Chapter 1

Basic results and definitions

1.1 Steffensen's inequality

Since its appearance in 1918 Steffensen's inequality has been a subject of investigation by many mathematicians. The book [82] is devoted to generalizations and refinements of Steffensen's inequality and its connection to other inequalities, such as Gauss', Jensen-Steffensen's, Hölder's and Iyengar's inequality.

In this section we recall some important generalizations and refinements of Steffensen's inequality.

The original version from [85] has the following form.

Theorem 1.1 *Suppose that f and g are integrable functions defined on (a,b), f is nonincreasing and for each $t \in (a,b)$ $0 \leq g \leq 1$. Then*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt, \tag{1.1}$$

where

$$\lambda = \int_{a}^{b} g(t)dt. \tag{1.2}$$

Proof. The proof of the second inequality in (1.1) goes as follows.

$$\begin{aligned}
\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt &= \int_a^{a+\lambda} [1-g(t)]f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&\geq f(a+\lambda)\int_a^{a+\lambda} [1-g(t)]dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= f(a+\lambda)\left[\lambda - \int_a^{a+\lambda} g(t)dt\right] - \int_{a+\lambda}^b f(t)g(t)dt \\
&= f(a+\lambda)\int_{a+\lambda}^b g(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_{a+\lambda}^b g(t)[f(a+\lambda)-f(t)]dt \geq 0.
\end{aligned}$$

The first inequality in (1.1) is proved in a similar way, but it also follows from the second one. One merely sets $G(t) = 1-g(t)$ and $\Lambda = \int_a^b G(t)dt$. Since $0 \leq g(t) \leq 1$ on (a,b) implies $0 \leq G(t) \leq 1$ on (a,b) and $b-a = \lambda+\Lambda$. Using the second inequality in (1.1) we obtain

$$\begin{aligned}
\int_a^b f(t)G(t)dt &\leq \int_a^{a+\Lambda} f(t)dt, \\
\int_a^b f(t)[1-g(t)]dt &\leq \int_a^{b-\lambda} f(t)dt, \\
\int_a^b f(t)dt - \int_a^{b-\lambda} f(t)dt &\leq \int_a^b f(t)g(t)dt.
\end{aligned}$$

Hence,

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt,$$

which is the first inequality in (1.1). □

Mitrinović stated in [48] (see also [82, p. 15]) that inequalities in (1.1) follow from the identities

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} [f(t)-f(a+\lambda)][1-g(t)]dt + \int_{a+\lambda}^b [f(a+\lambda)-f(t)]g(t)dt \qquad (1.3)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\
&= \int_a^{b-\lambda} [f(t)-f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b [f(b-\lambda)-f(t)][1-g(t)]dt. \qquad (1.4)
\end{aligned}$$

Applying Steffensen's inequality to appropriate functions, in [45] Masjed-Jamei, Qi and Srivastava obtained the following Steffensen type inequalities:

Theorem 1.2 *If f and g are integrable functions such that f is nonincreasing and*

$$-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \leq g(x) \leq 1-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \tag{1.5}$$

on (a,b), where $q \neq 0$ and

$$\sigma = q\int_a^b g(x)dx,$$

then

$$\begin{aligned}\int_{b-\sigma}^b f(x)dx-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq \int_a^{a+\sigma} f(x)dx-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx.\end{aligned} \tag{1.6}$$

The inequalities (1.6) *are reversed for f nondecreasing.*

Identities (1.3) and (1.4) are starting points for researching the conditions of Steffensen's inequality and eventually changing them. Milovanović and Pečarić in their paper [47], using integration by parts in identities (1.3) and (1.4), obtained weaker conditions on the function g. Vasić and Pečarić in paper [87] showed that this weaker conditions are necessary and sufficient. Hence, we have the following theorem.

Theorem 1.3 *Let f and g be integrable functions on $[a,b]$ and let $\lambda = \int_a^b g(t)dt$.*

(a) The second inequality in (1.1) *holds for every nonincreasing function f if and only if*

$$\int_a^x g(t)dt \leq x-a \text{ and } \int_x^b g(t)dt \geq 0, \quad \text{for every } x \in [a,b]. \tag{1.7}$$

(b) The first inequality in (1.1) *holds for every nonincreasing function f if and only if*

$$\int_x^b g(t)dt \leq b-x \text{ and } \int_a^x g(t)dt \geq 0, \quad \text{for every } x \in [a,b]. \tag{1.8}$$

Using identities (1.3) and (1.4) and integration by parts, Pečarić in [55] also proved conditions for inverse inequalities in (1.1).

Theorem 1.4 *Let $f : I \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ ($[a,b] \subseteq I$ where I is an interval in $\mathbb{R}$) be integrable functions, and $a+\lambda \in I$ where λ is given by* (1.2)*. Then*

$$\int_a^{a+\lambda} f(t)dt \leq \int_a^b f(t)g(t)dt$$

holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt \geq x-a, \text{ for } x \in [a, a+\lambda] \quad \text{and} \quad \int_x^b g(t)dt \leq 0, \text{ for } x \in (a+\lambda, b],$$

and $0 \leq \lambda \leq b-a$;
or

$$\int_a^x g(t)dt \geq x-a, \quad \text{for } x \in [a,b],$$

and $\lambda > b-a$;
or

$$\int_x^b g(t)dt \leq 0, \quad \text{for } x \in [a,b]$$

and $\lambda < 0$.

Theorem 1.5 *Let $f : I \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ ($[a,b] \subseteq I$ where I is an interval in $\mathbb{R}$) be integrable functions, and $b-\lambda \in I$ where λ is given by* (1.2). *Then*

$$\int_{b-\lambda}^b f(t)dt \geq \int_a^b f(t)g(t)dt$$

holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt \leq 0, \text{ for } x \in [a, b-\lambda] \quad \text{and} \quad \int_x^b g(t)dt \geq b-x, \quad \text{for } x \in (b-\lambda, b],$$

and $0 \leq \lambda \leq b-a$;
or

$$\int_x^b g(t)dt \geq b-x, \quad \text{for } x \in [a,b],$$

and $\lambda > b-a$;
or

$$\int_a^x g(t)dt \leq 0, \quad \text{for } x \in [a,b]$$

and $\lambda < 0$.

In 1982 Pečarić proved the following generalization of Steffensen's inequality (see [56]).

Theorem 1.6 *Let h be a positive integrable function on $[a,b]$ and f be an integrable function such that f/h is nondecreasing on $[a,b]$. If g is a real-valued integrable function such that $0 \leq g \leq 1$, then*

$$\int_a^b f(t)g(t)dt \geq \int_a^{a+\lambda} f(t)dt \tag{1.9}$$

holds, where λ is the solution of the equation

$$\int_a^{a+\lambda} h(t)dt = \int_a^b h(t)g(t)dt. \tag{1.10}$$

If f/h is a nonincreasing function, then the reverse inequality in (1.9) *holds.*

Theorem 1.7 *Let the conditions of Theorem 1.6 be fulfilled. Then*

$$\int_a^b f(t)g(t)dt \le \int_{b-\lambda}^b f(t)dt,$$

where λ *is the solution of the equation*

$$\int_{b-\lambda}^b h(t)dt = \int_a^b h(t)g(t)dt. \tag{1.11}$$

For $h(x) = 1$ we have Steffensen's inequality.

In [46] Mercer proved the following generalization of Steffensen's inequality.

Theorem 1.8 *Let* f, g *and* h *be integrable functions on* (a,b) *with* f *nonincreasing and* $0 \le g \le h$. *Then*

$$\int_{b-\lambda}^b f(t)h(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt, \tag{1.12}$$

where λ *is given by*

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt. \tag{1.13}$$

Wu and Srivastava in [93] and Liu in [44] noted that the generalization due to Mercer is incorrect as stated. They have proved that it is true if we add the condition:

$$\int_{b-\lambda}^b h(t)dt = \int_a^b g(t)dt. \tag{1.14}$$

As proven by Pečarić, Perušić and Smoljak in [61], a corrected version of Mercer's result follows from Theorems 1.6 and 1.7, and is stated as following.

Theorem 1.9 *Let* h *be a positive integrable function on* $[a,b]$ *and* f, g *be integrable functions on* $[a,b]$ *such that* f *is nonincreasing on* $[a,b]$ *and* $0 \le g \le h$.

a) Then

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt,$$

where λ *is given by* (1.13).

b) Then

$$\int_{b-\lambda}^b f(t)h(t)dt \le \int_a^b f(t)g(t)dt,$$

where λ *is given by* (1.14).

In [46] Mercer also gave the following theorem.

Theorem 1.10 *Let f,g,h and k be integrable functions on (a,b) with $k>0$, f/k nonincreasing and $0\le g\le h$. Then*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt,$$

where λ is given by

$$\int_a^{a+\lambda} h(t)k(t)dt = \int_a^b g(t)k(t)dt. \tag{1.15}$$

As showed in [82, p. 57] Theorem 1.10 is equivalent to Theorem 1.6.

Next, we recall a corrected and refined version of Mercer's result given by Wu and Srivastava in [93].

Theorem 1.11 *Let f,g and h be integrable functions on $[a,b]$ with f nonincreasing and let $0\le g\le h$. Then the following integral inequalities hold true*

$$\begin{aligned}\int_{b-\lambda}^b f(t)h(t)dt &\le \int_{b-\lambda}^b (f(t)h(t)-[f(t)-f(b-\lambda)][h(t)-g(t)])dt\\ &\le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} (f(t)h(t)-[f(t)-f(a+\lambda)][h(t)-g(t)])dt\\ &\le \int_a^{a+\lambda} f(t)h(t)dt,\end{aligned}$$

where λ satisfies

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt = \int_{b-\lambda}^b h(t)dt. \tag{1.16}$$

Motivated by refinement of Steffensen's inequality given in [93], Pečarić, Perušić and Smoljak [61] obtained the following refined version of results given in Theorems 1.6 and 1.7.

Corollary 1.1 *Let h be a positive integrable function on $[a,b]$ and f,g be integrable functions on $[a,b]$ such that f/h is nonincreasing and $0\le g\le 1$. Then*

$$\begin{aligned}\int_a^b f(t)g(t)dt &\le \int_a^{a+\lambda} \left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(a+\lambda)}{h(a+\lambda)}\right]h(t)[1-g(t)]\right)dt\\ &\le \int_a^{a+\lambda} f(t)dt,\end{aligned} \tag{1.17}$$

where λ is given by (1.10).
If f/h is a nondecreasing function, then the reverse inequality in (1.17) *holds.*

Corollary 1.2 *Let h be a positive integrable function on $[a,b]$ and f,g be integrable functions on $[a,b]$ such that f/h is nonincreasing and $0 \leq g \leq 1$. Then*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{b-\lambda}^{b} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda)}{h(b-\lambda)} \right] h(t)[1-g(t)] \right) dt \\ \leq \int_{a}^{b} f(t)g(t)dt \tag{1.18}$$

where λ is given by (1.11).
If f/h is a nondecreasing function, then the reverse inequality in (1.18) *holds.*

Furthermore, in [93] Wu and Srivastava proved a new sharpened and generalized version of inequality (1.12). In [61] authors separated this result into two theorems to obtain weaker conditions on λ.

Theorem 1.12 *Let f,g,h and ψ be integrable functions on $[a,b]$ with f nonincreasing and let $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$, $t \in [a,b]$. Then*

$$\int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)h(t)dt - \int_{a}^{b} |f(t) - f(a+\lambda)|\, \psi(t)dt$$

where λ is given by (1.13).

Theorem 1.13 *Let f,g,h and ψ be integrable functions on $[a,b]$ with f nonincreasing and let $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$, $t \in [a,b]$. Then*

$$\int_{b-\lambda}^{b} f(t)h(t)dt + \int_{a}^{b} |f(t) - f(b-\lambda)|\, \psi(t)dt \leq \int_{a}^{b} f(t)g(t)dt$$

where λ is given by (1.14).

The following theorem is Cerone's generalization of Steffensen's inequality given in [15]. This generalization allows bounds that involve any two subintervals instead of restricting them to include the end points.

Theorem 1.14 *Let $f,g : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ and let f be nonincreasing. Further, let $0 \leq g \leq 1$ and*

$$\lambda = \int_{a}^{b} g(t)dt = d_i - c_i,$$

where $[c_i,d_i] \subseteq [a,b]$ for $i = 1,2$ and $d_1 \leq d_2$. Then

$$\int_{c_2}^{d_2} f(t)dt - r(c_2,d_2) \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{c_1}^{d_1} f(t)dt + R(c_1,d_1)$$

holds, where

$$r(c_2,d_2) = \int_{d_2}^{b} (f(c_2) - f(t))g(t)dt \geq 0$$

and

$$R(c_1,d_1) = \int_{a}^{c_1} (f(t) - f(d_1))g(t)dt \geq 0.$$

In 1959 Bellman gave an L^p generalization of Steffensen's inequality (see [11]). As noted by many mathematicians Bellman's result is incorrect as stated. A comprehensive survey of corrected versions and generalizations of Bellman's result can be found in [82]. In the following theorem we recall generalization of Bellman's result obtained by Pachpatte in [53].

Theorem 1.15 *Let f,g,h be real-valued integrable functions defined on $[0,1]$ such that $f(t) \geq 0$, $h(t) \geq 0$, $t \in [0,1]$, f/h is nonincreasing on $[0,1]$ and $0 \leq g \leq A$, where A is a real positive constant. If $p \geq 1$, then*

$$\left(\int_0^1 g(t)f(t)dt\right)^p \leq A^p \int_0^\lambda f^p(t)dt, \tag{1.19}$$

where λ is the solution of the equation

$$\int_0^\lambda h^p(t)dt = \frac{1}{A^p}\left(\int_0^1 h^p(t)g(t)dt\right)\left(\int_0^1 g(t)dt\right)^{p-1}.$$

In [24] Gauss mentioned the following inequality:

Theorem 1.16 *If f is a nonnegative nonincreasing function and $k > 0$, then*

$$\int_k^\infty f(x)dx \leq \frac{4}{9k^2}\int_0^\infty x^2 f(x)dx. \tag{1.20}$$

In [59] Pečarić proved the following result.

Theorem 1.17 *Let $G : [a,b] \to \mathbb{R}$ be an increasing function and let $f : I \to \mathbb{R}$ be a nonincreasing function (I is an interval from $\mathbb{R}$ such that $a,b,G(a),G(b) \in I$). If $G(x) \geq x$ then*

$$\int_{G(a)}^{G(b)} f(x)dx \leq \int_a^b f(x)G'(x)dx. \tag{1.21}$$

If $G(x) \leq x$, the reverse inequality in (1.21) is valid.
If f is a nondecreasing function and $G(x) \geq x$ then the inequality (1.21) is reversed.

Inequality (1.21) is usually called Gauss-Steffensen's inequality. As pointed out in [82] Gauss-Steffensen's inequality includes as special cases three famous inequalities: Volkov's, Steffensen's and Ostrowski's inequality.

In [9] Alzer gave a lower bound for Gauss' inequality (1.20). In fact, he proved the following theorem.

Theorem 1.18 *Let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable, and let $f : I \to \mathbb{R}$ be nonincreasing function. Then*

$$\int_a^b f(s(x))g'(x)dx \leq \int_{g(a)}^{g(b)} f(x)dx \leq \int_a^b f(t(x))g'(x)dx, \tag{1.22}$$

where

$$s(x) = \frac{g(b)-g(a)}{b-a}(x-a)+g(a), \tag{1.23}$$

and

$$t(x) = g'(x_0)(x - x_0) + g(x_0), \quad x_0 \in [a,b]. \tag{1.24}$$

(I is an interval containing $a, b, g(a), g(b), t(a)$ and $t(b)$.)

If either g is concave or f is nondecreasing, then the reversed inequalities hold.

Remark 1.1 If we consider only the left-hand side inequality in (1.22), interval I should only contain $a, b, g(a)$ and $g(b)$. When considering the right-hand side inequality in (1.22), interval I should also contain $t(a)$ and $t(b)$.

1.2 Convex functions

In this section we give definitions and some properties of convex functions. Convex functions are very important in the theory of inequalities. The third chapter of the classical book of Hardy, Littlewood and Pólya [27] is devoted to the theory of convex functions (see also [52]).

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $f\colon I \to \mathbb{R}$ is called* convex *if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.25}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex *if the inequality* (1.25) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If the inequality in (1.25) *is reversed, then f is said to be* concave. *It is called* strictly concave *if the reversed inequality* (1.25) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If f is both convex and concave, f is said to be affine.

Remark 1.2 (a) For $x, y \in I$, $p, q \ge 0$, $p + q > 0$, (1.25) is equivalent to

$$f\left(\frac{px+qy}{p+q}\right) \le \frac{pf(x)+qf(y)}{p+q}.$$

(b) The simple geometrical interpretation of (1.25) is that the graph of f lies below its chords.

(c) If x_1, x_2, x_3 are three points in I such that $x_1 < x_2 < x_3$, then (1.25) is equivalent to

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} = (x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \ge 0$$

which is equivalent to

$$f(x_2) \le \frac{x_2 - x_3}{x_1 - x_3} f(x_1) + \frac{x_1 - x_2}{x_1 - x_3} f(x_3),$$

or, more symmetrically and without the condition of monotonicity on x_1, x_2, x_3

$$\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)} + \frac{f(x_2)}{(x_2-x_3)(x_2-x_1)} + \frac{f(x_3)}{(x_3-x_1)(x_3-x_2)} \geq 0.$$

Proposition 1.1 *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 1.2 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* convex in the Jensen sense, *or* J-convex on I (midconvex, midpoint convex) *if for all points $x, y \in I$ the inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.26}$$

holds. A J-convex function is said to be strictly J-convex *if for all pairs of points $(x,y), x \neq y$, strict inequality holds in* (1.26).

In the context of continuity the following criteria of equivalence of (1.25) and (1.26) is valid.

Theorem 1.19 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is a convex function if and only if f is a J-convex function.*

Definition 1.3 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* Wright convex *function if for each $x \leq y$, $z \geq 0$, $x, y+z \in I$, the inequality*

$$f(x+z) - f(x) \leq f(y+z) - f(y)$$

holds.

Next, we want do define convex functions of higher order, but first we need to define divided differences.

Definition 1.4 *Let f be a function defined on $[a,b]$. The n*-th order divided difference *of f at distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$ is defined recursively by*

$$[x_j; f] = f(x_j), \quad j = 0, \ldots, n$$

and

$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{1.27}$$

Remark 1.3 The value $[x_0,x_1,\ldots,x_n;f]$ is independent of the order of the points $x_0,\ldots,x_n$. Previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \le \cdots \le x_k$ and letting

$$\underbrace{[x,\ldots,x}_{j+1 \text{ times}};f] = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}(x)$ exists. Note that (1.27) is equivalent to

$$[x_0,\ldots,x_n;f] = \sum_{k=0}^{n} \frac{f(x_k)}{\omega'(x_k)}, \text{ where } \omega'(x_k) = \prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j).$$

Definition 1.5 *Let $n \in \mathbb{N}$. Function $f : [a,b] \to R$ is said to be n-*convex *on $[a,b]$ if and only if for every choice of $n+1$ distinct points $x_0,x_1,\ldots,x_n$ in $[a,b]$*

$$[x_0,x_1,\ldots,x_n;f] \ge 0. \tag{1.28}$$

If the inequality in (1.28) *is reversed, function f is said to be n-*concave *on $[a,b]$. If the inequality is strict, f is said to be* strictly $n-$convex ($n-$concave) *function.*

Remark 1.4 Specially, $0-$convex function is nonnegative function, $1-$convex function is nondecreasing function, 2-convex function is convex function.

Theorem 1.20 *If $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \ge 0$.*

Definition 1.6 *A positive function f is said to be* logarithmically convex *on an interval $I \subseteq \mathbb{R}$ if $\log f$ is a convex function on I, or equivalently if for all $x,y \in I$ and all $\alpha \in [0,1]$*

$$f(\alpha x + (1-\alpha)y) \le f^{\alpha}(x) f^{1-\alpha}(y). \tag{1.29}$$

For such function f, we shortly say f is log-convex.
It is said to be log-concave *if the inequality in* (1.29) *is reversed.*

Definition 1.7 *A positive function f is said to be* log-convex in the Jensen sense *if for each $x,y \in I$*

$$f^2\left(\frac{x+y}{2}\right) \le f(x)f(y)$$

holds, i.e. if $\log f$ is convex in the Jensen sense.

As a consequence of results from Remark 1.2 (c) and Proposition 1.1 we get the following inequality for log-convex function:

$$[f(b)]^{c-a} \le [f(a)]^{c-b}\,[f(c)]^{b-a}. \tag{1.30}$$

Corollary 1.3 *For a log-convex function f on interval I and $p,q,r,s \in I$ such that $p \le r,\ q \le s,\ p \ne q,\ r \ne s$, it holds*

$$\left(\frac{f(p)}{f(q)}\right)^{\frac{1}{p-q}} \le \left(\frac{f(r)}{f(s)}\right)^{\frac{1}{r-s}}. \tag{1.31}$$

Inequality (1.31) is known as Galvani's theorem for log-convex functions $f\colon I \to \mathbb{R}$.

At the end of this introductory section we overview one subclass of convex functions, so-called $s-$convex functions (see [14]).

Definition 1.8 *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex if*

$$f(\alpha x + (1-\alpha)y) \le \alpha^s f(x) + (1-\alpha)^s f(y). \tag{1.32}$$

for all $x, y \in [0,\infty)$ and $\alpha \in [0,1]$

This class of function is recently even further refined (for details see [88]).

Definition 1.9 *Let J be an open interval and $h : J \to \mathbb{R}$ non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an $h-$convex function if f is non-negative and for all $x,y \in I$ and $\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \le h(\alpha) f(x) + h(1-\alpha) f(y). \tag{1.33}$$

1.3 Exponentially convex functions

In this section we introduce definition of exponential convexity as given by Bernstein in [12] (see also [7], [50], [51]). In this section I is an open interval in $\mathbb{R}$.

Definition 1.10 *A function $h : I \to \mathbb{R}$ is said to be* exponentially convex *on I if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \ge 0$$

holds for every $n \in \mathbb{N}$ and all sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in I$, $1 \le i, j \le n$.

The following Proposition follows directly from the previous Definition.

Proposition 1.2 *For function $h : I \to \mathbb{R}$ the following statements are equivalent:*

(i) *h is exponentially convex*

(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.34}$$

for all $n \in \mathbb{N}$, all sequences $(\xi_n)_{n\in\mathbb{N}}$ of real numbers, and all sequences $(x_n)_{n\in\mathbb{N}}$ in I.

Note that for $n = 1$, it follows from (1.34) that exponentially convex function is non-negative.

Directly from a definition of positive semi-definite matrix and inequality (1.34) we get the following result.

Corollary 1.4 *If h is exponentially convex on I, then the matrix*

$$\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}$$

is a positive semi-definite matrix. Specially,

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0, \tag{1.35}$$

for every $n \in \mathbb{N}$ and every choice $x_i \in I$, $i = 1, \ldots, n$.

Remark 1.5 Note that for $n = 2$ from (1.35) we obtain

$$h(x_1)h(x_2) - h^2\left(\frac{x_1 + x_2}{2}\right) \geq 0.$$

Hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also log-convex function.

We continue with the definition of n-exponentially convex function.

Definition 1.11 *A function $h : I \to \mathbb{R}$ is* n-exponentially convex in the Jensen sense *on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices of $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$.

A function $h : I \to \mathbb{R}$ is *n-exponentially convex* on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.6 It is clear from the definition that 1-exponentially convex functions in the Jensen sense are nonnegative functions.

Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \leq n$, $k \in \mathbb{N}$.

A function $h : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

One of the most important properties of exponentially convex functions is their integral representation.

Theorem 1.21 *The function $\psi : I \to \mathbb{R}$ is exponentially convex on I if and only if*

$$\psi(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in I$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof. See [7, p. 211]. □

Remark 1.7 A function $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense, i.e.

$$\psi\left(\frac{x+y}{2}\right)^2 \leq \psi(x)\psi(y), \qquad \text{for all } x, y \in I, \tag{1.36}$$

if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$, i.e., if and only if ψ is 2-exponentially convex in the Jensen sense. By induction from (1.36) we have

$$\psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) \leq \psi(x)^{\frac{1}{2^k}} \psi(y)^{1-\frac{1}{2^k}}.$$

Therefore, if ψ is continuous and $\psi(x) = 0$ for some $x \in I$, then from the last inequality and nonnegativity of ψ (see Remark 1.6) we get

$$\psi(y) = \lim_{k\to\infty} \psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) = 0 \quad \text{ for all } y \in I.$$

Hence, 2-exponentially convex function is either identically equal to zero or it is strictly positive and log-convex.

1.4 Functions convex at point c

In this section we introduce definition of a class of functions that extends the class of convex functions as given by Pečarić and Smoljak in [75]

Definition 1.12 *Let $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$. We say that f belongs to class $\mathcal{M}_1^c[a,b]$ (f belongs to class $\mathcal{M}_2^c[a,b]$) if there exists a constant A such that the function $F(x)=f(x)-Ax$ is nonincreasing (nondecreasing) on $[a,c]$ and nondecreasing (nonincreasing) on $[c,b]$.*

If $f\in\mathcal{M}_1^c[a,b]$ or $f\in\mathcal{M}_2^c[a,b]$ and $f'(c)$ exists, then $f'(c)=A$.
Let us show this for $f\in\mathcal{M}_1^c[a,b]$. Since F is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$ for any distinct points $x_1,x_2\in[a,c]$ and $y_1,y_2\in[c,b]$ we have

$$[x_1,x_2;F]=[x_1,x_2;f]-A\le 0\le[y_1,y_2;f]-A=[y_1,y_2;F].$$

Therefore, since $f'_-(c)$ and $f'_+(c)$ exist, letting $x_1=y_1=c$, $x_2\nearrow c$ and $y_2\searrow c$ we get

$$f'_-(c)\le A\le f'_+(c). \tag{1.37}$$

Remark 1.8 We mention here that Florea and Păltănea recently introduced (see [21]) the following more general definition of the convexity of a function $f:[a,b]\to\mathbb{R}$ at a point $c\in(a,b)$:

$$f(c)+f(x+y-c)\le f(x)+f(y),$$

for all $x,y\in[a,b]$ such that $x\le c\le y$. This property is denoted by $f\in\mathrm{Conv}_c([a,b])$. We can easily state that $\mathcal{M}_1^c[a,b]\subset\mathrm{Conv}_c([a,b])$, but the two classes of punctual convex functions are not equal. For example, consider the function

$$f(x)=\begin{cases}|x|, & x\in[-1,1];\\ 2-|x|, & x\in[-2,2]\setminus[-1,1].\end{cases}$$

We have $f\in\mathrm{Conv}_0([-2,2])$ (see Example 2 in [21]). On the other hand, clearly $f\notin\mathcal{M}_1^0[-2,2]$.

In the following lemma and theorem we give a connection between the class of functions $\mathcal{M}_1^c[a,b]$ and the class of convex functions which was obtained in [75].

Lemma 1.1 *If $f:[a,b]\to\mathbb{R}$ is convex (concave), then $f\in\mathcal{M}_1^c[a,b]$ ($f\in\mathcal{M}_2^c[a,b]$) for every $c\in(a,b)$.*

Proof. If f is convex, then f'_- and f'_+ exist (see [71]). Hence, for every $x_1,x_2\in[a,c]$ and $y_1,y_2\in[c,b]$ it holds

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\le f'_-(c)\le f'_+(c)\le\frac{f(y_2)-f(y_1)}{y_2-y_1}.$$