Chapter 1

Introduction

In this chapter, a brief review of some fundamental results on the topics in the sequel is given and a several basic motivating ideas are presented.

1.1 Convex Functions

Definition 1.1 *Let I be an real interval. Then $\phi : I \to \mathbb{R}$ is said to be* ***convex function*** *on I if for all $x, y \in I$ and every $\lambda \in [0,1]$, we have*

$$\phi\left((1-\lambda)x+\lambda y\right) \leq (1-\lambda)\phi(x)+\lambda\phi(y). \tag{1.1}$$

If (1.1) is strict for all $x, y \in I$, $x \neq y$ and every $\lambda \in (0,1)$, then ϕ is said to be ***strictly convex***.
If in (1.1) the reverse inequality holds, then ϕ is said to be ***concave function***. *If it is strict for all $x, y \in I$, $x \neq y$ and every $\lambda \in (0,1)$, then ϕ is said to be* ***strictly concave***.

For convex functions the following propositions are valid which exactly define convex functions on equivalent ways.

Remark 1.1 *a) The inequality (1.1), for $x_1, x_2, x_3 \in I$, such that $x_1 \leq x_2 \leq x_3$, $x_1 \neq x_3$, we can write in the form*

$$\phi\left(x_2\right) \leq \frac{x_3-x_2}{x_3-x_1}\phi\left(x_1\right)+\frac{x_2-x_1}{x_3-x_1}\phi\left(x_3\right), \tag{1.2}$$

i.e.

$$(x_3 - x_2)\,\phi(x_1) + (x_1 - x_3)\,\phi(x_2) + (x_2 - x_1)\,\phi(x_3) \geq 0, \tag{1.3}$$

setting $x = x_1$, $y = x_3$, $\lambda = (x_2 - x_1)/(x_3 - x_1)$. *This inequality is often used as alternative definition of convexity.*
b) Another way of writing (1.3) is

$$\frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \leq \frac{\phi(y_2) - \phi(y_1)}{y_2 - y_1}, \tag{1.4}$$

where $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$ *and* $y_1 \neq y_2$.

The following two theorems concern derivatives of convex functions.

Theorem 1.1 *(see [144, p. 4]) Let I be an real interval. Let* $\phi : I \to \mathbb{R}$ *be convex. Then*

(i) ϕ *is Lipschitz on any closed interval in I;*

(ii) ϕ'_- *and* ϕ'_+ *exist and are increasing on I, and* $\phi'_- \leq \phi'_+$ *(if* ϕ *is strictly convex, then these derivatives are strictly increasing);*

(iii) ϕ' *exists, except possibly on a countable set, and on the complement of which it is continuous.*

Remark 1.2 *a) If* $\phi : I \to \mathbb{R}$ *is derivable function, then* ϕ *is convex iff a function* ϕ' *is increasing.*
b) If $\phi : I \to \mathbb{R}$ *is twice derivable function, then* ϕ *is convex iff* $\phi''(x) \geq 0$ *for all* $x \in I$. *If* $\phi''(x) > 0$, *then* ϕ *is strictly convex.*

Theorem 1.2 *(see [144, p. 5]) Let I be an open interval in* $\mathbb{R}$.

(i) $\phi : I \to \mathbb{R}$ *is convex iff there is at least one line of support for* ϕ *at each* $x_0 \in I$, *i.e. for all* $x \in I$ *we have*

$$\phi(x) \geq \phi(x_0) + \lambda(x - x_0),$$

where $\lambda \in \mathbb{R}$ *depends on* x_0 *and is given by* $\lambda = \phi'(x_0)$ *when* $\phi'(x_0)$ *exists, and* $\lambda \in [\phi'_-(x_0), \phi'_+(x_0)]$ *when* $\phi'_-(x_0) \neq \phi'_+(x_0)$.

(ii) $\phi : I \to \mathbb{R}$ *is convex if the function* $x \longmapsto \phi(x) - \phi(x_0) - \lambda(x - x_0)$, *(the difference between the function and its support) is decreasing for* $x < x_0$ *and increasing for* $x > x_0$.

Definition 1.2 *Let* $\phi : I \to \mathbb{R}$ *be a convex function. Then the subdifferential of* ϕ *at* x, *denoted by* $\partial\phi(x)$ *is defined by*

$$\partial\phi(x) = \{\alpha \in \mathbb{R} : \phi(y) - \phi(x) - \alpha(y - x) \geq 0,\ y \in I\}.$$

There is a connection between a convex function and its subdifferential. It is well-known that $\partial\phi(x) \neq 0$ for all $x \in IntI$. More precisely, at each point $x \in IntI$ we have $-\infty < \phi'_-(x) \leq \phi'_+(x) < \infty$ and

$$\partial\phi(x) \in \left[\phi'_-(x_0), \phi'_+(x_0)\right],$$

while the set on which ϕ is not differentiable is at most countable. Moreover, each function $\varphi : I \to \mathbb{R}$ such that $\varphi(x) \in \partial\phi(x)$, whenever $x \in IntI$, is increasing on $IntI$. For any such function φ and arbitrary $x \in IntI$, $y \in I$, we have

$$\phi(y) - \phi(x) - \varphi(x)(y-x) \geq 0$$

and

$$\begin{aligned}\phi(y) - \phi(x) - \varphi(x)(y-x) &= |\phi(y) - \phi(x) - \varphi(x)(y-x)| \\ &\geq \|\phi(y) - \phi(x)| - |\varphi(x)| \cdot |(y-x)\|.\end{aligned}$$

J. L. Jensen is considered generally as being the first mathematician whostudied convex functions in a systematic way. He defined the concept of convex functions using the inequality (1.5) that are listed in the following definition.

Definition 1.3 *A function $\phi : I \to \mathbb{R}$ is called **Jensen-convex** or **J-convex** if for all $x, y \in I$ we have*

$$\phi\left(\frac{x+y}{2}\right) \leq \frac{\phi(x)+\phi(y)}{2}. \tag{1.5}$$

Remark 1.3 *It can be easily proved that a convex function is J-convex. If $\phi : I \to \mathbb{R}$ is continuos function, then ϕ is convex iff it is J-convex.*

Inequality (1.1) can be extended to the convex combinations of finitely many points in I by mathematical induction. These extensions are known as **discrete Jensen's inequality**.

Theorem 1.3 (JENSEN'S INEQUALITY) *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ be a positive n-tuple. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i), \tag{1.6}$$

where

$$W_k = \sum_{i=1}^{k} w_i, \quad k = 1, \ldots, n. \tag{1.7}$$

If f is strictly convex, then inequality (1.6) is strict unless $x_1 = \cdots = x_n$.

The condition "$\boldsymbol{w}$ is a positive n-tuple" can be replaced by "$\boldsymbol{w}$ is a non-negative n-tuple and $W_n > 0$". Note that the Jensen inequality (1.6) can be used as an alternative definition of convexity.

It is reasonable to ask whether the condition "$\boldsymbol{w}$ is a non-negative n-tuple" can be relaxed at the expense of restricting $\boldsymbol{x}$ more severely. An answer to this question was given by Steffensen [161] (see also [144, p.57]).

Theorem 1.4 (THE JENSEN-STEFFENSEN INEQUALITY) *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{w} = (w_1, \ldots, w_n)$ a real n-tuple such that*

$$0 \leq W_k \leq W_n\ , \quad k = 1, \ldots, n-1, \quad W_n > 0, \tag{1.8}$$

is satisfied, where W_k are as in (1.7), then (1.6) holds. If f is strictly convex, then inequality (1.6) is strict unless $x_1 = \cdots = x_n$.

Inequality (1.6) under conditions from Theorem 1.4 is called **the Jensen-Steffensen inequality**.

1.2 Space of Integrable, Continuous and Absolutely Continuous Functions

Let $[a,b]$ be a finite interval in $\mathbb{R}$, where $-\infty \leq a < b \leq \infty$. We denote by $L_p[a,b]$, $1 \leq p < \infty$, the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|^p dt < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}},$$

and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$ with

$$\|f\|_\infty = ess\sup\{|f(x)| : x \in [a,b]\}.$$

Theorem 1.5 (HOLDER'S INEQUALITY) *Let $p,q \in \mathbb{R}$ be such that $1 \leq p,q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f,g : [a,b] \to \mathbb{R}$ be integrable functions such that $f \in L_p[a,b]$ and $g \in L_q[a,b]$. Then*

$$\int_a^b |f(t)g(t)|dt \leq \|f\|_p \|g\|_q\ . \tag{1.9}$$

The equality in (1.9) holds iff $A|f(t)|^p = B|g(t)|^q$ almost everywhere (shortened to a.e.), where A and B are constants.

We denote by $C^n([a,b]), n \in \mathbb{N}_0$, the space of functions which are n times continuously differentiable on $[a,b]$, that is

$$C^n([a,b]) = \left\{ f : [a,b] \to \mathbb{R} : f^{(k)} \in C([a,b]), k = 0,1,\ldots,n \right\}.$$

In particular, $C^0([a,b]) = C([a,b])$ is the space of continuous functions on $[a,b]$ with the norm

$$\|f\|_{C^n} = \sum_{k=0}^n \left\| f^{(k)} \right\|_C = \sum_{k=0}^n \max_{x\in[a,b]} \left| f^{(k)}(x) \right|,$$

and for $C([a,b])$

$$\|f\|_C = \max_{x\in[a,b]} |f(x)|.$$

Lemma 1.1 *The space $C^n([a,b])$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} c_k(x-a)^k, \tag{1.10}$$

where $\varphi \in C([a,b])$ and c_k are arbitrary constants ($k = 0,1,\dots,n-1$).
Moreover,

$$\varphi(t) = f^{(n)}(t), \quad c_k = \frac{f^{(k)}(a)}{k!} \quad (k = 0,1,\dots,n-1). \tag{1.11}$$

The space of **absolutely continuous functions** on an interval $[a,b]$ is denote by $AC([a,b])$. It is known that $AC([a,b])$ coincides with the space of primitives of Lebesgue integrable functions $L_1[a,b]$ (see [100]):

$$f \in AC([a,b]) \quad \Leftrightarrow \quad f(x) = f(a) + \int_a^x \varphi(t)dt, \quad \varphi \in L_1[a,b].$$

Therefore, an absolutely continuous function f has an integrable derivatives $f'(x) = \varphi(x)$ almost everywhere on $[a,b]$. We denote by $AC^n([a,b]), n \in \mathbb{N}$, the space

$$AC^n([a,b]) = \{f \in C^{n-1}([a,b]) : f^{(n-1)} \in AC([a,b])\}.$$

In particular, $AC^1([a,b]) = AC([a,b])$.

Lemma 1.2 *The space $AC^n([a,b])$ consists of those and only those functions which can be represented in the form (1.10), where $\varphi \in L_1[a,b]$ and c_k are arbitrary ($k = 0,1,..,n-1$). Moreover, (1.11) holds.*

The next theorem has numerous applications involving multiple integrals.

Theorem 1.6 (FUBINI'S THEOREM) *Let $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ be σ-finite measure space and f be $\mu\times\nu$-measurable function on $X\times Y$. If $f \geq 0$, then the next integrals are equal*

$$\int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y), \quad \int_X \left(\int_Y f(x,y)d\nu(y)\right)d\mu(x), \quad \int_Y \left(\int_X f(x,y)d\mu(x)\right)d\nu(y).$$

Remark 1.4 *The next equalities*

$$\int_a^b \left(\int_c^d f(x,y)dy\right)dx = \int_c^d \left(\int_a^b f(x,y)dx\right)dy,$$
$$\int_a^b \left(\int_c^x f(x,y)dy\right)dx = \int_a^b \left(\int_y^b f(x,y)dx\right)dy,$$

are consequences of the previous theorem.

Theorem 1.7 (INTEGRAL JENSEN'S INEQUALITY) *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $\phi : \Omega \to \mathbb{R}$ be μ-integrable function. Let $f : I \to \mathbb{R}$ be a convex function such that $Im\,\phi \subseteq I$ anf $f \circ \phi$ is a μ – integrable function. Then*

$$f\left(\frac{1}{\mu(\Omega)}\int_{\Omega}\phi(x)d\mu(x)\right) \leq \frac{1}{\mu(\Omega)}\int_{\Omega} f(\phi(x))d\mu(x). \tag{1.12}$$

If f is strictly convex, then (1.12) becomes equality iff ϕ is a constant μ-almost everywhere on Ω. If f is concave, then (1.12) is reversed.

Remark 1.5 *The discrete Jensen inequality (1.6) is obtained by means of the discrete measure μ on $\Omega = \{1,\ldots,n\}$, with $\mu(\{i\}) = p_i$ and $\phi(i) = x_i$.*

Another integral version of jensen's inequality is based on the notation of the Riemann-Stieltjes integral for which a brief outline is given here. One can find more information on the Riemann-Stieltjes integral in [153].

Let $[a,b] \subset \mathbb{R}$ and let $f, \phi : [a,b] \to \mathbb{R}$ be bounded functions. The each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$, such that $t_0 < t_1 < \cdots < t_{n-1} < t_n$, Stieltjes' integral sum

$$\sigma(f,\phi;D,\gamma_1,\ldots,\gamma_n) = \sum_{i=1}^{n} f(\gamma_i)(\phi(t_i) - \phi(t_{i-1}))$$

is assigned, where $\gamma_i \in [t_{i-1}, t_i]$, $i = 1,\ldots,n$. These sums will be denoted with $\sigma(f,\phi;D)$ in the sequel.

Definition 1.4 *Let $f, \phi : [a,b] \to \mathbb{R}$ be bounded functions. A function f is said to be Riemann-Stieltjes integrable regardung a function ϕ if there exists $I_f \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a decomposition D_0 of $[a,b]$ such that for every decomposition $D \supseteq D_0$ of $[a,b]$ anf for every sum $\sigma(f,\phi;D)$*

$$|\sigma(f,\phi;D) - I_f| < \varepsilon$$

holds. The unuque I_f is the Riemann-Stieltjes integral of the function F regarding the function ϕ and is denoted with

$$\int_a^b f(t)d\phi(t).$$

The Riemann-Stieltjes integral is a generalization of the Riemann integral and coincides with it when ϕ is an identity.

The notation of the Riemann-Stieltjes integral is narrowly related to the class of the function of bounded variation.

Definition 1.5 *Let $\phi : [a,b] \to \mathbb{R}$ be a real function. To each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$, such that $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, belongs the sum*

$$V(\phi;D) = \sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})|,$$

which is said to be a variation of the function ϕ regarding decomposition D.
A function ϕ is said to be a function of bounded variation if the set $\{V(\phi;D) : D \in \mathscr{D}\}$ is bounded, where $\mathscr{D}$ is a family of all decompositions of the interval $[a,b]$. Number

$$V(\phi) = \sup\{V(\phi;D) : D \in \mathscr{D}\}$$

is called a total variation of a function ϕ.

Theorem 1.8 *The following assertions hold:*

(i) Every monotonic function $\phi : [a,b] \to \mathbb{R}$ is a function of bounded variation on $[a,b]$ and $V(\phi) = |\phi(b) - \phi(a)|$;

(ii) Every function of bounded variation is a bounded function;

(iii) If f and g are functions of bounded variation on $[a,b]$, then $f+g$ is a function of bounded variation on $[a,b]$.

Theorem 1.9 *Let ϕ be a function of bounded variation on $[a,b]$. then:*

(i) ϕ has at most countably many of step discontinuities on $[a,b]$;

(ii) ϕ can be presented as $\phi = s_\phi + g$, where step function s_ϕ and continuous function g are both functions of bounded variation on $[a,b]$.

At the end of this section, we introduce two recently obtained results involving Čebyšev's functional that involve the Grüss and Ostrowski type inequalities.

Definition 1.6 *For two Lebesgue integrable functions $f,g : [\alpha,\beta] \to \mathbb{R}$, we define* ***Čebyšev's functional*** *as*

$$T(f,g) := \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)g(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt.$$

Theorem 1.10 *[57, Theorem 1] Let $f : [\alpha,\beta] \to \mathbb{R}$ be Lebesgue integrable and $g : [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(g')^2 \in L_1[\alpha,\beta]$. Then*

$$|T(f,g)| \leq \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (x-\alpha)(\beta-x)[g'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{1.13}$$

The constant $\frac{1}{\sqrt{2}}$ in (1.13) is the best possible.

Theorem 1.11 *[57, Theorem 2] Let $g : [\alpha,\beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha,\beta]$. Then*

$$|T(f,g)| \leq \frac{1}{2(\beta-\alpha)} \|f'\|_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dg(x). \tag{1.14}$$

The constant $\frac{1}{2}$ in (1.14) is the best possible.

1.3 About Majorization

In this section, we introduce the concepts of majorization and Schur-convexity in order to give some basic results from the theory of majorization that give an important characterization of convex functions. Majorization theorem for convex functions and the classical concept of majorization, due to Hardy et al. [79], have numerous applications in different fields of applied sciences (see the monograph [117]). In recent times, majorization type results has attracted the interest of several mathematicians which resulting with interesting generalizations and applications (see for example [4], [6], [5], [52], [137]-[136]). A complete and superb reference on the subject is the book by Marshall and Olkin [123]. The book by Bhatia (1997) [45] contains significant material on majorization theory as well. Other textbooks on matrix and multivariate analysis also include a section on majoriztion theory, e.g., [82, Sec.4.3], [24, Sec.8.10] and [144].

Majorization makes precise the vague notion that the components of a vector $\boldsymbol{y}$ are "less spread out" or "more nearly equal" than the components of a vector $\boldsymbol{x}$.
For fixed $n \geq 2$, let

$$\boldsymbol{x} = (x_1, \ldots, x_n), \;\; \boldsymbol{y} = (y_1, \ldots, y_n)$$

denote two n-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq \ldots \geq y_{[n]},$$

$$x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}, \quad y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(n)}$$

be their ordered components.

Definition 1.7 ***Majorization***: *(see [144, p.319])* $\mathbf{x}$ *is said to majorize* $\mathbf{y}$ *(or* $\mathbf{y}$ *is said to be majorized by* $\mathbf{x}$*), in symbol,* $\mathbf{x} \succ \mathbf{y}$*, if*

$$\sum_{i=1}^{m} y_{[i]} \leq \sum_{i=1}^{m} x_{[i]} \tag{1.15}$$

holds for $m = 1, 2, \ldots, n-1$ *and*

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i.$$

Note that (1.15) is equivalent to

$$\sum_{i=n-m+1}^{n} y_{(i)} \leq \sum_{i=n-m+1}^{n} x_{(i)}$$

holds for $m = 1, 2, \ldots, n-1$.

The following notion of Schur-convexity generalizes the definition of convex function via the notion of majorization.

Definition 1.8 ***Schur-convexity***: *A function $F : S \subseteq \mathbb{R}^m \to \mathbb{R}$ is called Schur-convex on S if*

$$F(\mathbf{y}) \leq F(\mathbf{x}) \tag{1.16}$$

for every $\mathbf{x}, \mathbf{y} \in S$ such that

$$\mathbf{y} \prec \mathbf{x}.$$

Definition 1.9 ***(Weakly Majorization)***: *For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say $\boldsymbol{y}$ is weakly majorized by $\boldsymbol{x}$ or $\mathbf{x}$ weakly majorizes $\mathbf{y}$ (denoted by $\mathbf{x}\ {}^{w}\!\succ \mathbf{y}$ or $\mathbf{x} \succ^{w} \mathbf{y}$) if*

$$\sum_{i=1}^{m} y_{(i)} \geq \sum_{i=1}^{m} x_{(i)}$$

holds for $m = 1, 2, \ldots, n-1, n$, or, equivalently,

$$\sum_{i=m}^{n} y_{[i]} \geq \sum_{i=m}^{n} x_{[i]}$$

holds for $m = 1, 2, \ldots, n-1, n$.

Note that $\boldsymbol{x} \succ \boldsymbol{y}$ implies $\boldsymbol{x}\ {}^{w}\!\succ \boldsymbol{y}$; in other words, majorization is a more restrictive definition than weakly majorization.
Observe that the original order of the elements of $\mathbf{x}$ and $\mathbf{y}$ plays no role in the definition of majorization. In other words,

$$\boldsymbol{x} \prec \Pi\boldsymbol{x}$$

for all permutation matrices Π.

Parallel to the concept of additive majorization is the notion of multiplicative majorization (also termed log-majorization).

Definition 1.10 ***(Multiplicative Majorization)***: *[139] Let $\mathbf{x}$, $\mathbf{y}$ be two positive n-tuples, $\mathbf{y}$ is said to be multiplicatively majorized by $\mathbf{x}$, denoted by $\mathbf{y} \prec_{\times} \mathbf{x}$ if*

$$\prod_{i=1}^{m} y_{[i]} \leq \prod_{i=1}^{m} x_{[i]} \tag{1.17}$$

holds for $m = 1, 2, \ldots, n-1$ and

$$\prod_{i=1}^{n} y_i = \prod_{i=1}^{n} x_i.$$

Note that (1.17) is equivalent to

$$\prod_{i=n-m+1}^{n} y_{(i)} \leq \prod_{i=n-m+1}^{n} x_{(i)}$$

holds for $m = 1, 2, \ldots, n-1$.
To differentiate the two types of majorization, we sometimes use the symbol $\prec_{+}$ rather than $\prec$ to denote (additive) majorization.

There are several equivalent characterizations of the majorization relation $\boldsymbol{x} \succ \boldsymbol{y}$ in addition to the conditions given in definition of majorization. One is actually the answer of a question posed and answered in 1929 by Hardy, Littlewood and Polya [80, 79] in the form of the following theorem well-known as **Majorization theorem** (see [123, p.11], [144, p.320]).

Theorem 1.12 (MAJORIZATION THEOREM) *Let I be an interval in $\mathbb{R}$, and let $\mathbf{x}$, $\mathbf{y}$ be two n-tuples such that x_i, $y_i \in I$ $(i = 1, \ldots, n)$. Then*

$$\sum_{i=1}^{n} \phi(y_i) \leq \sum_{i=1}^{n} \phi(x_i) \tag{1.18}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ if and only if $\mathbf{x} \succ \mathbf{y}$.
If ϕ is a strictly convex function, then equality in (1.18) is valid iff $x_{[i]} = y_{[i]}$, $i = 1, \ldots, n$.

Another interesting characterization of $\boldsymbol{x} \succ \boldsymbol{y}$, also by Hardy, Littlewood, and Polya [80, 79], is that $\boldsymbol{y} = \mathbf{P}\boldsymbol{x}$ for some double stochastic matrix $\mathbf{P}$. In fact, the previous characterization implies that the set of vectors $\boldsymbol{y}$ that satisfy $\boldsymbol{x} \succ \boldsymbol{y}$ is the convex hull spanned by the $n!$ points formed from the permutations of the elements of $\boldsymbol{y}$.

The previous Majorization theorem can be be slightly preformulate in the following form which gives a relation between one-dimensional convex function and m-dimensional Schur-convex function (see [144, p. 333]).

Theorem 1.13 *Let $I \subset \mathbb{R}$ be an interval and $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m) \in I^m$. Let $\phi : I \to \mathbb{R}$ be continuous function. Then a function $F : I^m \to \mathbb{R}$, defined by*

$$F(\mathbf{x}) = \sum_{i=1}^{m} \phi(x_i),$$

is Schur-convex on I^m iff ϕ is convex on I.

The following theorem can be regarded as a weighted version of Theorem 1.13 and is proved by Fuchs in ([74], [144, p.323]).

Theorem 1.14 (FUCHS'S THEOREM) *Let $\mathbf{x}$, $\mathbf{y}$ be two decreasing real n-tuples, $\mathbf{x}$, $\mathbf{y} \in I^n$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a real n-tuple such that*

$$\sum_{i=1}^{k} w_i y_i \leq \sum_{i=1}^{k} w_i x_i \text{ for } k = 1, \ldots, n-1, \tag{1.19}$$

and

$$\sum_{i=1}^{n} w_i y_i = \sum_{i=1}^{n} w_i x_i. \tag{1.20}$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$, we have

$$\sum_{i=1}^{n} w_i \phi(y_i) \leq \sum_{i=1}^{n} w_i \phi(x_i). \tag{1.21}$$

Remark 1.6 *Throughout this book, if in some results we have* $\boldsymbol{x} = (x_1, x_2, \ldots, x_t)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_t)$ *and* $\boldsymbol{w} = (w_1, w_2, \ldots, w_t)$ *are t-tuples and g is associated function and we say that that these tuples are satisfying conditions (1.19), (1.20) and (1.21) holds, then we take* $n = t$ *and* $\phi = g$ *in (1.19), (1.20) and (1.21).*

The following theorem is valid ([133, p.32]):

Theorem 1.15 ([108]) *Let* $\phi : I \to \mathbb{R}$ *be a continuous convex function on an interval I,* $\mathbf{w}$ *be a positive n-tuple and* $\mathbf{x}, \mathbf{y} \in I^n$ *such that satisfying (1.19) and (1.20)*

(i) If $\mathbf{y}$ *is decreasing n-tuple, then* (1.21) *holds.*

(ii) If $\mathbf{x}$ *is increasing n-tuple, then reverse inequality in* (1.21) *holds.*

If ϕ *is strictly convex and* $\boldsymbol{x} \neq \boldsymbol{y}$*, then* (1.21) *and reverse inequality in* (1.21) *are strict.*

Proof. As in [161] (see [133, p.32]), because of the convexity of ϕ

$$\phi(u) - \phi(v) \geq \phi'_+(v)\,(u - v)\,.$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{n} w_i \left[\phi(x_i) - \phi(y_i)\right] \\
&\quad \geq \sum_{i=1}^{n} w_i\, \phi'_+(y_i)\,(x_i - y_i) \\
&\quad = \phi'_+(y_n)\,(X_n - Y_n) \\
&\quad + \sum_{k=1}^{n-1} (X_k - Y_k) \left[\phi'_+(y_k) - \phi'_+(y_{k+1})\right] \geq 0.
\end{aligned} \tag{1.22}$$

where $X_k = \sum_{i=1}^{k} w_i x_i$ and $Y_k = \sum_{i=1}^{k} w_i y_i$.
The last inequality follows from (1.19) and (1.20), $\boldsymbol{y}$ is decreasing and the convexity of ϕ.
Similarly, we can prove the case when $\boldsymbol{x}$ is increasing.
If ϕ is strictly convex and $\boldsymbol{x} \neq \boldsymbol{y}$, then

$$\phi(x_i) - \phi(y_i) > \phi'_+(y_i)\,(x_i - y_i)\,,$$

for at least one $i = 1, \ldots, n$. Which gives strict inequality in (1.21) and reverse inequality in (1.21). □

The following theorem is a slight extension of Theorem 1.14 proved by J. Pečarić and S. Abramovich [161].

Theorem 1.16 ([108]) *Let* $\mathbf{w}$, $\mathbf{x}$ *and* $\mathbf{y}$ *be an positive n-tuples. Suppose* $\psi, \phi : [0, \infty) \to \mathbb{R}$ *are such that* ψ *is a strictly increasing function and* ϕ *is a convex function with respect to* ψ *i.e.,* $\phi \circ \psi^{-1}$ *is convex. Also suppose that*

$$\sum_{i=1}^{k} w_i\, \psi(y_i) \leq \sum_{i=1}^{k} w_i\, \psi(x_i)\,, \quad k = 1, \ldots, n-1, \tag{1.23}$$

and

$$\sum_{i=1}^{n} w_i \,\psi(y_i) = \sum_{i=1}^{n} w_i \,\psi(x_i). \tag{1.24}$$

(i) If $\mathbf{y}$ *is a decreasing n-tuple, then* (1.21) *holds.*

(ii) If $\mathbf{x}$ *is an increasing n-tuple, then the reverse inequality in* (1.21) *holds.*

If $\phi \circ \psi^{-1}$ *is strictly convex and* $\mathbf{x} \neq \mathbf{y}$*, then the strictly inequality holds in* (1.21).

Definition 1.11 ***(Integral majorization)*** *Let x, y be real valued functions defined on an interval* $[a,b]$ *such that* $\int_a^s x(\tau)d\tau$, $\int_a^s y(\tau)d\tau$ *both exist for all* $s \in [a,b]$. *[144, p.324]* $x(\tau)$ *is said to majorize* $y(\tau)$*, in symbol,* $x(\tau) \succ y(\tau)$*, for* $\tau \in [a,b]$ *if they are decreasing in* $\tau \in [a,b]$ *and*

$$\int_a^s y(\tau)\,d\tau \leq \int_a^s x(\tau)\,d\tau \quad for\, s \in [a,b], \tag{1.25}$$

and equality in (1.25) holds for $s = b$.

The following theorem can be regarded as **integral majorization theorem** [144, p.325].

Theorem 1.17 (INTEGRAL MAJORIZATION THEOREM) $x(\tau) \succ y(\tau)$ *for* $\tau \in [a,b]$ *iff they are decreasing in* $[a,b]$ *and*

$$\int_a^b \phi\,(y(\tau))\,d\tau \leq \int_a^b \phi\,(x(\tau))\,d\tau \tag{1.26}$$

holds for every ϕ *that is continuous and convex in* $[a,b]$ *such that the integrals exist.*

The following theorem is a simple consequence of Theorem 1 in [140] (see also [144, p.328]):

Theorem 1.18 *Let* $x(\tau), y(\tau) : [a,b] \to \mathbb{R}$*,* $x(\tau)$ *and* $y(\tau)$ *are continuous and increasing and let* $\mu : [a,b] \to \mathbb{R}$ *be a function of bounded variation.*

(a) If

$$\int_\nu^b y(\tau)\,d\mu(\tau) \leq \int_\nu^b x(\tau)\,d\mu(\tau) \;\; for\ every \;\; \nu \in [a,b], \tag{1.27}$$

and

$$\int_a^b y(\tau)\,d\mu(\tau) = \int_a^b x(\tau)\,d\mu(\tau) \tag{1.28}$$

hold, then for every continuous convex function ϕ*, we have*

$$\int_a^b \phi\,(y(\tau))\,d\mu(\tau) \leq \int_a^b \phi\,(x(\tau))\,d\mu(\tau). \tag{1.29}$$

(b) If (1.27) *holds, then* (1.29) *holds for every continuous increasing convex function* ϕ.

Definition 1.12 *Let* $F(\tau)$, $G(\tau)$ *be two continuous and increasing functions for* $\tau \geq 0$ *such that* $F(0) = G(0) = 0$ *and define*

$$\overline{F}(\tau) = 1 - F(\tau), \quad \overline{G}(\tau) = 1 - G(\tau) \quad for \quad \tau \geq 0. \tag{1.30}$$

(cf.[144], p.330) $\overline{F}(\tau)$ *is said to majorize* $\overline{G}(\tau)$, *in symbol,* $\overline{F}(\tau) \succ \overline{G}(\tau)$, *for* $\tau \in [0, +\infty)$ *if*

$$\int_0^s \overline{G}(\tau)\,d\tau \leq \int_0^s \overline{F}(\tau)\,d\tau \quad \text{for all } s > 0,$$

and

$$\int_0^\infty \overline{G}(\tau)\,d\tau = \int_0^\infty \overline{F}(\tau)\,d\tau < \infty.$$

The following result was obtained by Boland and Proschan (1986) [47] (see [144], p.331):

Theorem 1.19 $\overline{F}(\tau) \succ \overline{G}(\tau)$ *for* $\tau \in [0, +\infty)$ *holds iff*

$$\int_0^\infty \phi(\tau)\,dF(\tau) \leq \int_0^\infty \phi(\tau)\,dG(\tau) \tag{1.31}$$

holds for all convex functions ϕ, *provided the integrals are finite.*

The following theorem is a slight extension of Lemma 2 in [120] which is proved by L. Maligranda, J. Pečarić, L. E. Persson (1995):

Theorem 1.20 ([109]) *Let* w, x *and* y *be positive functions on* $[a,b]$. *Suppose that* ϕ : $[0,\infty) \to \mathbb{R}$ *is a convex function and that*

$$\int_a^\nu y(t)\,w(t)\,dt \leq \int_a^\nu x(t)\,w(t)\,dt, \;\; \nu \in [a,b] \quad and$$

$$\int_a^b y(t)\,w(t)\,dt = \int_a^b x(t)\,w(t)\,dt.$$

(i) If y *is a decreasing function on* $[a,b]$, *then*

$$\int_a^b \phi\left(y(t)\right) w(t)\,dt \leq \int_a^b \phi\left(x(t)\right) w(t)\,dt. \tag{1.32}$$

(i) If x *is an increasing function on* $[a,b]$, *then*

$$\int_a^b \phi\left(x(t)\right) w(t)\,dt \leq \int_a^b \phi\left(y(t)\right) w(t)\,dt. \tag{1.33}$$

If ϕ *is strictly convex function and* $x \neq y$ $(a.e.)$, *then* (1.32) *and* (1.33) *are strict.*

Proof. As in [120], if we prove the inequalities for $\phi \in C^1[0,\infty)$, then the general case follows from the pointwise approximation of ϕ by smooth convex functions.
Since ϕ is a convex function on $[0,\infty)$, it follows that

$$\phi(u_1) - \phi(u_2) \geq \phi'(u_2)(u_1 - u_2).$$

If we set

$$F(v) = \int_a^v [x(t) - y(t)]\, w(t)\, dt,$$

then $F(v) \geq 0$, $v \in [a,b]$, and $F(a) = F(b) = 0$.
Then

$$\begin{aligned}
&\int_a^b [\phi[x(t)] - \phi[y(t)]]\, w(t)\, dt \\
&\geq \int_a^b \phi'[y(t)]\,[x(t) - y(t)]\, w(t)\, dt \\
&= \int_a^b \phi'[y(t)]\, dF(t) \\
&= \left[\phi'[y(t)]\, F(t)\right]_a^b - \int_a^b F(t)\, d\left[\phi'[y(t)]\right] \\
&= -\int_a^b F(t)\, \phi''[y(t)]\, f'(t)\, dt \geq 0.
\end{aligned}$$

The last inequality follows from the convexity of ϕ and y being decreasing.
Similarly, we can prove the case when x is increasing.
If ϕ is strictly convex function and $x \neq y\ (a.e.)$, then

$$\phi[x(t)] - \phi[y(t)] > \phi'[y(t)]\,[x(t) - y(t)] \quad \text{(a. e.)}.$$

Which gives strict inequality in (1.32) and (1.33). □

The following theorem (see [109]) is a slight extension of Theorem 2 in [161] which is proved by J. Pečarić and S. Abramovich (1997):

Theorem 1.21 ([109]) *Let w be a weight function on $[a,b]$ and let x and y be positive functions on $[a,b]$. Suppose $\phi, \psi : [0,\infty) \to \mathbb{R}$ are such that ψ is a strictly increasing function and ϕ is a convex function with respect to ψ i.e., $\phi \circ \psi^{-1}$ is convex. Suppose also that*

$$\int_a^v \psi(y(t))\, w(t)\, dt \leq \int_a^v \psi(x(t))\, w(t)\, dt, \quad v \in [a,b] \tag{1.34}$$

and

$$\int_a^b \psi(y(t))\, w(t)\, dt = \int_a^b \psi(x(t))\, w(t)\, dt. \tag{1.35}$$

(i) If y is a decreasing function on $[a,b]$, then (1.32) holds.

(ii) If x is an increasing function on $[a,b]$, then (1.33) holds.

If $\phi \circ \psi^{-1}$ is strictly convex function and $x \neq y\ (a.e.)$, then the strict inequality holds in (1.32) *and* (1.33).

1.4 Mean Value Theorems

A mean on I^n, where $I \subseteq \mathbb{R}$ is an interval, is every function $M : I^n \to \mathbb{R}$, with property

$$\min\{x_1, x_2, \ldots, x_n\} \le M(x_1, x_2, \ldots, x_n) \le \max\{x_1, x_2, \ldots, x_n\}$$

that holds for every choice of all $x_1, \ldots, x_n \in I$. For mean M we said that is symmetric if for every permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ we have $M(x_1, x_2, \ldots, x_n) = M(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$.

As examples, we present classes of means that follows from the well-known mean value theorems.

Theorem 1.22 (LAGRANGE'S MEAN VALUE THEOREM) *If a function $\varphi : [x, y] \to \mathbb{R}$ is continuous on a closed interval $[x, y]$ and differentiable on the open interval (x, y), then there is at least one point $\xi \in (x, y)$ such that*

$$\varphi'(\xi) = \frac{\varphi(y) - \varphi(x)}{y - x}.$$

Under assumption that a function φ' is invertible, from Lagrange's theorem it follows that there is a unique number

$$\xi = (\varphi')^{-1} \left(\frac{\varphi(y) - \varphi(x)}{y - x} \right)$$

which we called **Lagrange's mean** of $[x, y]$.

Lagrange's mean we can generalize using Cauchy's mean value theorem.

Theorem 1.23 (CAUCHY'S MEAN VALUE THEOREM) *Let functions $\varphi, \psi : [x, y] \to \mathbb{R}$ be continuous on an interval $[x, y]$ and differentiable on (x, y) and let $\psi'(t) \neq 0$ for all $t \in (x, y)$. Then there is a point $\xi \in (x, y)$ such that*

$$\frac{\varphi'(\xi)}{\psi'(\xi)} = \frac{\varphi(y) - \varphi(x)}{\psi(y) - \psi(x)}.$$

Under assumption that a function $\frac{\varphi'}{\psi'}$ is invertible, from Cauchy's theorem it follows that there is a unique number

$$\xi = \left(\frac{\varphi'}{\psi'} \right)^{-1} \left(\frac{\varphi(y) - \varphi(x)}{\psi(y) - \psi(x)} \right).$$

which we called **Cauchy's mean** of interval $[x, y]$. Continuous expansion gives $\xi = x$ if $y = x$.

Remark 1.7 *If we take $\psi(x) = x$, as a special case of Cauchy's mean we get Lagrange's mean. Moreover, many well known means in mathematics we can get as special cases of Cauchy's mean. Under assumption that $x, y \in (0, \infty)$ and choosing $\varphi(x) = x^v$ and $\psi(x) = x^u$, $u, v \in \mathbb{R}$, $u \neq v$, $u, v \neq 0$, we obtain two-parameter mean*