)^s : i = 1,\dots,N\right\}.$$

□

Another type of conversion of the Hölder inequality is given in [13], Theorem 4.16, p. 115. Similarly, as in the proof of Theorem 1.4, using discreet version of a linear functional, we get the next theorem.

Theorem 1.5 *Under the same assumptions as in Theorem 1.4, the following result holds*

$$f(k,N,q,r+s) \leq \frac{\alpha^{-\frac{1}{\alpha}}\beta^{-\frac{1}{\beta}}(M^{\alpha} - m^{\alpha})}{(M-m)^{\frac{1}{\alpha}}(mM^{\alpha} - Mm^{\alpha})^{\frac{1}{\beta}}}(f(k,N,q,s\alpha))^{\frac{1}{\alpha}}(f(k,N,q,r\beta))^{\frac{1}{\beta}}. \tag{1.38}$$

1.2 Analytical properties of Zipf-Mandelbrot law and Hurwitz $\zeta-$function

For $N \in \mathbb{N},\ q \geq 0,\ s > 0,\ k \in \{1,2,\ldots,N\}$, we can rewrite Zipf-Mandelbrot law (probability mass function) in the following form

$$f(k,N,q,s) = \frac{1/(k+q)^s}{\zeta(N,s,q)}, \tag{1.39}$$

where

$$\zeta(N,s,q) = \sum_{i=1}^{N} \frac{1}{(i+q)^s}, \tag{1.40}$$

$N \in \mathbb{N},\ q \geq 0,\ s > 0,\ k \in \{1,2,\ldots,N\}$. If total number of words N tends to infinity we denote

$$f(k,q,s) = \frac{1/(k+q)^s}{\zeta(s,q)}, \tag{1.41}$$

where

$$\zeta(s,q) = \sum_{i=1}^{\infty} \frac{1}{(i+q)^s} \tag{1.42}$$

we recognize as Hurwitz $\zeta-$function. This infinite case, when total mass is spread over all set of positive integers, particularly, is studied in [9]. Note here, that we use more suitable version of Hurwitz ζ function (see also [1]), since in the classical definition sum starts from zero and $q > 0$. However, this fact does not alter our conclusions about Hurwitz $\zeta-$function.

The are also quite different interpretation of Zipf-Mandelbrot law. As it is pointed out in [11] (see also [3], [15]), parameters in (1.39) can be interpreted in the following way: N is the number of species present and the parameters q and s have an ecological interpretation: q represents the diversity of the environment and s the predictability of the ecosystem, i.e. the average probability of the appearance of a species.

1.2.1 Monotonicity properties

As starting point, we use the next proposition on inequalities for sums of positive order ([12, pp. 36] , [13, pp. 165]).

Proposition 1.2 *If* $a_i \geq 0,\ i \in \mathbb{N}$ *then for* $0 < t < s$

$$\left(\sum_{i=1}^{\infty} a_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i=1}^{\infty} a_i^t\right)^{\frac{1}{t}}. \tag{1.43}$$

Theorem 1.6

i) The function $s \mapsto [\zeta(N,s,q)]^{1/s}$ *is decreasing i.e. for* $s > t > 0$

$$[\zeta(N,s,q)]^{1/s} \le [\zeta(N,t,q)]^{1/t}.$$

ii) The function $s \mapsto [f(k,N,q,s)]^{1/s}$ *is increasing i.e. for* $s > t > 0$

$$[f(k,N,q,s)]^{1/s} \ge [(f(k,N,q,t)]^{1/t}.$$

iii) The function $s \mapsto [\zeta(s,q)]^{1/s}$ *is decreasing i.e. for* $s > t > 0$

$$[\zeta(s,q)]^{1/s} \le [\zeta(t,q)]^{1/t}.$$

iv) The function $s \mapsto [f(k,q,s)]^{1/s}$ *is increasing i.e. for* $s > t > 0$

$$[f(k,q,s)]^{1/s} \ge [(f(k,q,t)]^{1/t}.$$

Proof.
i) We use the Proposition 1.2, for

$$a_i = \begin{cases} \frac{1}{i+q}, & i = 1,\dots,N; \\ 0, & i > N. \end{cases}$$

ii) Follows from i)-part and

$$\frac{1}{f(k,N,q,s)} = \sum_{i=1}^{N} \left(\frac{k+q}{i+q}\right)^s = (k+q)^s \zeta(N,s,q). \tag{1.44}$$

iii) Use Proposition 1.2 for $a_i = \frac{1}{i+q}$, $i \in \mathbb{N}$.
iv) Follows from iii)-part and

$$\frac{1}{f(k,q,s)} = (k+q)^s \zeta(s,q). \tag{1.45}$$

□

Theorem 1.7 *The function*

$$s \mapsto (Nf(k,N,q,s))^{1/s} \tag{1.46}$$

is decreasing i.e. for $s > t > 0$

$$(Nf(k,N,q,s))^{1/s} \le (Nf(k,N,q,t))^{1/t}. \tag{1.47}$$

Proof. From (1.44) it follows

$$\frac{1}{Nf(k,N,q,s)} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{s}, \tag{1.48}$$

i.e.

$$(Nf(k,N,q,s))^{-1/s} = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{s}\right]^{1/s}. \tag{1.49}$$

Denote $x_i = \frac{k+q}{i+q},\ i = 1,\ldots,N$. Then the right-hand side of (1.49) is the power mean

$$M_N^{[s]}(x_{\overline{1,N}}) := \left[\frac{1}{N}\sum_{i=1}^{N} x_i^{\,s}\right]^{1/s}.$$

Using well-known fact, that $s \mapsto M_N^{[s]}(x_{\overline{1,N}})$ is increasing function (see for example [12, 13]) we conclude that the function

$$s \mapsto (Nf(k,N,q,s))^{1/s} \tag{1.50}$$

is decreasing. □

1.2.2 Log-convexity and exponential convexity

Let us recall well-known Lyapunov inequality, for sequences ([12, pp. 34], [13, pp. 117]).

Proposition 1.3 *If $a_i \geq 0,\ i \in \mathbb{N}$, then for $0 < r < s < t$*

$$\left(\sum_{i=1}^{\infty} a_i^s\right)^{t-r} \leq \left(\sum_{i=1}^{\infty} a_i^r\right)^{t-s}\left(\sum_{i=1}^{\infty} a_i^t\right)^{s-r}. \tag{1.51}$$

If we set $a_i = \frac{1}{i+q},\ i \in \mathbb{N}$ in (1.51) we get

Corollary 1.1 *For $1 < r < s < t$*

$$\zeta^{t-r}(s,q) \leq \zeta^{t-s}(r,q)\zeta^{s-r}(t,q). \tag{1.52}$$

In the next theorem we prove, log-concavity of $s \mapsto f(k,N,q,s)$ and log-convexity of $s \mapsto \zeta(s,q)$.

Theorem 1.8 *Let $\lambda \in (0,1)$.*

i) For $0 < r < t$,

$$\zeta(N,\lambda r + (1-\lambda)t,q) \leq \zeta^{\lambda}(N,r,q)\zeta^{1-\lambda}(N,t,q).$$

ii) For $0 < r < t$,

$$(f(k,N,q,\lambda r+(1-\lambda)t))^{-1} \leq (f(k,N,q,r))^{-\lambda}\,(f(k,N,q,t))^{-(1-\lambda)}.$$

iii) For $1 < r < t$,

$$\zeta(\lambda r+(1-\lambda)t,q) \leq \zeta^{\lambda}(r,q)\zeta^{1-\lambda}(t,q).$$

iv) For $1 < r < t$,

$$(f(k,q,\lambda r+(1-\lambda)t))^{-1} \leq (f(k,q,r))^{-\lambda}\,(f(k,q,t))^{-(1-\lambda)}.$$

Proof.
i) For $0 < r < t$ and $\lambda \in (0,1)$ we set

$$a_i = \begin{cases} \frac{1}{i+q}, & i = 1,\ldots,N; \\ 0, & i > N. \end{cases}$$

and $s = \lambda r + (1-\lambda)t$ in (1.51):

$$\left(\sum_{i=1}^{N}\left(\frac{1}{i+q}\right)^{\lambda r+(1-\lambda)t}\right)^{t-r} \leq \left(\sum_{i=1}^{N}\left(\frac{1}{i+q}\right)^{r}\right)^{\lambda(t-r)} \left(\sum_{i=1}^{N}\left(\frac{1}{i+q}\right)^{t}\right)^{(1-\lambda)(t-r)}.$$

ii) Follows from (1.44) and i)-part.
iii) We set $a_i = \frac{1}{i+q}$ and $s = \lambda r + (1-\lambda)t$ in (1.51).
iv) Follows from iii)-part and (1.45). □

We can conclude even more since this result can be extended to exponential convexity [4].

Definition 1.1 *A function $h : I \to \mathbb{R}$ is exponentially convex on an interval $I \subseteq \mathbb{R}$ if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $x_i \in I$, $i = 1,\ldots,n$.

Theorem 1.9 *The function $s \mapsto \zeta(s,q)$ is exponentially convex function on $(1,\infty)$.*

Proof. For a given $n \in \mathbb{N}$ let $\xi_m \in \mathbb{R}$, $s_m \in (1,\infty)$ $(m = 1,\ldots,n)$ we have

$$\sum_{l,m=1}^{n} \xi_l \xi_m \zeta\left(\frac{s_l + s_m}{2}, q\right) = \sum_{l,m=1}^{n} \xi_l \xi_m \sum_{i=1}^{\infty} \frac{1}{(i+q)^{\frac{s_l+s_m}{2}}} \tag{1.53}$$

$$= \sum_{i=1}^{\infty} \sum_{l,m=1}^{n} \xi_l \xi_m \frac{1}{(i+q)^{\frac{s_l+s_m}{2}}} \tag{1.54}$$

$$= \sum_{i=1}^{\infty} \left(\sum_{m=1}^{n} \frac{1}{(i+q)^{\frac{s_m}{2}}}\right)^2 \geq 0. \tag{1.55}$$

Since the function $s \mapsto \zeta(s,q)$ is continuous function on $(1,\infty)$, we conclude its exponential convexity on $(1,\infty)$. □

Using (1.45) we have also the next corollary.